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ON A NEW CLASS OF INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION ${}_4F_3$

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Abstract. The main aim of this research paper is to evaluate a general integral of the form

$$\int_0^1 x^{d-1}(1-x)^{d+\ell}\ [1+\alpha x+\beta(1-x)]^{-2d-\ell-1}$$
$$\times\ {}_4F_3\left[\begin{matrix} a, & b, & 2d+\ell+1, & c \\ \frac{1}{2}(a+b+i+1), & d, & 2c+j \end{matrix};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right]dx$$

in the most general form for any $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$. The results are established with the help of generalized Watson's summation theorem due to Lavoie, *et al.* More than fifty interesting general integrals have also been obtained as special cases of our main findings.

Keywords: Generalized hypergeometric function, Watson's Theorem, Definite integral.

AMS Subject Classification: 33C05, 33C20, 33C70.

1. Introduction

The generalization of well known Gauss's hypergeometric function ${}_2F_1$ known as the generalized hypergeometric function ${}_pF_q$ with p numerator parameters and q denominator parameters is defined as [1, 2]

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix};z\right] &= {}_pF_q\left[a_1,\cdots,a_p;b_1,\cdots,b_q;z\right] \\ &= \sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{z^n}{n!} \end{aligned} \tag{1}$$

where $(a)_n$ is the well known Pochhammer's symbol (or the shifted or raised factorial) defined for every complex number a by

$$(a)_n=\begin{cases} a(a+1)\ldots(a+n-1), & n\in\mathbb{N} \\ 1, & n=0. \end{cases} \tag{2}$$

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Regarding the convergence of the series (1), we refer [1, 10].

It is interesting to mention here that whenever hypergeometric function ${}_2F_1$ and generalized hypergeometric functions ${}_pF_q$ expressed in terms of gamma function, the results are very important from the application point of view. Thus the classical summation theorem such as those of Gauss, Gauss's second, Kummer, and Bailey for the series ${}_2F_1$, Watson, Dixon, and Whipple for the series ${}_3F_2$ and others play an important role.

During 1992-1996, in a series of three interesting research papers, Lavoie, *et al.* [6, 7, 8] have generalized the above mentioned classical summation theorems.

However, in our present investigation, we are interested in the following classical Watson's summation theorem [1, 2, 10]

$$ {}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+1), \quad 2c \end{matrix};1\right] = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+c\right)\,\Gamma\left(\frac{a+b+1}{2}\right)\,\Gamma\left(c+\frac{1-a-b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\,\Gamma\left(\frac{1+b}{2}\right)\,\Gamma\left(c+\frac{1-a}{2}\right)\,\Gamma\left(c+\frac{1-b}{2}\right)} \tag{3} $$

provided $Re(2c-a-b) > -1$ and its following generalization due to Lavoie, *et al.* [6]

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ \frac{1}{2}(a+b+i+1), \quad 2c+j \end{matrix};1\right] = A_{i,j}\,2^{a+b+i-2} \\ &\times \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\left(a+b+|i+j|-j-1\right)\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\ &\times \left\{ \frac{B_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1-(-1)^i\right)\right)\,\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)} \right. \\ &\left. + \frac{C_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1+(-1)^i\right)\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)} \right\} \\ &= \Omega_{i,j} \end{aligned} \tag{4} $$

for $i, j = 0, \pm1, \pm2$, where $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are same as given in the paper [6].

Here, $[x]$ denotes the greatest integer less than or equal to x and it's modulus is denoted by $|x|$.

For $i = j = 0$, the result (4) reduce to classical Watson's summation theorem (3).

In addition to this, we also require the following interesting integral due to MacRobert [9]

$$ \int_0^1 x^{a-1}(1-x)^{b-1}\ \left[1+\alpha x+\beta(1-x)\right]^{-a-b}\,dx = \frac{\Gamma(a)\,\Gamma(b)}{(1+\alpha)^a\,(1+\beta)^b\,\Gamma(a+b)} \tag{5} $$

provided $Re(a) > 0$, $Re(b) > 0$ and α, β are the constants such that none of the expressions $1+\alpha$, $1+\beta$ and $1+\alpha x+\beta(1-x)$, where $0 \le x \le 1$ is zero.

The main aim of this research paper is to evaluate a general integral of the form

$$\int_0^1 x^{d-1}(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-1} \times {}_4F_3\left[\begin{matrix} a, & b, & 2d+\ell+1, & c \\ \frac{1}{2}(a+b+i+1), & d, & 2c+j & \end{matrix};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right] dx$$

in the most general form for any $\ell \in \mathbb{Z}$ and $i, j = 0, \pm 1, \pm 2$. The results are established with the help of generalized Watson's summation theorem (4) due to Lavoie, *et al.* More than fifty interesting general integrals have also been obtained as special cases of our main findings.

2. Main integral formula

In this section, we present a class of integral formulas involving the generalized hypergeometric functions ${}_4F_3$, which is asserted by the following theorem.

Theorem 2.1. *The following general integral formula containing twenty-five results holds true:*

$$\begin{aligned}\int_0^1 & x^{d-1}(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-1} \qquad (6)\\ &\times {}_4F_3\left[\begin{matrix} a, & b, & 2d+\ell+1, & c \\ \frac{1}{2}(a+b+i+1), & d, & 2c+j & \end{matrix};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right] dx \\ &= \frac{\Gamma(d)\,\Gamma(d+\ell+1)}{(1+\alpha)^d\,(1+\beta)^{d+\ell+1}\,\Gamma(2d+\ell+1)}\,\Omega_{i,j}\end{aligned}$$

where $\Omega_{i,j}$ is given in (4), $\ell \in \mathbb{Z}$, $i, j = 0, \pm 1, \pm 2$ *and* $Re(d) > 0$ *for* $\ell = 0, 1, 2, \ldots$; $Re(d) > -\ell$, *for* $\ell = -1, -2, \ldots$ *and* $Re(2d-a-b+i+2j+1) > 0$. *The coefficients* α *and* β *are the constants such that none of the expressions* $1+\alpha$, $1+\beta$ *and* $1+\alpha x+\beta(1-x)$, *where* $0 \le x \le 1$ *is zero.*

Proof. The proof of our theorem is quite straight forward. For this, we proceed as follows. Denoting the left-hand side of (6) by I, expressing the ${}_4F_3$ function as a series, changing the order of integration and summation which is easily seen to be justified due to the uniform convergence of the series in the interval (0,1), we have

$$\begin{aligned} I = & \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(2d+\ell+1)_n\,(c)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(2c+j)_n\,(d)_n\,n!} \\ & \times \int_0^1 x^{d+n-1}\,(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-n-1}\,dx. \end{aligned}$$

Evaluating the integral and after some simplification, we have

$$\begin{aligned} I = & \frac{\Gamma(d)\,\Gamma(d+\ell+1)}{(1+\alpha)^d\,(1+\beta)^{d+\ell+1}\,\Gamma(2d+\ell+1)} \\ & \times \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\,(c)_n}{\left(\frac{1}{2}(a+b+i+1)\right)_n\,(2c+j)_n\,n!}. \end{aligned}$$

Summing up the series, we have

$$I = \frac{\Gamma(d)\,\Gamma(d+\ell+1)}{(1+\alpha)^d\,(1+\beta)^{d+\ell+1}\,\Gamma(2d+\ell+1)} \times {}_3F_2\left[\begin{array}{cc} a, & b, \quad c \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{array};1\right].$$

Finally, evaluating ${}_3F_2$ using (4), we are led to the right-hand side of (6). This completes the proof of (6). □

We conclude this section by remarking that more than fifty interesting special cases in the form of two corollaries and some other known results will be given in the next section.

3. Special Cases

In this section, we shall mention more than fifty interesting special cases in the form of two integrals, which are also general in nature.

In (6), let $b=-2n$ and replace a by $a+2n$ or let $b=-2n-1$ and replace a by $a+2n+1$, where n is zero or a positive integer. In each case, one of the two terms appearing on the right-hand sides of (6) will vanish and under the same conditions of convergence, we get fifty interesting special cases, which are given below in the form of two corollaries.

Corollary 3.1. *For $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$, the following 25 results hold true.*

$$\begin{aligned}&\int_0^1 x^{d-1}\,(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-1} \\ &\times {}_4F_3\left[\begin{array}{c} -2n,\ a+2n,\ 2d+\ell+1,\ c \\ \frac{1}{2}(a+i+1),\ d,\ 2c+j \end{array};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right]dx \\ &= \frac{D_{i,j}}{(1+\alpha)^d\,(1+\beta)^{d+\ell+1}}\,\frac{\Gamma(d)\,\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ &\quad\times\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)_n},\end{aligned}\tag{7}$$

where the coefficients, $D_{i,j}$ are same as given in the paper [6].

Corollary 3.2. *For $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$, the following 25 results hold true.*

$$\begin{aligned}&\int_0^1 x^{d-1}\,(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-1} \\ &\times {}_4F_3\left[\begin{array}{c} -2n-1,\ a+2n+1,\ 2d+\ell+1,\ c \\ \frac{1}{2}(a+i+1),\ d,\ 2c+j \end{array};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right]dx \\ &= \frac{E_{i,j}}{(1+\alpha)^d\,(1+\beta)^{d+\ell+1}}\,\frac{\Gamma(d)\,\Gamma(d+\ell+1)}{\Gamma(2d+\ell+1)} \\ &\quad\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(3-(-1)^i\right)\right)_n},\end{aligned}\tag{8}$$

where the coefficients, $E_{i,j}$ are are same as given in the paper [6].

In particular, in (7), if we take $i = j = 0$, we get the following result.

$$\begin{aligned}
&\int_0^1 x^{d-1}\,(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-1} \qquad (9)\\
&\times\ {}_4F_3\left[\begin{matrix} -2n,\ a+2n,\ 2d+\ell+1,\ c \\ \frac{1}{2}(a+1),\quad d,\quad 2c \end{matrix};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right]dx\\
&=\frac{\Gamma(d)\,\Gamma(d+\ell+1)}{(1+\alpha)^d\,(1+\beta)^{d+\ell+1}\,\Gamma(2d+\ell+1)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n}.
\end{aligned}$$

Further, if we take $\ell = -1$, it reduces to

$$\begin{aligned}
&\int_0^1 x^{d-1}\,(1-x)^{d-1}\,[1+\alpha x+\beta(1-x)]^{-2d} \qquad (10)\\
&\times\ {}_4F_3\left[\begin{matrix} -2n,\ a+2n,\ 2d,\ 2c \\ \frac{1}{2}(a+1),\ d,\ 2c \end{matrix};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right]dx\\
&=\frac{\Gamma(d)\,\Gamma(d)}{(1+\alpha)^d\,(1+\beta)^{d}\,\Gamma(2d)}\,\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n\left(\frac{1}{2}a+\frac{1}{2}\right)_n}.
\end{aligned}$$

Similarly, in (8), if we take $i = j = 0$, we get the following elegant result.

$$\begin{aligned}
&\int_0^1 x^{d-1}\,(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-1} \qquad (11)\\
&\times\ {}_4F_3\left[\begin{matrix} -2n-1,\ a+2n+1,\ 2d+\ell+1,\ c \\ \frac{1}{2}(a+1),\qquad d,\qquad 2c \end{matrix};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right]dx=0.
\end{aligned}$$

We observe that the result (11) is interesting.

Corollary 3.3. *(a) In* (6)*, if we take $d = c$, we get results very recently obtained by Kim, et al.* [5].

(b) In (6)*, if we take $\beta = \alpha$ and $d = c$, we get results obtained by Choi and Rathie* [3].

Remark. For double finite integrals of this type, see recent papers by Choi and Rathie [4]

4. Conclusions

We evaluate a general integral of the form

$$\begin{aligned}
&\int_0^1 x^{d-1}(1-x)^{d+\ell}\,[1+\alpha x+\beta(1-x)]^{-2d-\ell-1}\\
&\qquad\times\ {}_4F_3\left[\begin{matrix} a,\quad b,\quad 2d+\ell+1,\quad c \\ \frac{1}{2}(a+b+i+1),\ d,\ 2c+j \end{matrix};\frac{(1+\alpha)x}{1+\alpha x+\beta(1-x)}\right]dx
\end{aligned}$$

for any $\ell \in \mathbb{Z}$ and $i, j = 0, \pm1, \pm2$. More than fifty interesting general integrals have also been obtained as special cases of our main findings.

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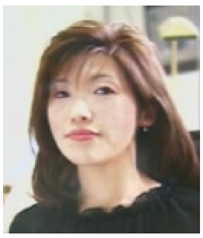

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