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SOME PARAMETERS OF THE IDENTITY GRAPH OF MULTIGROUP

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Abstract. B. Sharada et. al. [17], have introduced the concept of the identity graph of a multigroup $\Gamma(G, E)$, which derived a representation of the multigroup as a graph. In this paper, we study some parameters of $\Gamma(G, E)$.

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1. Introduction

A mathematical structure known as multiset (mset, for short) is obtained if the restriction of distinctness on the nature of the objects forming a set is relaxed. Unlike classical set theory which assumes that mathematical objects occur without repetition. However, the situation in science and in ordinary life is not like that. It is observed that there is enormous repetition in the physical world. For example, consideration of repeated roots of polynomial equation, repeated observations in statistical sample, repeated hydrogen atoms in a water molecule H_2O, etc., do play a significant role. The challenging task of formulating sufficiently rich mathematics of multiset started receiving serious attention from beginning of the 1970s.

In set theory, repetition of objects are not allowed in a collection. This perspective rendered set almost irrelevant because many real life problems admit repetition. To remedy the handicap in the idea of sets, the concept of multiset was introduced in [13] as a generalization of set wherein objects repeat in a collection. Multiset is very promising in mathematics, computer science, website design, etc. For more details see [18, 19].

Since algebraic structures like groupoids, semigroups, monoids and groups were built from the idea of sets, it is then natural to introduce the algebraic notions of multiset. In [15], the term multigroup was proposed as a generalization of group in analogous to some non-classical groups such as fuzzy groups [16], intuitionistic fuzzy groups [3], etc. Although the term multigroup was earlier used in [5, 14] as an extension of group theory, it is only the idea of multigroup in [15] that captures multiset and relates to other non-classical groups. In fact, every multigroup is a multiset but the converse is not necessarily true and the concept of classical groups is a specialize multigroup with a unit count [6].

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In furtherance of the study of multigroups, some properties of multigroups and the analogous of isomorphism theorems were presented in [2]. Subsequently, in [1], the idea of order of an element with respect to multigroup and some of its related properties were discussed. A complete account on the concept of multigroups from different algebraic perspectives was outlined in [11]. The notions of upper and lower cuts of multigroups were proposed and some of their algebraic properties were explicated in [6]. In continuation to the study of homomorphism in multigroup setting (cf. [2, 15]), some homomorphic properties of multigroups were explored in [7]. In [12], the notion of multigroup actions on multiset was proposed and some results were established. An extensive work on normal submultigroups and comultisets of a multigroup were presented in [8].

The concept of identity graph of a multigroup have been introduced by B. Sharada et. al., which is so-called identity graph of a multigroup and denoted by $\Gamma(G, E)$. They discussed some properties of $\Gamma(G, E)$. Also, they derived the concept of twin multigroups and discussed some properties over twin multigroup. For further information about the identity graph of a multi group see [17].

A graph is a pair of sets $\Gamma(V, E)$, where V is a finite set called the set of vertices and E is a set of 2-element subsets of V, called the set of edges. A graph $\Gamma(V, E)$ is a simple graph, that is having no loops, no multiple and directed edges. We denote n and m to be the order and the size of the graph Γ, respectively. The degree of a vertex $d(v)$, is the number of vertices that adjacent to $v \in V$, where the maximum degree is $\Delta(\Gamma) = max\{d(v) : v \in V\}$ and the minimum degree is $\delta(\Gamma) = min\{d(v) : v \in V\}$. As usual we denote K_n and $K_{a,b}$ for the complete and the complete bipartite graphs, respectively. A graph $\acute{\Gamma}$ whose vertices and edges form subsets of the graph Γ is said to be a subgraph and written $\acute{\Gamma} \subseteq \Gamma$. An induced subgraph $\Gamma[H]$ of a graph Γ is another graph formed from a subset of the vertices of Γ and all of the edges connecting pair of vertices in that subset. Let Γ_1 and Γ_2 have disjoint sets $V_1,\ V_2$ and line sets $E_1,\ E_2$, respectively. Their union $\Gamma = \Gamma_1 \cup \Gamma_2$ has $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Their join is denoted by $\Gamma_1 + \Gamma_2$ and consists of $\Gamma_1 \cup \Gamma_2$ and all lines joining V_1 and V_2.

A graph $\Gamma(V, E)$ is said to be embedded in a surface S when it is drawn on S so that no two edges intersect. A graph is planar if it can be embedded in the plane. The thickness $\theta(\Gamma)$ of a graph is the minimum number of planar subgraphs whose union is Γ. A vertex and an edge are said to cover each other if they are incident. A set of vertices which covers all the edges of a graph Γ is called a vertex cover for Γ. The smallest number of vertices in any vertex cover for Γ is called vertex covering number and denoted by $\alpha_0(\Gamma)$ or α_0. A set of vertices in Γ is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independent number of Γ and is denoted by $\beta_0(\Gamma)$ or β_0. A coloring of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. An n-coloring of a graph Γ uses n colors. The chromatic number $\chi(\Gamma)$ is defined as the minimum n for which Γ has n-coloring. A clique of a simple graph $\Gamma(V, E)$ is a subset S of V such that $\Gamma[S]$ is complete. The clique number denoted by $\omega(\Gamma)$ is the order of the maximum clique of Γ. All the definitions and terminologies about the graph in this paragraph available in [4, 10].

Definition 1.1. [9] *A multiset M (mset) drawn from the set X is represented by a count function C_M defined as $C_M :\ X \to \mathbb{N}$, where $\mathbb{N}$ represent the set of non negative integers.*

For $x \in X$, $C_M(x)$ denotes the number of occurrence of the element x in the mset M. The representation of the mset M drawn from $X = \{x_1,\ x_2,\ ...,\ x_n\}$ is

$$M(X) = [x_1,\ x_2,\ ...,\ x_n]_{C_M(x_1),\ C_M(x_2),\ ...,\ C_M(x_n)} = [x_1^{C_M(x_1)},\ x_2^{C_M(x_2)},\ ...,\ x_n^{C_M(x_n)}]$$

where $C_M(x_i)$ is the number of times that x_i occurs in M.

The set of all msets over X is denoted by $MS(X)$.

Definition 1.2. [15] *Let X be a group. A mset G is said to be a multigroup (mgroup) over X if the count function C_G satisfies the following two conditions.*
(1) $C_G(xy) \geq [C_G(x) \wedge C_G(y)],\ \forall x, y \in X$, *where $\wedge$ denotes the minimum.*
(2) $C_G(x^{-1}) = C_G(x),\ \forall x \in X$.

The set of all mgroups over X is denoted by $MG(X)$.
Easily, we can see that $C_G(e) \geq C_G(x),\ \forall x \in X$, where e is the identity element of X.

The group is represented as a graph in many ways, one of these representations is the identity graph of group, which is explained in the following definition.

Definition 1.3. [20] *Let $(X, *)$ be a finite group, we say two elements x, y in the group are adjacent or can be joined by an edge if $x * y = e$, where e is the identity element of X. It is by convention every element is adjoined with the identity of the group X.*

The following is to classify the mgroup as a set of msets.

Let $(X, *)$ be a finite group where $X = \{x_1,\ x_2,\ ...,\ x_n\}$, with $x_1 = e$ is the identity element and let $G \in MG(X)$. Then we write G as

$$G = [X_1,\ X_2,\ ...,\ X_n]$$

where the mset $X_i = [x_i^{C_G(x_i)}] = [x_{i1},\ x_{i2},\ ...,\ x_{iC_G(x_i)}]$, with $x_{i1} =\ x_{i2} =\ ... =\ x_{iC_G(x_i)} = x_i$.

Definition 1.4. [17] *Let $(X, *)$ be a finite group and let $G \in MG(X)$. If e is the identity element of X, then the identity graph $\Gamma(G, E)$ of a mgroup is a simple graph whose vertex set is the elements of G and satisfies the following*
(1) $xe \in E,$ *for all* $x \in G$.
(2) $xy \in E,$ *if* $x * y = e,$ *for all* $x, y \in E$.

Since $\Gamma(G, E)$ is simple, then there is no loops for the elements that has the property $x^{-1} = x,$ for all $x \in X$.

As usual for any group if we let $k = |\{x_i : x_i^{-1} = x_i\}|,\ i = 1,\ 2,\ ...,\ n$, then we denote $Y_1 = \{x_i : x_i^{-1} = x_i\}$ and $Y_2 = \{x_i : x_i^{-1} \neq x_i\}$. The cardinality of $Y_1,\ Y_2$ is $|Y_1| = k$ and $|Y_2| = n - k$ respectively.

2. LEMMAS

We state here some previously known results that are needed in the later sections.

Lemma 2.1. [10] *Let Γ be a graph with n vertices and m edges. Then*
(1) $\alpha_0(K_n) = n - 1$.

(2) $\alpha_0(K_{a,b}) = max\{a, b\}$.

(3) $\alpha_0(\Gamma) + \beta_0(\Gamma) = |V(\Gamma)|$.

Lemma 2.2. [10] *The chromatic number for the complete graph K_n is $\chi(K_n) = n$. Also, for the complete bipartite graph $K_{a,b}$ is $\chi(K_{a,b}) = 2$.*

Lemma 2.3. [10] *Let $\theta(\Gamma)$ be the thickness of the graph Γ. Then*

(1) $\theta(K_n) = \lfloor \frac{n+7}{6} \rfloor$, $n > 2$, $n \neq 9, 10$ *and* $\theta(K_9) = \theta(K_{10}) = 3$.

(2) $\theta(K_{a,a}) = \lfloor \frac{a+5}{4} \rfloor$.

Lemma 2.4. [10] *Let $\Gamma = K_n$ be a complete graph. Then Γ is planar if and only if $n \leq 4$.*

Corollary 2.1. *Let $\Gamma = K_n$ be a complete graph. Then $\theta(\Gamma) = 1$ if and only if $n \leq 4$. Also, if $n > 4$, then Γ is not planar, thus $\theta(\Gamma) \geq 2$.*

Theorem 2.1. [17] *Let $(X, *)$ be a finite group and let $G \in MG(X)$. Then $\Gamma(G, E)$ is complete if and only if X contains at most two elements, or $X = \{x_1, x_2, x_3\}$, such that $C_G(x_2) = C_G(x_3) = 1$ and $x_2^{-1} = x_3$.*

Theorem 2.2. [17] *Let $(X, *)$ be a finite group and let $G \in MG(X)$. Then $\Gamma(G, E)$ is a star if and only if $C_G(x_1) = 1$ and $x_i^{-1} = x_i$, for all $i = 2, 3, ..., n$.*

3. SOME PARAMETERS OF $\Gamma(G, E)$

There are a lot of parameters of the graph, in this section we will study some of these parameters for $\Gamma(G, E)$.

Theorem 3.1. *Let $(X, *)$ be a finite group and let $G \in MG(X)$. Then the clique number $\omega(\Gamma)$ of the identity graph of the mgroup G is*

$$\omega(\Gamma) = \begin{cases} C_G(x_1) + max\{C_G(x_i)\}, & x_i^{-1} = x_i \text{ for some } i = 2, 3, ..., n \text{ and } C_G(x_i) \geq 2, \\ C_G(x_1) + 2, & x_i^{-1} \neq x_i \text{ or } C_G(x_i) = 1 \text{ for all } i = 2, 3, ..., n. \end{cases}$$

Proof. Let G be a mgroup over the finite group X. Then we have the following two cases.
Case 1: There exists $x_i^{-1} = x_i$, for some $i = 2, 3, ..., n$. Since $\Gamma[X_1]$ is complete and $\Gamma[X_i]$ is also complete, then by using the fact that the join of two complete graphs is complete, we get $\Gamma[X_1] + \Gamma[X_i]$ is complete. So the maximum induced complete subgraph is $\Gamma[X_1] + max\{\Gamma[X_i]\}$ for some $i = 2, 3, ..., n$. Thus

$$\begin{aligned} \omega(\Gamma) &= |\Gamma[X_1] + max\{\Gamma[X_i]\}| \\ &= C_G(x_1) + max\{C_G(x_i)\}, \end{aligned}$$

for $i = 2, 3, ..., n$.
Case 2: If $x_i^{-1} \neq x_i$ for all $i = 2, 3, ..., n$, then $\Gamma[X_1]$ is complete and since the vertices in $\Gamma[X_1]$ is adjacent to all the vertices in $\Gamma - \Gamma[X_1]$, then $\Gamma[X_1] + x_{ij} + x_{kg}$, with $x_i^{-1} = x_k$ is complete. If we assume that there exists another vertex $x_{it} \in \Gamma[X_i]$, then x_{it} is not adjacent to x_{ij}, so it is not complete, thus

$$\begin{aligned} \omega(\Gamma) &= |\Gamma[X_1] + x_{ij} + x_{kg}| \\ &= C_G(x_1) + 2, \end{aligned}$$

which completes the proof. □

Example 3.1. *Let $G \in MG(\mathbf{Z}_n, \bigoplus_n)$ and let $\Gamma(G, E)$ be the identity graph of the mgroup G. Then*

$$\omega(\Gamma) = \begin{cases} C_G(0) + 2, & n \text{ is odd}, \\ C_G(0) + C_G(\frac{n}{2}), & n \text{ is even}. \end{cases}$$

It is clear that if n is odd, then $x_i^{-1} = x_i$, $i = 2,\ 3,\ ...,\ n$ and if n is even, then $x_i^{-1} = x_i$ if and only if $i = 1$ and $\frac{n}{2} + 1$ which holds at $x_i = 0$ and $\frac{n}{2}$, respectively.

Theorem 3.2. *Let G be a mgroup over the finite group $(X,\ *)$ and let Γ be the identity graph of the mgroup G. Then the chromatic number of Γ is given by*

$$\chi(\Gamma) = \begin{cases} C_G(x_1) + max\{C_G(x_i)\}, & x_i^{-1} = x_i \ for\ some\ i = 2, 3, ..., n\ and\ C_G(x_i) \geq 2, \\ C_G(x_1) + 2, & x_i^{-1} \neq x_i,\ for\ all\ i = 2, 3, ..., n\ or\ C_G(x_i) = 1,\ for\ all\ x_i^{-1} = x_i. \end{cases}$$

Proof. Let $(X,\ *)$ G be a finite group and let $G \in MG(X)$. Assume that Γ is the identity graph of the m group G. Then we have three cases.
Case 1: There exists $x_i^{-1} = x_i$ for some $i = 2,\ 3,\ ...,\ n$; with $C_G(x_i) \geq 2$; then by using Theorem 3.1, we get that

$$\Gamma[X_1 \cup X_i],$$

where $C_G(x_i) \geq C_G(x_j)$, $j = 2,\ 3,\ ...,\ i-1,\ i+1,\ ...,\ n$, is the maximum complete induced subgraph of Γ.
Hence, by Lemma 2.2, we get

$$\chi(\Gamma[X_1 \cup X_i]) = C_G(x_1) + C_G(x_i)$$

where $C_G(x_i) \geq C_G(x_j)$, $j = 2,\ 3,\ ...,\ i-1,\ i+1,\ ...,\ n$.
Since all the induced subgraphs $\Gamma[X_j]$, $j = 2,\ 3,\ ...,\ i-1,\ i+1,\ ...,\ n$ are independent than $\Gamma[X_i]$, then we have two subcases.
Subcase 1.1: $\Gamma[X_i]$ is complete, then $C_G(x_i) \geq C_G(x_j)$, and since they are independent, we can use same colors from the set of colors that used in $\Gamma[X_i]$.
Subcase 1.2: $\Gamma[X_j \cup X_k]$ is not complete, then $\Gamma[X_j \cup X_k]$ is complete bipartite graph. Thus, by Lemma 2.2, we get

$$\chi(\Gamma[X_j \cup X_k]) = 2.$$

Since $\Gamma[X_j \cup X_k]$ is independent from $\Gamma[X_i]$, then we can use two colors from the set of the colors that used in $\Gamma[X_i]$.
Hence

$$\chi(\Gamma) = C_G(x_1) + max\{C_G(x_i)\},\ x_i^{-1} = x_i\ and\ C_G(x_i) \geq 2.$$

Case 2: $x_i^{-1} = x_i$ for all $i = 2,\ 3,\ ...,\ n$, then $\Gamma[X_j \cup X_k]$ is regular complete bipartite graph, so by using Theorem 3.1 and Lemma 2.2, we get

$$\chi(\Gamma) = C_G(x_1) + 2.$$

Case 3: $C_G(x_i) = 1$ for all $x_i^{-1} = x_i$, then by using Theorem 3.1 and from the above case we get that

$$\chi(\Gamma) = C_G(x_1) + 2.$$

□

Theorem 3.3. *Let G be a mgroup over the finite group $(X,\ *)$ and let Γ be the identity graph of the mgroup G. Then the covering number of Γ is*

$$\alpha_0(\Gamma) = \sum_{i=1}^{k} C_G(x_i) + \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} - (k-1),$$

where $k = |Y_1|$.

Proof. Let G be a mgroup over the finite group $(X, *)$ and let Γ be the identity graph of the mgroup G. We divide Γ into induced subgraphs $\Gamma[X_i]$, $i = 1, 2, ..., n$. As $|Y_1| = k$ and $|Y_2| = n - k$, so we have k of complete induced subgraphs $K_{C_G(x_i)}$, $i = 1, 2, ..., k$ and $\frac{n-k}{2}$ induced regular complete bipartite subgraphs $K_{C_G(x_i),C_G(x_i)}$, $i = k+1, ..., n$. Using Lemma 2.1, parts 1,2, and since each vertex in $\Gamma[X_1]$ is adjacent to all other vertices in Γ, we get

$$\begin{aligned}\alpha_0(\Gamma) &= C_G(x_1) + \sum_{i=2}^{k}(C_G(x_i) - 1) + \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} \\ &= C_G(x_1) + \sum_{i=2}^{k} C_G(x_i) - (k-1) + \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} \\ &= \sum_{i=1}^{k} C_G(x_i) + \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} - (k-1).\end{aligned}$$

□

Theorem 3.4. *Let G be a mgroup over the finite group $(X, *)$ and let Γ be the identity graph of the mgroup G. Then the vertex matching of Γ is*

$$\beta_0(\Gamma) = \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} + k - 1,$$

where $k = |Y_1|$.

Proof. Let G be an mgroup over the finite group $(X, *)$, Γ be the identity graph of the mgroup G and $\beta_0(\Gamma)$ be the vertex matching of Γ. Then by using Lemma 2.1 part 3, we get

$$\begin{aligned}\beta_0(\Gamma) &= \sum_{i=1}^{n} C_G(x_i) - \Big(\sum_{i=1}^{k} C_G(x_i) + \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} - (k-1)\Big) \\ &= \sum_{i=k+1}^{n} C_G(x_i) - \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} + k - 1 \\ &= \frac{\sum\limits_{i=k+1}^{n} C_G(x_i)}{2} + k - 1.\end{aligned}$$

□

Theorem 3.5. *Let G be a mgroup over the finite group $(X, *)$ and let Γ be the identity graph of the mgroup G. If Γ is complete graph, then the thickness $\theta(\Gamma)$ is given by*

$$\theta(\Gamma) = \begin{cases} \lfloor \frac{C_G(x_1)+C_G(x_2)+7}{6} \rfloor, & X \textit{ contains at most two elements and } C_G(x_1)+C_G(x_2) \neq 9, 10, \\ \lfloor \frac{C_G(x_1)+9}{6} \rfloor, & X = \{x_1, x_2, x_3\},\ x_2^{-1} = x_3,\ C_G(x_2) = C_G(x_3) = 1 \textit{ and } C_G(x_1) + 2 \neq 9, 10, \\ 3, & C_G(x_1) + C_G(x_2) = 9, 10 \textit{ in first axiom and } C_G(x_1) + 2 = 9, 10 \textit{ in second axiom.} \end{cases}$$

Proof. Let Γ be a complete graph, by using Lemma 2.3 and Theorem 2.1, we have three cases.
Case 1: X contains at most two elements and $C_G(x_1) + C_G(x_2) \neq 9, 10$. Then we have two subcases.
Subcase 1.1: X contains only one element. Then Γ is complete by Theorem 2.1, moreover $\Gamma = K_{C_G(x_1)}$.
Thus by Lemma 2.3, we get

$$\theta(\Gamma) = \lfloor \frac{C_G(x_1) + 7}{6} \rfloor.$$

Subcase 1.2: X contains two elements i.e.$X = \{x_1,\ x_2\}$. Then by Theorem 2.1, Γ is complete, moreover $\Gamma = K_{C_G(x_1)+C_G(x_2)}$.
Thus by using Lemma 2.3, we get

$$\theta(\Gamma) = \lfloor \frac{C_G(x_1) + C_G(x_2) + 7}{6} \rfloor.$$

So if we assume by default that $C_G(x_2) = 0$, then we can generalize the formula for the two subcases 1.1, 1.2 as

$$\theta(\Gamma) = \lfloor \frac{C_G(x_1) + C_G(x_2) + 7}{6} \rfloor.$$

Case 2: $X = \{x_1,\ x_2,\ x_3\},\ x_2^{-1} = x_3$ with $C_G(x_2) = C_G(x_3) = 1$ and $C_G(x_1) + 2 \neq 9, 10$. Then by using Lemma 2.3, we get that Γ is complete with $\Gamma = K_{C_G(x_1)+C_G(x_2)+C_G(x_3)} = K_{C_G(x_1)+2}$.
Thus by using Lemma 2.3, we get

$$\theta(\Gamma) = \lfloor \frac{C_G(x_1) + 2 + 7}{6} \rfloor = \lfloor \frac{C_G(x_1) + 9}{6} \rfloor.$$

Case 3: $C_G(x_1) + C_G(x_2) = 9$ *or* 10, in the first case or $C_G(x_1) + 2 = 9$ *or* 10 in the second case. Then it is clear by employing Lemma 2.3 that

$$\theta(\Gamma) = 3.$$

□

Theorem 3.6. *Let $(X,\ *)$ be a finite group and let $G \in MG(X)$. Then*

$$\theta(\Gamma) \geq max\{\lfloor \frac{C_G(x_1) + 7}{6} \rfloor,\ \lfloor \frac{C_G(x_i) + 5}{4} \rfloor\},$$

where $x_i \in Y_2$.

Proof. Since the graph Γ can be derived into independent induced subgraphs; which is complete is $x_i \in Y_1$ or complete bipartite if $x_j, x_k \in Y_2$ with $x_j^{-1} = x_k$; then by using Lemma 2.3, we get

$$\theta(\Gamma[X_i]) = \begin{cases} \lfloor \frac{C_G(x_i)+7}{6} \rfloor, & x_i \in Y_1 \text{ and } C_G(x_i) \neq 9, 10, \\ 3, & C_G(x_i) = 9 \text{ or } 10. \end{cases}$$

And $\theta(\Gamma[X_j \cup X_k]) = \lfloor \frac{C_G(x_j)+5}{4} \rfloor$ if $x_j, x_k \in Y_2$.
Also, each $\Gamma[X_i],\ x_i \in Y_1$ and $\Gamma[X_j \cup x_k],\ x_j, x_k \in Y_2$ is independent than each other, so we can treat the proper induced subgraphs $\Gamma[X_i],\ \Gamma[X_j \cup X_k]$, separately.
The thickness of any graph is greater than or equal the thickness of any proper induced subgraph of the graph itself. Thus, we will compute the maximum thickness of an induced subgraph of Γ.

Since $\Gamma[X_i],\ x_i \in Y_1$ is a complete graph and $C_G(x_1) \geq C_G(x_i),\ x_i \neq x_1$ *and* $x_i \in Y_1$ and since the function $g(x) = \frac{x+7}{6}$ is increasing function, then

$$\theta(\Gamma[X_1]) \geq \theta(\Gamma(X_i)),\ i = 2,\ 3,\ ...,\ h,$$

where $h = |Y_1|$.

Also, the induced subgraph $\Gamma[X_j \cup x_k],\ x_j, x_k \in Y_2$ is regular complete bipartite. So, by using Lemma 2.3, we get

$$\theta(\Gamma[X_j \cup x_k]) = \lfloor \frac{C_G(x_i) + 5}{4} \rfloor.$$

Hence,

$$\theta(\Gamma) \geq max\{\lfloor \frac{C_G(x_1) + 7}{6} \rfloor,\ \lfloor \frac{C_G(x_i) + 5}{4} \rfloor\},$$

where $x_i \in Y_2$. □

Theorem 3.7. *Let $(X,\ *)$ be a finite group, $G \in MG(X)$, and let Γ be the identity graph of G. If Γ is not complete graph and $C_G(x_1) \geq 3$, then Γ is not planar graph.*

Proof. Let $(X,\ *)$ be a finite group and let $G \in MG(X)$, assume that Γ is the identity graph of G, and let Γ be not complete graph with $C_G(x_1) \geq 3$. Take the least number for $C_G(x_1) = 3$.

Since Γ is not complete, then there exist at least two vertices outside $\Gamma[X_1]$. If the two vertices lies in $\Gamma[X_2]$ with $x_2^{-1} = x_2$, then Γ is not complete, which contradict that Γ is not complete, so we assume that the two vertices lies in $\Gamma[X_2]$ and $\Gamma[X_3]$, with $x_2^{-1} = x_3$, respectively.

Since all the vertices in $\Gamma[X_1]$ is adjacent to all the vertices in $\Gamma - \Gamma[X_1]$, and x_{21} is adjacent to x_{31}, then $\Gamma = K_5$, so by using Corollary 2.1, we get that Γ is not planar.

Hence, if $C_G(x_1) \geq 3$, then Γ is not planar. □

The above theorem explain the case, if $C_G(x_1) \geq 3$, then Γ is not a planar graph. Here the following question directly bring to mind,

Q : What about the cases if $C_G(x_1) = 1,\ 2$?

A : In fact, the case, if $C_G(x_1) = 1$, then by using Theorem 2.2, we have two cases.

Case 1: If X is a 2-group (a group $(X,\ *)$ that have $x_i^{-1} = x_i$ for all $x_i \in X$), then Γ is a star, which is planar.

Case 2: If X is not a 2-group, then there exists $x_j,\ x_k \in X$ such that $x_j^{-1} = x_k$, and since $C_G(x_1) = 1 \geq C_G(x_i),\ i = 2,\ 3,\ ...,\ n$, then $C_G(x_j) = C_G(x_k) = 1$, so Γ contains pendant vertices and triangles, so easily we can get that Γ is planar.

The second case, if $C_G(x_1) = 2$, then Γ is not necessary to be planar, so the following theorem explain some cases at which Γ is planar.

Theorem 3.8. *Let $(X,\ *)$ be a finite group, $G \in MG(X)$ and let Γ be the identity graph of the mgroup G. If $C_G(x_1) = 2$ and $C_G(x_i) = 1$, for all $i = 2,\ 3,\ ...,\ n$, then Γ is planar graph.*

Proof. Let $(X,\ *)$ be a finite group, $G \in MG(X)$ and let Γ be the identity graph of the mgroup G. Suppose that $C_G(x_1) = 2$ and $C_G(x_i) = 1$, for all $i = 2,\ 3,\ ...,\ n$, then $\Gamma[X_1]$ consists of x_{11} *and* x_{12}, hence $\Gamma - x_{11}$ is a star or consists from pendent vertices and triangles which is independent, thus $\Gamma - x_{11}$ is planar.

So we want to show that the graph $\Gamma \cong \Gamma - x_{11} + x_{11}$ is a planar graph.

Easily we can connect all the pendent vertices of the star to the vertex x_{11} without intersection; which is also do not make a closed cycle around x_{11} and x_{12}; hence, easily we can connect the vertices x_{11} *and* x_{12} by an edge that is not intersect any other edges. thus Γ is a planar graph. The following figure explain the proof.

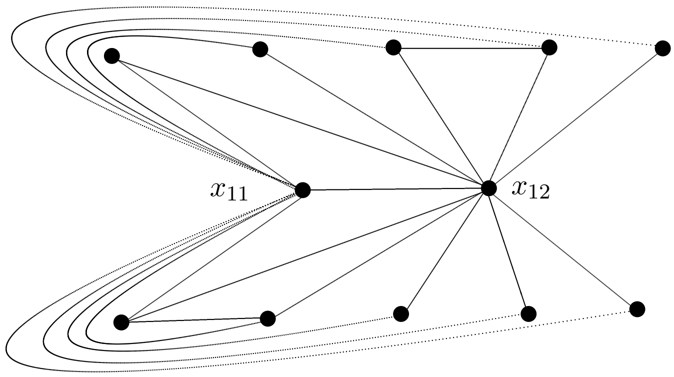

Figure : 1

Note in the figure that the two vertices that makes triangle with x_{12} is the vertices that liess in Y_2 and that not makes triangle lies on Y_1. □

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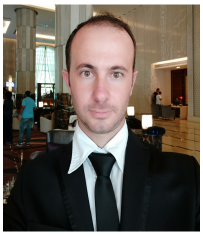

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