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SOLVING OF A NEUMANN BOUNDARY VALUE PROBLEM THROUGH VARIATIONAL METHODS

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Abstract. In this work, applying the multiple critical points theorems, we obtain the existence results of two and three classical solutions for a Neumann boundary value problem with the Sturm-Liouville equation.

Keywords: Neumann boundary value problem; Sturm-Liouville equation; Critical points theory; Variational method.

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1. Introduction.

The multiplicity results for a two-point boundary value problem obtained in [3] have inspired a large number of authors in the study of several kinds of boundary value problems (BVPs). In these latest years the study of BVPs increased because it arises in many fields, such as physical problems, nonlinear elasticity theory or mechanics, and engineering topics. For instance, vibrations of a guy-wire with uniform cross section possessing parts of different densities can be modeled with a multi-point BVP [13]; moreover, certain problems belonging to the theory of elastic stability can be set up as multi-point BVPs [18].

Our aim in this paper is to establish multiple solutions for a particular case of the following Neumann boundary value problem

$$\begin{cases} -(\overline{p}u')' + \overline{r}u' + \overline{q}u = \lambda f(x,u), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\overline{p} \in C^1([0,1])$, $\overline{q}, \overline{r} \in C^0([0,1])$, with $\overline{p}$ and $\overline{q}$ positive functions, and λ is a positive parameter.

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There is extensive literature that deals with multiplicity results for such a problem (see [6, 16, 17] and references therein) and, in the last few years, the existence of infinitely many solutions to Neumann problems has been widely investigated, for instance, in [7, 8, 9, 10, 11, 12, 14]. Also, in [16, 17], for the case $\overline{p} = 1$, $\overline{q} = 1$, $\overline{r} = 0$ and $\lambda = 1$, by using fixed point theorems, the existence of three solutions is established under a suitable behaviour of nonlinear term f which, in addition, must be sublinear at infinity. In this paper, we use some ideas of [5, 6].

2. Preliminaries

The variational approach, together with the critical point theory, is one of the important methods in the study of two-point boundary value problems of ordinary differential equation. Our main tools, in this paper, are critical point theorems. Here we recall them.

Theorem 2.1. *(*[2, Theorem B]*). Let X be a reflexive real Banach space, $\phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteauxderivative is compact. Put, for each $r > inf_X(\phi)$,*

$$\varphi_1(r) = \inf_{x\in\phi^{-1}(]-\infty,r[)} \frac{\psi(x) - \inf_{\overline{\phi^{-1}(]-\infty,r[)}^w} \psi}{r - \phi(x)},$$

$$\varphi_2(r) = \inf_{x\in\phi^{-1}(]-\infty,r[)} \sup_{y\in\phi^{-1}([r,+\infty[)} \frac{\psi(x) - \psi(y)}{\phi(y) - \phi(x)},$$

where $\overline{\phi^{-1}(]-\infty,r[)}^w$ is the closure of $\phi^{-1}(]-\infty,r[)$ in the weak topology, and assume that

(i) *There is $r \in \mathbb{R}$ such that $inf_X\phi < r$ and $\varphi_1(r) < \varphi_2(r)$.*

Further, assume that:

(ii) *$lim_{\|x\|\to+\infty}(\phi(x) + \lambda\psi(x)) = +\infty$, for all $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$.*

Then, for each $\lambda \in]\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}[$ the functional $\phi + \lambda\psi$ admits at least three distinct critical point in X.

We also use the following theorem concerning two critical points.

Theorem 2.2. *(*[4, Theorem 1.1]*). Let X be a reflexive real Banach space, and let $\phi, \psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differential functionals. Assume that ϕ is (strongly) continuous and $lim_{\|x\|\to+\infty}\phi(x) = +\infty$. Assume also that there exist two constants r_1 and r_2 such that*

(j) *$inf_X\phi < r_1 < r_2$;*

(jj) *$\varphi_1(r_1) < \varphi_2^*(r_1, r_2)$;*

(jjj) *$\varphi_1(r_2) < \varphi_2^*(r_1, r_2)$,*

where φ_1 is defined as in Theorem 2.1 and

$$\varphi_2^*(r_1, r_2) = \inf_{x\in\phi^{-1}(]-\infty,r_1[)} \sup_{y\in\phi^{-1}([r_1,r_2[)} \frac{\psi(x) - \psi(y)}{\phi(y) - \phi(x)}.$$

Then, for each $\lambda \in]\frac{1}{\varphi_2^*(r_1,r_2)}, min\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\}[$, *the functional* $\phi + \lambda\psi$ *admits at least two critical points which lie in* $\phi^{-1}(]-\infty, r_1[)$ *and* $\phi^{-1}([r_1, r_2[)$ *respectively.*

We recall that Theorem 2.1 and Theorem 2.2 are based on the variational principle stated by Ricceri [15].

3. Main results

Consider the Neumann boundary value problem with Sturm-Liouville equation

$$\begin{cases} -(pu')' + qu = \lambda f(x,u), \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $p \in C^1([0,1])$, $q \in C^0([0,1])$, with $p_0 = min_{x\in[0,1]} p(x) > 0$ and $q_0 = min_{x\in[0,1]} q(x) > 0$, and λ is a positive parameter.

Throughout the paper, we put

$$F(x,t) = \int_0^t f(x,\xi)d\xi, \text{ for all } (x,t) \in [0,1] \times \mathbb{R}. \tag{3.2}$$

Also, let X be the Sobolev space $W^{1,2}([0,1])$ equipped with the norm

$$\|u\| = \Big(\int_0^1 p(x)|u'(x)|^2 dx + \int_0^1 q(x)|u(x)|^2 dx \Big)^{\frac{1}{2}}, \tag{3.3}$$

and, for each $u \in X$, consider

$$\phi(u) = \frac{1}{2}\|u\|^2, \ \psi(u) = -\int_0^1 F(x, u(x))dx. \tag{3.4}$$

Remark 3.1. It is known that the critical points of the functional $\phi + \lambda\psi$ in X, are exactly the classical solutions of problem (3.1).

In the sequel, let $m = min\{p_0, q_0\}$ and define

$$k = \frac{m}{\|q\|_1}, \ \|q\|_1 = \int_0^1 q(x)dx. \tag{3.5}$$

Remark 3.2. If c and d are two positive constants such that $d > c$, we can easily check that $d^2 > c^2 k$.

Remark 3.3. It is well known that $(X, \|.\|)$ is compactly embedded in $C^0([0,1], \mathbb{R})$ (see for instance [1]). Hence

$$\widetilde{k} = \sup_{u \in X\setminus\{0\}} \frac{\|u\|_{C^0}}{\|u\|} < \infty, \tag{3.6}$$

and so

$$\|u\|_{C^0} \leq \widetilde{k}\|u\|, \ for \ all \ u \in X. \tag{3.7}$$

Theorem 3.4. *Assume that:*

(i) *there exist two positive constants c and d with d > c, such that*

$$\frac{\int_0^1 \max\limits_{t\in[-c,c]} F(x,t)dx}{c^2} < \frac{k}{3} \frac{\int_0^1 F(x,d)dx}{d^2}, \tag{3.8}$$

(ii) *there exist two positive constants* a *and* s *with* $s < 2$ *and* $a < \frac{2}{m(c\widetilde{k})^2}\int_0^1 \max_{t\in[-c,c]} F(x,t)dx$ *such that*

$$F(x,t) \le a(1+|t|^s), \text{ for all } (x,t) \in [0,1]\times\mathbb{R}. \tag{3.9}$$

Then, for each $\lambda \in \left]\frac{3}{4}\frac{d^2\|q\|_1}{\int_0^1 F(x,d)dx}, \frac{mc^2}{4\int_0^1 \max_{t\in[-c,c]} F(x,t)dx}\right[$, *problem* (3.1) *admits at least three classical solutions.*

Proof. We prove this theorem, applying Theorem 2.1. Let $X = W^{1,2}([0,1])$ and ϕ,ψ are the functionals defined in (3.4). It is known that X is a reflexive Banach space, $\phi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable, coercive and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and $\psi : X \to \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. We have $inf_X(\phi) = 0$. For each $r > 0$, consider the functions $\varphi_1(r)$ and $\varphi_2(r)$ in Theorem 2.1. We show that there exits $r_0 > 0$ such that $\varphi_1(r_0) < \varphi_2(r_0)$.
Claim: $r_0 = m(\frac{c}{2})^2$.
We have $|u(t)| \le \sqrt{\frac{2}{m}}\|u\|$, for all $u \in X$ and all $t \in [0,1]$. Since $inf_X(\phi) = \phi(0) = 0$ and for each $r > 0$

$$0 \in \phi^{-1}(]-\infty,r[),\ \overline{\phi^{-1}(]-\infty,r[)}^{w} = \phi^{-1}(]-\infty,r]), \tag{3.10}$$

for any fixed $r > 0$ we have

$$\begin{aligned}\varphi_1(r) &= \inf_{u\in\phi^{-1}(]-\infty,r[)} \frac{\psi(u) - \inf_{\overline{\phi^{-1}(]-\infty,r[)}^{w}}\psi}{r-\phi(u)}\\ &\le \frac{-\inf_{\|u\|^2\le 2r}\Big(-\int_0^1 F(x,u(x))dx\Big)}{r}\\ &= \frac{\sup_{\|u\|^2\le 2r}\int_0^1 F(x,u(x))dx}{r}.\end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned}\varphi_1(r_0) &\le \frac{\int_0^1 \max_{|t|\le c} F(x,t)dx}{r_0}\\ &= \frac{4\int_0^1 \max_{|t|\le c} F(x,t)dx}{mc^2}\\ &< \frac{4}{3}\frac{k}{m}\frac{\int_0^1 F(x,d)dx}{d^2}.\end{aligned} \tag{3.12}$$

From here, we conclude that $\frac{1}{\varphi_1(r_0)} > \frac{mc^2}{4\int_0^1 \max_{|t|\le c} F(x,t)dx}$. Now, we claim that

$$\varphi_2(r_0) > \frac{4}{3}\frac{k}{m}\frac{\int_0^1 F(x,d)dx}{d^2}, \tag{3.13}$$

which from (3.13) one has $\frac{1}{\varphi_2(r_0)} < \frac{3}{4}\frac{d^2\|q\|_1}{\int_0^1 F(x,d)dx}$. In order to prove (3.13), we fix $v = d$. Clearly $v \in X$ and one has

$$\int_0^1 F(x,v(x))dx = \int_0^1 F(x,d)dx, \tag{3.14}$$

$$\|v\|^2 = \int_0^1 p(x)|v'(x)|^2dx + \int_0^1 q(x)|v(x)|^2dx = d^2\|q\|_1 = d^2\frac{m}{k}. \tag{3.15}$$

Hence, for each $u \in X$ with $\|u\|^2 < 2r$ one has

$$\begin{aligned}\frac{\int_0^1 F(x,v(x))dx - \int_0^1 F(x,u(x))dx}{\|v\|^2 - \|u\|^2} &\geq \frac{\int_0^1 F(x,d)dx - \int_0^1 F(x,u(x))dx}{\|v\|^2} \\ &= \frac{k}{md^2}\Big(\int_0^1 F(x,d)dx - \int_0^1 F(x,u(x))dx\Big),\end{aligned} \tag{3.16}$$

and therefore

$$\begin{aligned}\varphi_2(r_0) &\geq 2\inf_{\|u\|^2\leq 2r_0}\frac{\int_0^1 F(x,v(x))dx - \int_0^1 F(x,u(x))dx}{\|v\|^2 - \|u\|^2} \\ &> \frac{4}{3}\frac{k}{m}\frac{\int_0^1 F(x,d)dx}{d^2},\end{aligned} \tag{3.17}$$

since in the case $r = r_0$, as we see in above, for any $u \in X$ with $\|u\|^2 \leq 2r_0$ one has

$$\begin{aligned}\int_0^1 F(x,u(x))dx &\leq \sup_{\|u\|^2\leq 2r_0}\int_0^1 F(x,u(x))dx \\ &\leq \int_0^1 \max_{|t|\leq c} F(x,t)dx \\ &< \frac{c^2k}{3}\frac{\int_0^1 F(x,d)dx}{d^2} \\ &< \frac{1}{3}\int_0^1 F(x,d)dx.\end{aligned} \tag{3.18}$$

Fix $\lambda \in \left]\frac{3}{4}\frac{d^2\|q\|_1}{\int_0^1 F(x,d)dx}, \frac{mc^2}{4\int_0^1 \max\limits_{|t|\leq c} F(x,t)dx}\right[$. Since

$$\int_0^1 \big(1 + |u(x)|^s\big)dx \leq \int_0^1 \big(1 + |u(x)|^2\big)dx \leq 1 + \|u\|^2_{C^0} \leq 1 + \tilde{k}^2\|u\|^2, \tag{3.19}$$

we have

$$\begin{aligned}\phi(u)+\lambda\psi(u) &= \frac{1}{2}\|u\|^2-\lambda\int_0^1 F(x,u(x))dx\\ &\geq \frac{1}{2}\|u\|^2-\lambda a\int_0^1 \big(1+|u(x)|^s\big)dx\\ &\geq \frac{1}{2}\|u\|^2-\lambda a\big(1+\tilde{k}^2\|u\|^2\big)\\ &= \frac{1}{2}\big(1-2\lambda a\tilde{k}^2\big)\|u\|^2-\lambda a\\ &> \frac{1}{2}\Big(1-\frac{mc^2a\tilde{k}^2}{2\int_0^1 \max_{|t|\leq c} F(x,t)dx}\Big)\|u\|^2-\lambda a,\end{aligned} \tag{3.20}$$

which shows that $lim_{\|u\|\to+\infty}\big(\phi(u)+\lambda\psi(u)\big)=+\infty$, and the conclusion is achieved. □

The next theorem guarantee two solutions for problem (3.1)

Theorem 3.5. *Assume that there exist three positive constants c_1, c_2, d with $c_1 < d < c_2$ such that*

$$max\left\{\frac{\int_0^1 \max_{|t|\leq c_1} F(x,t)dx}{c_1^2},\frac{\int_0^1 \max_{|t|\leq c_2} F(x,t)dx}{c_2^2}\right\}\leq \frac{k}{3}\frac{\int_0^1 F(x,d)dx}{d^2}. \tag{3.21}$$

Then for all $\lambda \in \left]\frac{3}{4}\frac{d^2\|q\|_1}{\int_0^1 F(x,d)dx},\min\left\{\frac{mc_1^2}{4\int_0^1 \max_{|t|\leq c_1} F(x,t)dx},\frac{mc_2^2}{4\int_0^1 \max_{|t|\leq c_2} F(x,t)dx}\right\}\right[$, the problem (3.1) admits two classical solutions u_1, u_2 such that $|u_1(t)| < c_1$ and $|u_2(t)| < c_2$ for all $t \in [0,1]$.

Proof. We prove this theorem applying Theorem 2.2. Let $X = W^{1,2}([0,1])$ and ϕ, ψ are the functionals defined in (3.4). It is known that X, ϕ and ψ are suitable. Namely, X is a reflexive Banach space and ϕ, ψ are two sequentially weakly lower semicontinuous and Gâteaux differential functionals. Also, ϕ is (strongly) continuous and $lim_{\|x\|\to+\infty}\phi(x) = +\infty$. We have $inf_X(\phi) = 0$.
Claim: $r_1 = m(\frac{c_1}{2})^2$ and $r_2 = m(\frac{c_2}{2})^2$. Since $c_1 < c_2$, it is obvious that $inf_X\phi < r_1 < r_2$. Similar to (3.12), we obtain the following statements:

$$\varphi_1(r_1)\leq \frac{4\int_0^1 \max_{|t|\leq c_1} F(x,t)dx}{mc_1^2}, \tag{3.22}$$

$$\varphi_1(r_2)\leq \frac{4\int_0^1 \max_{|t|\leq c_2} F(x,t)dx}{mc_2^2}, \tag{3.23}$$

and therefore by (3.21) one has

$$\max\{\varphi_1(r_1),\varphi_1(r_2)\}\leq \frac{4}{3}\frac{k}{m}\frac{\int_0^1 F(x,d)dx}{d^2}. \tag{3.24}$$

In what follows, we want to prove that

$$\varphi_2^*(r_1,r_2) > \frac{4}{3}\frac{k}{m}\frac{\int_0^1 F(x,d)dx}{d^2}. \tag{3.25}$$

As the proof of Theorem 3.4, if $v = d$ then $v \in X$ and $\|v\|^2 = d^2 \frac{m}{k}$. Hence, for each $u \in X$ with $\|u\|^2 < 2r$ one has

$$\frac{\int_0^1 F(x, v(x))dx - \int_0^1 F(x, u(x))dx}{\|v\|^2 - \|u\|^2} \geq \frac{k}{md^2}\Big(\int_0^1 F(x, d)dx - \int_0^1 F(x, u(x))dx\Big), \tag{3.26}$$

and therefore

$$\begin{aligned} \varphi_2^*(r_1, r_2) &\geq \inf_{u \in \phi^{-1}(-\infty, r_1)} \frac{\psi(u) - \psi(v)}{\phi(v) - \phi(u)} \\ &= 2 \inf_{\|u\|^2 < 2r_1} \frac{\int_0^1 F(x, v(x))dx - \int_0^1 F(x, u(x))dx}{\|v\|^2 - \|u\|^2} \\ &> \frac{4}{3}\frac{k}{m}\frac{\int_0^1 F(x, d)dx}{d^2}, \end{aligned} \tag{3.27}$$

since, for each $u \in X$ with $\|u\|^2 < 2r_1$ we have

$$\begin{aligned} \int_0^1 F(x, u(x))dx &\leq \sup_{\|u\|^2 < 2r_1} \int_0^1 F(x, u(x))dx \\ &\leq \int_0^1 \max_{|t| \leq c_1} F(x, t)dx \\ &< \frac{c_1^2 k}{3}\frac{\int_0^1 F(x, d)dx}{d^2} \\ &< \frac{1}{3}\int_0^1 F(x, d)dx. \end{aligned} \tag{3.28}$$

Finally, from (3.27) one has

$$\frac{1}{\varphi_2^*(r_1, r_2)} < \frac{3}{4}\frac{d^2\|q\|_1}{\int_0^1 F(x, d)dx}. \tag{3.29}$$

Furthermore, by using (3.22) and (3.23) we have

$$\min\Big\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)}\Big\} > \min\Big\{\frac{mc_1^2}{4\int_0^1 \max_{|t| \leq c_1} F(x, t)dx}, \frac{mc_2^2}{4\int_0^1 \max_{|t| \leq c_2} F(x, t)dx}\Big\}. \tag{3.30}$$

As above, by Theorem 2.2, we conclude that for each λ that

$$\lambda \in \Big]\frac{3}{4}\frac{d^2\|q\|_1}{\int_0^1 F(x, d)dx}, \min\Big\{\frac{mc_1^2}{4\int_0^1 \max_{|t| \leq c_1} F(x, t)dx}, \frac{mc_2^2}{4\int_0^1 \max_{|t| \leq c_2} F(x, t)dx}\Big\}\Big[,$$

the problem (3.1) admits classical solutions u_1, u_2 such that $u_1 \in \Phi^{-1}(]-\infty, r_1[)$ and $u_2 \in \Phi^{-1}([r_1, r_2[)$. Therefore, $|u_1(t)| < c_1$ and $|u_2(t)| < c_2$ for all $t \in [0, 1]$. □

The following examples are the autonomous cases of problem (3.1).

Example 3.6. The problem

$$\begin{cases} -(pu')' + qu = \lambda f(u), \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.31}$$

where $p(x) = \frac{3}{2} + \arctan(x)$, $q(x) = \frac{3}{2} + \frac{\sin(x)}{2}$ and $f(u) = 3u^2$ admits at least three classical solutions for each $\lambda \in \left]0.75 - 0.18\cos(1), 0.75\right[$. In fact, if we choose $c = 0.5$ and $d = 2$, hypotheses of Theorem 3.4 are satisfied.

Example 3.7. Consider following problem

$$\begin{cases} -u'' + u = \lambda f(u), \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.32}$$

where

$$f(u) = \begin{cases} 1, & \text{if } u \in]-\infty, 1], \\ u^{10}, & \text{if } u \in]1, 2], \\ 2^5(u-3), & \text{if } u \in]2, 200], \\ 0, & \text{if } u \in]200, +\infty[. \end{cases}$$

Choosing $c_1 = 1$, $d = 2$ and $c_2 = 200$, thanks to Theorem 3.5, for each $\lambda \in \left]\frac{33}{2^{11}+10}, \frac{110\times 10^4}{2^{11}+2^7(53361)+10}\right[$, the given problem admits at least two classical solutions u_i such that $|u_i(t)| < 200$ for all $t \in [0, 1]$, $i = 1, 2$.

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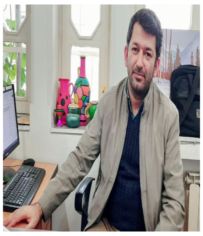

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