TWMS J. App. and Eng. Math. V.10, N.4, 2020, pp. 1055-1063

SOME QUADRATIC TRANSFORMATIONS MOTIVATED BY THE WORK OF KUMMER AND THEIR GENERALIZATIONS

M. I. QURESHI[1], KALEEM A. QURAISHI[2], M. KASHIF KHAN[1], §

Abstract. In this paper, we construct two quadratic transformations influenced by the work of Kummer and application of hypergeometric summation theorems of argument "two". Further, we establish some generalizations of these quadratic transformations in terms of double series identities having the bounded sequence. Three reduction formulas for Kampé de Fériet's double hypergeometric functions are also obtained as special cases.

Keywords: Generalized hypergeometric function, Hypergeometric summation theorems, Quadratic transformations, Bounded sequence.

AMS Subject Classification: Primary 33C05, Secondary 33C20

1. INTRODUCTION

Pochhammer's Symbol:
The Pochhammer's symbol or Appell's symbol or the shifted factorial or the rising factorial or generalized factorial function is given by

$$(b,\ k) = (b)_k := \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } \ k = 1, 2, 3, \cdots \\ 1 \qquad ; & \text{if } \ k = 0; b \in \mathbb{C}\backslash\{0\} \\ k! \qquad ; & \text{if } \ b = 1;\ k = 0, 1, 2, 3, \cdots \end{cases}, \tag{1}$$

where b is neither zero nor a negative integer and the notation Γ stands for Gamma function.

Generalized Hypergeometric Function [25, p. 42(1)]**:**
Generalized ordinary hypergeometric function of one variable is given by the following

---

[1] Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi-110025, India
e-mail: miqureshi_delhi@yahoo.co.in; ORCID: https://orcid.org/0000-0002-5093-1925.
e-mail: md.kashifkhan85@gmail.com; ORCID: https://orcid.org/0000-0001-6013-8026.

[2] Department of Applied Sciences and Humanities, Mewat Engineering College, Palla District, Nuh-122107, Haryana, India.
e-mail: kaleemspn@gmail.com; ORCID: https://orcid.org/0000-0001-6735-897X.

§ Manuscript received: February 15, 2019; accepted: August 31, 2019.

series

$$
{}_AF_B\left[\begin{array}{lc} (a_A) & ; \\ & z \\ (b_B) & ; \end{array}\right] \equiv {}_AF_B\left[\begin{array}{lc} a_1,\, a_2,\, \ldots,\, a_A & ; \\ & z \\ b_1,\, b_2,\, \ldots,\, b_B & ; \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}, \tag{2}
$$

where denominator parameters $b_1,\, b_2,\, \ldots,\, b_B$ are neither zero nor negative integers, and $A,\ B$ are non-negative integers and numerator parameters $a_1,\, a_2,\, \ldots,\, a_A$ may be zero or negative integers.

Convergence Conditions of Series (2):
Suppose that numerator parameters $a_1,\, a_2,\, \ldots,\, a_A$ are neither zero nor negative integers (otherwise the question of convergence will not arise), and denominator parameters $b_1,\, b_2,\, \ldots,\, b_B$ are also neither zero nor negative integers.

(i) If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex) i.e., $|z| < \infty$.
(ii) If $A = B + 1$ and $|z| < 1$, then series ${}_{B+1}F_B$ is convergent.
(iii) If $A = B + 1$ and $|z| = 1$, then series ${}_{B+1}F_B$ is absolutely convergent, when

$$
\Re\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{B+1} a_n \right\} > 0.
$$

(iv) If $A = B+1$ and $|z| = 1$, but $z \neq 1$, then series ${}_{B+1}F_B$ is conditionally convergent, when

$$
-1 < \Re\left\{ \sum_{m=1}^{B} b_m - \sum_{n=1}^{B+1} a_n \right\} \leq 0.
$$

Kampé De Fériet's Double Hypergeometric Function:

In 1921, Appell's four double hypergeometric functions $F_1,\ F_2,\ F_3,\ F_4$ [25, p. 53(4,5,6,7)] and their seven confluent forms $\Phi_1,\ \Phi_2,\ \Phi_3,\ \Psi_1,\ \Psi_2,\ \Xi_1,\ \Xi_2$ given by Humbert [9, 10, 11] were unified and generalized by Kampé de Fériet [13]. We recall the definition of general double hypergeometric function of Kampé de Fériet [1, p. 150(29); see also [2], p. 112] in the slightly modified notation of Srivastava and Panda [26, p. 423(26); see also [27], p. 23(1.2,1.3)] given by the following series

$$
F^{A:B;D}_{E:G;H}\left[\begin{array}{lc} (a_A):(b_B);(d_D) & ; \\ & x,\, y \\ (e_E):(g_G);(h_H) & ; \end{array}\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[(a_A)]_{m+n}[(b_B)]_m[(d_D)]_n\, x^m\, y^n}{[(e_E)]_{m+n}[(g_G)]_m[(h_H)]_n\, m!\, n!}, \tag{3}
$$

where (a_A) denotes the array of A number of parameters $a_1,\, a_2,\, \ldots,\, a_A$ and

$$
[(a_A)]_m = \prod_{j=1}^{A} (a_j)_m,
$$

with similar interpretation for others.

Convergence Conditions of Double Series (3)[26, p. 423-424(26-27)]:

(i) $A + B < E + G + 1,\ A + D < E + H + 1,\ \ |x| < \infty,\ |y| < \infty,$ or
(ii) $A + B = E + G + 1,\ A + D = E + H + 1,$ and

$$\left\{\begin{array}{ll} |x|^{\frac{1}{(A-E)}}+|y|^{\frac{1}{(A-E)}}<1 & ,\text{if } \ A>E \\ \\ \max\{|x|,|y|\}<1 & ,\text{if } \ A\leq E \end{array}\right\}.$$

For absolutely and conditionally convergence of double series (3), we refer to a research paper by Hài et al. [12, p.106-107].

In our analysis, we shall use the following standard results.

Summation Theorems Recorded by Prudnikov et al. [19, p. 493(7.3.8(1,2,3)), see also [21], p. 126(2), p. 127(10)]**:**

$${}_2F_1\left[\begin{array}{lll} -2n,\, a & ; & \\ & & 2 \\ 2a & ; & \end{array}\right]=\frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n},\quad n\in\mathbb{N}_0 \tag{4}$$

$${}_2F_1\left[\begin{array}{lll} -2n-1,\, a & ; & \\ & & 2 \\ 2a & ; & \end{array}\right]=0,\quad n\in\mathbb{N}_0 \tag{5}$$

$${}_2F_1\left[\begin{array}{lll} -2n,\, a & ; & \\ & & 2 \\ 2a+1 & ; & \end{array}\right]=\frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n},\quad n\in\mathbb{N}_0 \tag{6}$$

$${}_2F_1\left[\begin{array}{lll} -2n-1,\, a & ; & \\ & & 2 \\ 2a+1 & ; & \end{array}\right]=\frac{(\frac{3}{2})_n}{(2a+1)(a+\frac{3}{2})_n},\quad n\in\mathbb{N}_0 \tag{7}$$

$${}_2F_1\left[\begin{array}{lll} -2n,\, a & ; & \\ & & 2 \\ 2a-1 & ; & \end{array}\right]=\frac{(\frac{1}{2})_n}{(a-\frac{1}{2})_n},\quad n\in\mathbb{N}_0 \tag{8}$$

$${}_2F_1\left[\begin{array}{lll} -2n-1,\, a & ; & \\ & & 2 \\ 2a-1 & ; & \end{array}\right]=-\frac{(\frac{3}{2})_n}{(2a-1)(a+\frac{1}{2})_n},\quad n\in\mathbb{N}_0. \tag{9}$$

We have verified above results (4) to (9) by taking suitable numerical values of a and n.

Series Decomposition Identity [25, p. 200(1)]**:**

$$\sum_{n=0}^{\infty}\Lambda(n)=\sum_{n=0}^{\infty}\Lambda(2n)+\sum_{n=0}^{\infty}\Lambda(2n+1). \tag{10}$$

Series Rearrangement Formulas [25, p. 100(Lemmas 1 and 2)]**:**

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Xi(m,\,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\Xi(m,\,n-m), \tag{11}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Psi(m,\,n)=\sum_{m=0}^{\infty}\sum_{n=0}^{[\frac{m}{2}]}\Psi(m-2n,\,n), \tag{12}$$

provided that involved double series are absolutely convergent and $[k]$ denotes the greatest integer for $k\in\mathbb{R}$.

Motivated by the Kummer's transformation [14, p. 78(52)]

$$(1+z)^{-2a}\,{}_2F_1\left[\begin{array}{ccc}2a,\,c-\frac{1}{2} & ; & \\ & & \frac{2z}{1+z}\\ 2c-1 & ; & \end{array}\right]={}_2F_1\left[\begin{array}{ccc}a,\,a+\frac{1}{2} & ; & \\ & & z^2\\ c & ; & \end{array}\right], \tag{13}$$

$$\Big((2c-1)\in\mathbb{C}\backslash\mathbb{Z}_0^-\Big),$$

also recorded by Erdélyi et al. [6, p. 111 eq. (4), see also [21], p. 65 theorem 24], we construct two quadratic transformations in section 2 by means of series rearrangement technique and use of hypergeometric summation theorems having the argument "two" given by equations (6) to (9). Further, we establish generalizations of these quadratic transformations in terms of double series identities (20), (21) and (22) having the bounded sequence and three reduction formulas (23), (24) and (25) for Kampé de Fériet's double hypergeometric functions in terms of sum of two generalized hypergeometric functions of one variable.

2. TWO QUADRATIC TRANSFORMATIONS

When values of parameters and variables leading to the results which do not make sense are tacitly excluded.Then

$$(1+z)^{-2a}\,{}_2F_1\left[\begin{array}{ccc}2a,\,c-\frac{1}{2} & ; & \\ & & \frac{2z}{1+z}\\ 2c-2 & ; & \end{array}\right] = {}_2F_1\left[\begin{array}{ccc}a,\,a+\frac{1}{2} & ; & \\ & & z^2\\ c-1 & ; & \end{array}\right]+ \\ +\frac{az}{c-1}\,{}_2F_1\left[\begin{array}{ccc}a+1,\,a+\frac{1}{2} & ; & \\ & & z^2\\ c & ; & \end{array}\right], \tag{14}$$

$$\Big((2c-2)\in\mathbb{C}\backslash\mathbb{Z}_0^-\Big)$$

and

$$(1+z)^{-2a}\,{}_2F_1\left[\begin{array}{ccc}2a,\,c-\frac{1}{2} & ; & \\ & & \frac{2z}{1+z}\\ 2c & ; & \end{array}\right] = {}_2F_1\left[\begin{array}{ccc}a,\,a+\frac{1}{2} & ; & \\ & & z^2\\ c & ; & \end{array}\right]- \\ -\frac{az}{c}\,{}_2F_1\left[\begin{array}{ccc}a+1,\,a+\frac{1}{2} & ; & \\ & & z^2\\ c+1 & ; & \end{array}\right], \tag{15}$$

$$\Big((2c)\in\mathbb{C}\backslash\mathbb{Z}_0^-\Big).$$

Proof of Transformation (14):
Denoting the series expansion of left hand side of transformation (14) by Ω and after some

simplifications, we have

$$\begin{aligned}\Omega &= \sum_{m=0}^{\infty} \frac{(2a)_m\,(c-\frac{1}{2})_m\,2^m\,z^m}{(2c-2)_m\,m!}\,{}_1F_0\left[\begin{array}{lc} 2a+m & ; \\ & & -z \\ \text{———} & ; \end{array}\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2a)_{n+m}\,(c-\frac{1}{2})_m\,2^m\,(-1)^n\,z^{n+m}}{(2c-2)_m\,m!\,n!} \\
&= \sum_{n=0}^{\infty} \frac{(2a)_n\,(-1)^n\,z^n}{n!} \sum_{m=0}^{n} \frac{(c-\frac{1}{2})_m\,2^m\,(-n)_m}{(2c-2)_m\,m!} \\
&= \sum_{n=0}^{\infty} \frac{(2a)_n\,(-1)^n\,z^n}{n!}\,{}_2F_1\left[\begin{array}{lcc} -n,\,c-\frac{1}{2} & ; & \\ & & 2 \\ 2c-2 & ; & \end{array}\right]. \end{aligned} \tag{16}$$

Now, using infinite series decomposition identity (10) in equation (16), yields

$$\begin{aligned}\Omega \;=\; & \sum_{n=0}^{\infty} \frac{(2a)_{2n}\,(-1)^{2n}\,z^{2n}}{(2n)!}\,{}_2F_1\left[\begin{array}{lcc} -2n,\,c-\frac{1}{2} & ; & \\ & & 2 \\ 2c-2 & ; & \end{array}\right] + \\
& + \sum_{n=0}^{\infty} \frac{(2a)_{2n+1}\,(-1)^{2n+1}\,z^{2n+1}}{(2n+1)!}\,{}_2F_1\left[\begin{array}{lcc} -2n-1,\,c-\frac{1}{2} & ; & \\ & & 2 \\ 2c-2 & ; & \end{array}\right]. \end{aligned} \tag{17}$$

By applying summation theorems (8) and (9) in equation (17) and after some simplifications, we obtain the right hand side of transformation (14). This completes the proof of transformation (14).

Proof of Transformation (15):
Denoting the series expansion of left hand side of transformation (15) by $\mho$ and after some simplifications, we have

$$\begin{aligned}\mho &= \sum_{m=0}^{\infty} \frac{(2a)_m\,(c-\frac{1}{2})_m\,2^m\,z^m}{(2c)_m\,m!}\,{}_1F_0\left[\begin{array}{lc} 2a+m & ; \\ & & -z \\ \text{———} & ; \end{array}\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(2a)_{m+n}\,(c-\frac{1}{2})_m\,2^m\,(-1)^n\,z^{n+m}}{(2c)_m\,m!\,n!} \\
&= \sum_{n=0}^{\infty} \frac{(2a)_n\,(-1)^n\,z^n}{n!} \sum_{m=0}^{n} \frac{(c-\frac{1}{2})_m\,2^m\,(-n)_m}{(2c)_m\,m!} \\
&= \sum_{n=0}^{\infty} \frac{(2a)_n\,(-1)^n\,z^n}{n!}\,{}_2F_1\left[\begin{array}{lcc} -n,\,c-\frac{1}{2} & ; & \\ & & 2 \\ 2c & ; & \end{array}\right]. \end{aligned} \tag{18}$$

Now, using infinite series decomposition identity (10) in equation (18), yields

$$\begin{aligned} \mho &= \sum_{n=0}^{\infty} \frac{(2a)_{2n}\,(-1)^{2n}\,z^{2n}}{(2n)!}\; {}_2F_1\left[\begin{array}{ccc} -2n,\, c-\frac{1}{2} & ; & \\ & & 2 \\ 2c & ; & \end{array}\right] + \\ &+ \sum_{n=0}^{\infty} \frac{(2a)_{2n+1}\,(-1)^{2n+1}\,z^{2n+1}}{(2n+1)!}\; {}_2F_1\left[\begin{array}{ccc} -2n-1,\, c-\frac{1}{2} & ; & \\ & & 2 \\ 2c & ; & \end{array}\right]. \end{aligned} \tag{19}$$

By applying summation theorems (6) and (7) in equation (19) and after some simplifications, we obtain the right hand side of transformation (15). Similarly, on the same parallel lines of the derivations of (14) and (15), we can derive Kummer's transformation (13) with the help of summation theorems (4) and (5).

3. GENERALIZATIONS OF QUADRATIC TRANSFORMATIONS

Theorem 3.1. *Let $\{\Phi(n)\}_{n=0}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers. Then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m+n)\frac{(c-\frac{1}{2})_m\,(2z)^m\,(-z)^n}{(2c-1)_m\,m!\,n!} = \sum_{n=0}^{\infty} \Phi(2n)\frac{(\frac{z^2}{4})^n}{(c)_n\,n!}, \tag{20}$$

provided that single and double series involved are absolutely convergent and $(2c-1) \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 3.2. *Let $\{\Phi(n)\}_{n=0}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers. Then*

$$\begin{aligned} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m+n)\frac{(c-\frac{1}{2})_m\,(2z)^m\,(-z)^n}{(2c-2)_m\,m!\,n!} &= \sum_{n=0}^{\infty} \Phi(2n)\frac{(\frac{z^2}{4})^n}{(c-1)_n\,n!} + \\ &+ \frac{z}{2(c-1)}\sum_{n=0}^{\infty} \Phi(2n+1)\frac{(\frac{z^2}{4})^n}{(c)_n\,n!}, \end{aligned} \tag{21}$$

provided that single and double series involved are absolutely convergent and $(2c-2) \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Theorem 3.3. *Let $\{\Phi(n)\}_{n=0}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers. Then*

$$\begin{aligned} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Phi(m+n)\frac{(c-\frac{1}{2})_m\,(2z)^m\,(-z)^n}{(2c)_m\,m!\,n!} &= \sum_{n=0}^{\infty} \Phi(2n)\frac{(\frac{z^2}{4})^n}{(c)_n\,n!} - \\ &- \frac{z}{2c}\sum_{n=0}^{\infty} \Phi(2n+1)\frac{(\frac{z^2}{4})^n}{(c+1)_n\,n!}, \end{aligned} \tag{22}$$

provided that single and double series involved are absolutely convergent and $(2c) \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

We can derive above generalizations of quadratic transformations (13), (14) and (15) in the aforementioned manner.

4. REDUCTION FORMULAS FOR KAMPÉ DE FÉRIET FUNCTIONS

In the assertions (20), (21) and (22) put $\Phi(n) = \left(\prod_{j=1}^{D}(d_j)_n\right)\left(\prod_{j=1}^{E}(e_j)_n\right)^{-1}$; $n \in \mathbb{N}_0$, after simplification we obtain presumably new hypergeometric reduction formulas for Kampé de Fériet double hypergeometric functions. Under the common convergence conditions of reduction formulas (23), (24) and (25) given below
(i) When $2D \leq 2E+1$, then $|z| < \infty$,
(ii) When $D = E+1$ and $D > E$ then $|z| < \frac{1}{3}$ and
(iii) When $D = E+1$ and $D \leq E$ then $|z| < \frac{1}{2}$, the following hypergeometric reduction formulas hold true for Kampé de Fériet double hypergeometric functions and generalized hypergeometric functions:

Reduction Formula 4.1.

$$F^{D:1;0}_{E:1;0}\left[\begin{array}{l}(d_D):\, c-\frac{1}{2};\text{—};\\ \\ (e_E):2c-1;\text{—};\end{array}\ 2z,\,-z\right] = {}_{2D}F_{2E+1}\left[\begin{array}{l}\frac{(d_D)}{2},\,\frac{1+(d_D)}{2}\quad;\\ \\ c,\,\frac{(e_E)}{2},\,\frac{1+(e_E)}{2}\quad;\end{array}\ 4^{(D-E-1)}\,z^2\right], \tag{23}$$

$$\left(d_1,\,d_2,\,\ldots,\,d_D,\,e_1,\,e_2,\,\ldots,\,e_E,\,(2c-1)\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Reduction Formula 4.2.

$$F^{D:1;0}_{E:1;0}\left[\begin{array}{l}(d_D):\, c-\frac{1}{2};\text{—};\\ \\ (e_E):2c-2;\text{—};\end{array}\ 2z,\,-z\right] = {}_{2D}F_{2E+1}\left[\begin{array}{l}\frac{(d_D)}{2},\,\frac{1+(d_D)}{2}\quad;\\ \\ c-1,\,\frac{(e_E)}{2},\,\frac{1+(e_E)}{2}\quad;\end{array}\ 4^{(D-E-1)}\,z^2\right]+$$
$$+\frac{z\prod_{i=1}^{D}(d_i)}{2(c-1)\prod_{i=1}^{E}(e_i)}\,{}_{2D}F_{2E+1}\left[\begin{array}{l}\frac{1+(d_D)}{2},\,\frac{2+(d_D)}{2}\quad;\\ \\ c,\,\frac{1+(e_E)}{2},\,\frac{2+(e_E)}{2}\quad;\end{array}\ 4^{(D-E-1)}\,z^2\right], \tag{24}$$

$$\left(d_1,\,d_2,\,\ldots,\,d_D,\,e_1,\,e_2,\,\ldots,\,e_E,\,(2c-2)\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

Reduction Formula 4.3.

$$F^{D:1;0}_{E:1;0}\left[\begin{array}{l}(d_D):\, c-\frac{1}{2};\text{—};\\ \\ (e_E):2c;\text{—};\end{array}\ 2z,\,-z\right] = {}_{2D}F_{2E+1}\left[\begin{array}{l}\frac{(d_D)}{2},\,\frac{1+(d_D)}{2}\quad;\\ \\ c,\,\frac{(e_E)}{2},\,\frac{1+(e_E)}{2}\quad;\end{array}\ 4^{(D-E-1)}\,z^2\right]-$$
$$-\frac{z\prod_{i=1}^{D}(d_i)}{2c\prod_{i=1}^{E}(e_i)}\,{}_{2D}F_{2E+1}\left[\begin{array}{l}\frac{1+(d_D)}{2},\,\frac{2+(d_D)}{2}\quad;\\ \\ c+1,\,\frac{1+(e_E)}{2},\,\frac{2+(e_E)}{2}\quad;\end{array}\ 4^{(D-E-1)}\,z^2\right], \tag{25}$$

$$\left(d_1,\,d_2,\,\ldots,\,d_D,\,e_1,\,e_2,\,\ldots,\,e_E,\,(2c)\in\mathbb{C}\backslash\mathbb{Z}_0^-\right).$$

5. CONCLUSION

We conclude our present analysis by observing that several interesting quadratic, cubic and higher order reduction formulas (see [3, 4, 5, 7, 8, 15, 16, 18, 20, 22, 23, 24]), corresponding multiple series identities and their hypergeometric representations can be derived in an analogous manner. Moreover, presented quadratic transformations and their generalizations should be (potentially)beneficial to those who are interested in the field of Applied Mathematics and Applied Physics.

Acknowledgement. The authors wish to thank the referees for their valuable suggestions in the preparation of the revised version of the manuscript.

References

[1] Appell, P. and Kampé de Fériet, J., (1926), Fonctions Hypergéométriques et Hypersphériques Polynômes d' Hermite, Gauthier-Villars, Paris.

[2] Burchnall, J. L. and Chaundy, T. W., (1941), Expansions of Appell's Double Hypergeometric Functions-II, Quart. J. Math. (Oxford) Ser., 12, pp. 112-128.

[3] Buschman, R. G. and Srivastava, H. M., (1982), Series Identities and Reducibility of Kampé de Fériet Functions, Math. Proc. Cambridge Philos. Soc., 91, pp. 435-440.

[4] Chen, K.-Y. and Srivastava, H. M. (2005), Series Identities and Associated Families of Generating Functions, J. Math. Anal. Appl., 311, pp. 582-599.

[5] Chen, K.-Y., Liu, S.-J. and Srivastava, H. M. (2006), Some Double-Series Identities and Associated Generating-Function Relationships, Appl. Math. Lett. 19, pp. 887-893.

[6] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G.,(1953), Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, New York, Toronto and London.

[7] Gosper, R. W. Jr. (1976), A Calculus of Series Rearrangements, Algorithms and Complexity, J. F. Traub, ed., Academic Press , New York .

[8] Gupta, K. C. (1965), A Reduction Formula for Appell's Function F_4, Ganita 15, pp. 81-82.

[9] Humbert, P. (1920), La Fonction $W_k, \mu_1, \mu_2 ....\mu_n(x_1, x_2, ..., x_n)$. C. R. Acad. Sci. Paris, 171, pp. 428-430.

[10] Humbert, P. (1920-21), The Confluent Hypergeometric Functions of Two Variables, Proc. Royal Soc. Edinburgh. Sec. A. 41, pp. 73-96.

[11] Humbert, P. (1921), The Confluent Hypergeometriques d' Order Superieur a Deux Variables, C. R. Acad. Sci Paris, 173, pp. 73-96.

[12] Hài, N. T., Marichev, O. I. and Srivastava, H. M.(1992) A Note on the Convergence of Certain Families of Multiple Hypergeometric Series, J. Math. Anal. Appl., 164, pp. 104-115.

[13] Kampé de Fériet, J. (1921), Les Fonctions Hypergéométriques d' Ordre Supérieur à Deux Variables, C. R. Acad. Sci. Paris, 173, pp. 401-404.

[14] Kümmer, E. E. (1836), Üeber Die Hypergeometrische Reihe

$$1 + \frac{\alpha.\beta}{1.\gamma}\, x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}\, x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}\, x^3 + \cdots,$$

J. für die Reine und Angewandte Math., 15, pp. 39-83 and pp. 127-172.

[15] Lee, Keumsik, Cho, Young Joon and Seo, Tae Young (2000), A Transformation Formula Associated with the Generalized Hypergeometric Series, Comm. Korean Math. Soc., 15(4), pp. 707-714.

[16] Lin, S.-D., Srivastava, H. M. and Wang, P.-Y. (2002), Some Families of Hypergeometric Transformations and Generating Relations, Math. Comput. Modelling, 36, pp. 445-459.

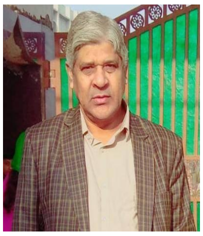
[17] Meethal, S., Rathie, A. K. and Paris, R. B. (2015), A Derivation of Two Quadratic Transformations Contiguous to that of Gauss via a Differential Equation Approach, Applied Mathematical Sciences, 9(17), pp. 845-851.

[18] Panda, R (1974), Some Multiple Series Transformations, *Jñānābha Sect. A*, 4, pp. 165-168.

[19] Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I. (1990), Integrals and Series, Vol. 3: More Special Functions, Nauka, Moscow, 1986 (In Russian), (Translated from the Russian by G. G. Gould), Gorden and Breach Science Publishers, New York.

[20] Qureshi, M. I., Khan, M. S. and Pathan, M. A. (2005), Some Reduction Formulae Involving Srivastava-Daoust Doube Hypergeometric Function, Italian Journal of Pure and Applied Mathematics, 18, pp. 85-88.

[21] Rainville, E. D. (1960), Special Functions, The Macmillan Co. Inc., New York.
[22] Rathie, A. K. and Kim, Young Sup (2001), On Two Results Contiguous to a Quadratic Transformation Formula due to Gauss, Far East Journal of Mathematical Sciences, 3(1), pp. 51-58.
[23] Saxena, R. K. (1966), On the Reducibility of Appell's Function F_4, Canad. Math. Bull., 9, pp. 215-222.
[24] Srivastava, H. M. (1973), On the Reducibility of Appell's Function F_4, Canad. Math. Bull., 16, pp. 295-298.
[25] Srivastava, H. M. and Manocha, H. L. (1984), A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto.
[26] Srivastava, H. M. and Panda, R. (1976), An Integral Representation for the Product of Two Jacobi Polynomials, J. London Math. Soc., 12(2), pp. 419-425.
[27] Srivastava, H. M. and Pathan, M. A. (1979), Some Bilateral Generating Functions for the Extended Jacobi Polynomial-I, Comment. Math. Uni. St. Paul., 28(1), pp. 23-30.



M. I. QURESHI is Professor of Mathematics at the Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia (A Central University), New Delhi, India. He received the Ph.D. degree (in year 1984) in Mathematics from Department of Mathematics, Aligarh Muslim University, Aligarh (U.P.), India. His research interests are in the areas of Integral Transforms, Special Functions, Ramanujan's Mathematics and Multiple Hypergeometric Functions. He has published more than 130 research articles in reputed national and international journals of Mathematics. He has guided more than 20 Ph.D. scholars.

KALEEM A. QURAISHI is Associate Professor in Mathematics at Mewat Engineering College, Nuh, Haryana, India, since 2010. Dr. Quraishi was awarded Ph.D. degree in Mathematics from Jamia Millia Islamia (A Central University), New Delhi in 2011. He passed M.Sc. (Mathematics with Computer Science) form Jamia Millia Islamia in 2005. He has published 52 research papers in reputed international/national journals. He has guided 2 Ph.D. scholars.

M. KASHIF KHAN has received his M.Sc. degree in Mathematics from Jamia Millia Islamia (A Central University), New Delhi, India in 2010. He is doing Ph.D. in Mathematics at Department of Applied Sciences and Humanities, Faculty of Engineering and Technology, Jamia Millia Islamia, New Delhi. His research interests are in the areas of Integral Transforms, Special Functions and Multiple Hypergeometric Functions.