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ON ZAGREB INDICES OF DOUBLE VERTEX GRAPHS

S. SAMPATH KUMAR[1], R. SUNDARESWARAN[1], M. SUNDARAKANNAN[1]

Abstract. Let $G = (V, E)$ be a graph with at least 2 vertices, then the double vertex graph $U_2(G)$ is the graph whose vertex set consists of all 2-subsets of V such that two distinct vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. Similarly, the complete double vertex graph, denoted by $CU_2(G)$, has vertex set consists of all unordered pairs of elements of V and two distinct vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G. In this work, we compute the zagreb indices of double vertex and complete double vertex graphs.

Keywords: Double vertex graph, Complete Double vertex graph, Zagreb Index, Hyper-Zagreb index.
AMS Subject Classification: 05C72

1. Introduction

We consider, a finite undirected and connected graph $G = (V, E)$ with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. The *degree* of a vertex v in G is the number of edges incident to v and is denoted by $deg_G(v)$. For graph theoretic definitions which are not seen here can be found in [14].

In [1, 3], the authors introduced the following definition. The *double vertex graph* (resp. *complete double vertex graph*) $U_2(G)$ (resp. $CU_2(G)$) of G is the graph whose vertex set consists of all 2-subsets of V (resp. all unordered pairs of elements of V) such that two distinct vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, then y and v are adjacent in G, see Fig. 1.

Let x and y be any two vertices of the connected graph G. Then

$$deg_{U_2(G)} = \begin{cases} deg_G(x) + deg_G(y), & \text{x not adjacent to y} \\ deg_G(x) + deg_G(y) - 2, & \text{x adjacent to y} \end{cases}$$

Similarly, if x and y are any two vertices of the connected graph G, then

$$deg_{CU_2(G)} = \begin{cases} deg_G(x), & x = y \\ deg_G(x) + deg_G(y), & x \neq y \end{cases}$$

[1] Department of Mathematics,SSN College of Engineering, Chennai, India.
e-mail: ssamkumar.2008@gmail.com; ORCID: https://orcid.org/0000-0001-7234-26147.
e-mail: sundareswaranr@ssn.edu.in; ORCID: https://orcid.org/0000-0002-0439-695X.
e-mail: sundarakannanm@ssn.edu.in; ORCID: https://orcid.org/0000-0001-8902-1307.

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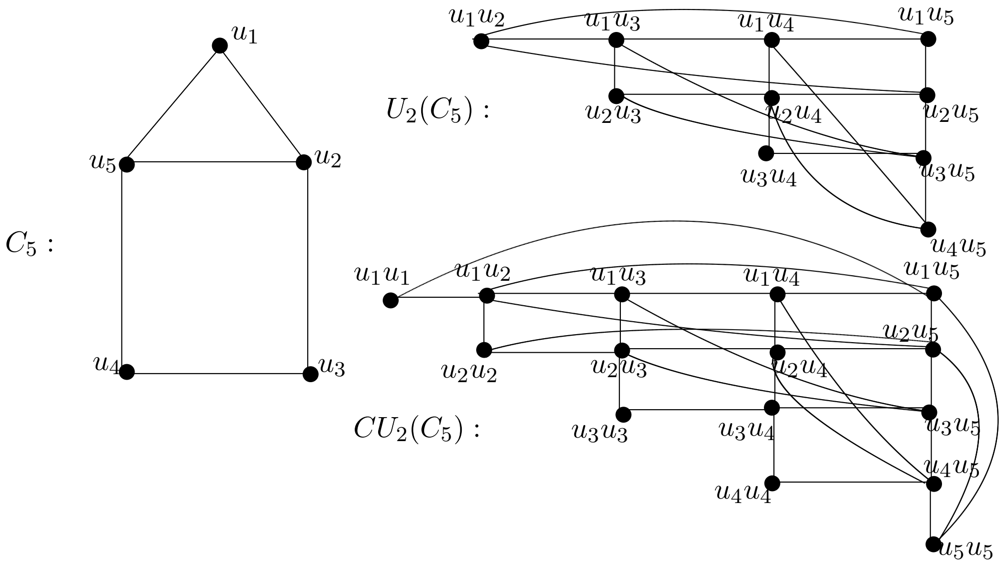

Figure 1. Double Vertex Graph and Complete Double Vertex Graph of C_5

In [11], Gutman and Trinajstić introduced *Zagreb indices.* The first and second Zagreb indices are

$$M_1(G) = \sum_{uv\in E(G)} deg_G(u) + deg_G(v) = \sum_{u\in V(G)} deg_G(u)^2$$

$$M_2(G) = \sum_{uv\in E(G)} deg_G(u)deg_G(v)$$

Some recent results on the Zagreb indices were reported in [4–7,12], where also references to the previous mathematical research n this area can be found. These indices , reflect the extent of branching of the molecular carbon-atom skeleton, and can thus be viewed as molecular structure-descriptors [2, 13].

Do šlić [8] introduced the *first and second Zagreb co-indices* as

$$\overline{M_1}(G) = \sum_{uv\notin E(G)} deg_G(u) + deg_G(v)$$

$$\overline{M_2}(G) = \sum_{uv\notin E(G)} deg_G(u)deg_G(v)$$

The *Forgotten topological index* [9] is defined as

$$F(G) = \sum_{u\in V(G)} deg_G(u)^3 = \sum_{uv\in E(G)} [deg_G(u)^2 + deg_G(v)^2].$$

Shirdel et al. [15] introduced The modified version of Zagreb indices, named "hyper-Zagreb index" and is defined as:

$$HM(G) = \sum_{uv\in E(G)} [deg_G(u) + deg_G(v)]^2.$$

In this paper, we calculate first and second Zagreb indices of the $U_2(G)$ and $CU_2(G)$. Throughout this paper, we denote $\{u_i, u_j\}$ as u_iu_j.

2. Zagreb indices of Double vertex and Complete Double vertex Graphs

Theorem 2.1. *Let G be a graph with $n \geq 2$ with m edges. Then* $M_1(U_2(G)) = (n-6)M_1(G) + 4m(m+1)$.

Proof: By definition,

$$
\begin{aligned}
M_1(U_2(G)) &= \sum_{u_iu_j \in V(U_2(G))} deg_{U_2(G)}(u_iu_j)^2 \\
&= \sum_{u_iu_j \in E(G)} deg_{U_2(G)}(u_iu_j)^2 + \sum_{u_iu_j \notin E(G)} deg_{U_2(G)}(u_iu_j)^2 \\
&= \sum_{u_iu_j \in E(G)} (deg_G(u_i) + deg_G(u_j) - 2)^2 + \sum_{u_iu_j \notin E(G)} (deg_G(u_i) + deg_G(u_j))^2 \\
&= \sum_{u_i,u_j \in V(G), i<j} [deg_G(u_i)^2 + deg_G(u_j)^2 + 2deg_G(u_i)deg_G(u_j)] \\
&- 4 \sum_{u_iu_j \in E(G)} [deg_G(u_i) + deg_G(u_j)] + 4 \sum_{u_iu_j \in E(G)} 1 \\
&= (n-1) \sum_{u_i \in V(G)} deg_G(u_i)^2 + \sum_{u_i,u_j \in V(G), i<j} [2deg_G(u_i)deg_G(u_j)] \\
&- 4M_1(G) + 4m \\
&= (n-5)M_1(G) + 4m + 2M_2(G) + 2\overline{M_2}(G) \\
&= (n-5)M_1(G) + 4m + 4m^2 - M_1(G) \\
&= (n-6)M_1(G) + 4m(m+1).
\end{aligned}
$$

Theorem 2.2. *Let G be a graph with $n \geq 2$, then $M_2(U_2(G)) = (3m+4)M_1(G) + (n-12)M_2(G) - 2F(G) + 12t(G)$, where $F(G)$ is the Forgotten topological index and $t(G)$ is the number of triangles in G.*

Proof:

Let G be a graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $V(U_2(G)) = \{u_iu_j : 1 \leq i < j \leq n\}$. Then by definition,

$$
\begin{aligned}
M_2(U_2(G)) &= \sum_{(u_iu_j)\sim(u_iu_\ell)} deg_{U_2(G)}(u_iu_j)deg_{U_2(G)}(u_iu_\ell) \\
&= \sum_{u_i \in V(G)} \sum_{u_ju_\ell \in E(G-u_i)} deg_{U_2(G)}(u_iu_j)deg_{U_2(G)}(u_iu_\ell)
\end{aligned}
$$

First we partition the edge set of $G - u_i$ into three sets R_1, R_2 and R_3 as follows, see Fig. 2.

Let $R_1 = \{u_ju_\ell \in E(G) \,|\, (u_iu_j)(u_iu_\ell) \in E(U_2(G)),\ u_iu_j,\, u_iu_\ell \notin E(G)\}$,
$R_2 = \{u_ju_\ell \in E(G) \,|\, (u_iu_j)(u_iu_\ell) \in E(U_2(G)),\ u_iu_j \in E(G) \text{ and } u_iu_\ell \notin E(G)\}$
$R_3 = \{u_ju_\ell \in E(G) \,|\, (u_iu_j)(u_iu_\ell) \in E(U_2(G)),\ u_iu_j,\, u_iu_\ell \in E(G)\}$.

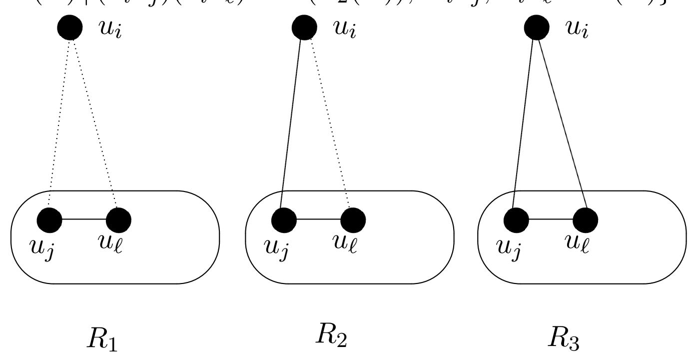

Figure 2. Edges of type R_1, R_2, R_3 of $E(G - u_i)$
(dotted lines and bold lines indicate non-adjacency and adjacency, respectively.)

It is straightforward to see that the edge induced subgraph $\langle R_1 \cup R_2 \cup R_3\rangle$ is isomorphic to $G - u_i$, $1 \le i \le |V(G)|$.

Hence,

$$M_2(U_2(G)) = \sum_{u_i \in V(G)} \Bigg\{ \sum_{\substack{u_ju_\ell \in R_1 \\ i\neq j,\ell}} \{(deg_G(u_i) + deg_G(u_j))(deg_G(u_i) + deg_G(u_\ell))\}$$
$$+ \sum_{\substack{u_ju_\ell \in R_2 \\ i\neq j,\ell}} \{(deg_G(u_i) + deg_G(u_j) - 2)(deg_G(u_i) + deg_G(u_\ell))\}$$
$$+ \sum_{\substack{u_ju_\ell \in R_3 \\ i\neq j,\ell}} \{(deg_G(u_i) + deg_G(u_j) - 2)(deg_G(u_i) + deg_G(u_\ell) - 2)\} \Bigg\}$$

$$= \sum_{u_i \in V(G)} \Bigg\{ \sum_{\substack{u_ju_\ell \in R_1 \\ i\neq j,\ell}} \big\{deg_G(u_i)^2 + deg_G(u_i)deg_G(u_j) + deg_G(u_i)deg_G(u_\ell) + deg_G(u_j)deg_G(u_\ell)\big\}$$
$$+ \sum_{\substack{u_ju_\ell \in R_2 \\ i\neq j,\ell}} \Big\{deg_G(u_i)^2 + deg_G(u_i)deg_G(u_j) + deg_G(u_i)deg_G(u_\ell) + deg_G(u_j)deg_G(u_\ell)$$
$$-2deg_G(u_i) - 2deg_G(u_\ell)\Big\}$$
$$+ \sum_{\substack{u_ju_\ell \in R_3 \\ i\neq j,\ell}} \Big\{deg_G(u_i)^2 + deg_G(u_i)deg_G(u_j) + deg_G(u_i)deg_G(u_\ell) + deg_G(u_j)deg_G(u_\ell)$$
$$-2deg_G(u_i) - 2deg_G(u_j) - 2deg_G(u_i) - 2deg_G(u_\ell) + 4\Big\}\Bigg\}$$

Combing the first four terms of each of the sums over R_1, R_2 and R_3, we have

$$M_2(U_2(G)) = \sum_{u_i \in V(G)} \Bigg\{ \sum_{u_ju_\ell \in E(G-u_i)} \Big\{deg_G(u_i)^2 + deg_G(u_i)deg_G(u_j) + deg_G(u_i)deg_G(u_\ell)$$
$$+deg_G(u_j)deg_G(u_\ell)\Big\} + \sum_{\substack{u_ju_\ell \in R_2 \\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G(u_\ell)\}$$
$$+ \sum_{\substack{u_ju_\ell \in R_3 \\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G(u_j) - 2deg_G(u_i) - 2deg_G(u_\ell) + 4\}\Bigg\}$$

$$M_2(U_2(G)) = \sum_{u_i \in V(G)} \Big\{|E(G-u_i)|deg_G(u_i)^2 + deg_G(u_i)M_1(G-u_i) + M_2(G-u_i)\Big\}$$
$$+ \sum_{u_i \in V(G)} \Bigg\{ \sum_{\substack{u_ju_\ell \in R_2 \\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G(u_\ell)\}$$
$$+ \sum_{\substack{u_ju_\ell \in R_3 \\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G(u_j) - 2deg_G(u_i) - 2deg_G(u_\ell) + 4\}\Bigg\}$$

$= S_1 + S_2$, where

$$S_1 = \sum_{u_i \in V(G)} \Big\{|E(G-u_i)|deg_G(u_i)^2 + deg_G(u_i)M_1(G-u_i) + M_2(G-u_i)\Big\} \text{ and}$$

$$S_2 = \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_ju_\ell\in R_2\\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G u_\ell)\} + \sum_{\substack{u_ju_\ell\in R_3\\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G(u_j) - 2deg_G(u_i) - 2deg_G(u_\ell) + 4\} \Bigg\}$$

Now,

$$S_1 = \sum_{u_i\in V(G)} \Bigg\{ \big(|E(G)| - deg_G(u_i)\big) deg_G(u_i)^2 + deg_G(u_i) \Bigg[M_1(G) - \sum_{u_r\in N(u_i)} (deg_G(u_i) + deg_G(u_r)) \Bigg] + M_2(G) - \sum_{u_r\in N(u_i)} \Big\{ deg_G(u_i) deg_G(u_r) \Big\} \Bigg\}$$

$$= |E(G)|M_1(G) - \sum_{u_i\in V(G)} deg_G(u_i)^3 + 2|E(G)|M_1(G) - \sum_{u_i\in V(G)} deg_G(u_i)^3 - \sum_{u_i\in V(G)} deg_G(u_i) \sum_{u_r\in N(u_i)} deg_G(u_r) + |V(G)|M_2(G) - \sum_{u_i\in V(G)} deg_G(u_i) \sum_{u_r\in N(u_i)} deg_G(u_r)$$

$$= 3|E(G)|M_1(G) - 2\sum_{u_i\in V(G)} deg_G(u_i)^3 + |V(G)|M_2(G) - 2\sum_{u_i\in V(G)} deg_G(u_i) \sum_{u_r\in N(u_i)} deg_G(u_r)$$

$$= 3|E(G)|M_1(G) - 2\sum_{u_i\in V(G)} deg_G(u_i)^3 + |V(G)|M_2(G) - 4M_2(G),$$

since $\sum_{u_i\in V(G)} deg_G(u_i) \sum_{u_r\in N(u_i)} deg_G(u_r) = 2M_2(G)$

$$= 3mM_1(G) - 2F(G) + (n-4)M_2(G).$$

$$S_2 = \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_ju_\ell\in R_2\\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G(u_\ell)\} + \sum_{\substack{u_ju_\ell\in R_3\\ i\neq j,\ell}} \Big\{ -2deg_G(u_i) - 2deg_G(u_j) - 2deg_G(u_i) - 2deg_G(u_\ell) + 4 \Big\} \Bigg\}$$

$$= \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_ju_\ell\in R_2\\ i\neq j,\ell}} \{-2deg_G(u_i) - 2deg_G(u_\ell)\} + \sum_{\substack{u_ju_\ell\in R_3\\ i\neq j,\ell}} \Big\{ -2deg_G(u_i) - 2deg_G(u_j) - 2deg_G(u_i) - 2deg_G(u_\ell) \Big\} + 4t(u_i) \Bigg\},$$

where $t(u_i)$ is the number of triangles containing u_i.

In the sums over R_2 and R_3, we investigate how many times, the term $deg_G(u_i)$ is counted. Hence,

$$S_2 = \sum u_i \in V(G) \Bigg\{ \sum_{u_r\in N(u_i)} \sum_{\substack{u_ru_\ell\in E(G)\\ u_\ell\neq u_i}} (-2deg_G(u_\ell)) + \sum_{u_r\in N(u_i)} \sum_{\substack{u_ru_\ell\in E(G)\\ u_\ell\neq u_i}} (-2deg_G(u_\ell)) + 4t(u_i) \Bigg\}$$

$$= 2 * \sum_{u_i\in V(G)} \Bigg\{ \sum_{u_r\in N(u_i)} (-2deg_G(u_r)deg_G(u_i) + 2deg_G(u_i)) \Bigg\} + 12\ t(G),$$

$$= -4 \sum_{u_i\in V(G)} \Big\{ \sum_{u_r\in N(u_i)} deg_G(u_i)deg_G(u_r) \Big\} + 4 \sum_{u_i\in V(G)} \sum_{u_r\in N(u_i)} \big(deg_G(u_i)\big) + 12\ t(G)$$

$$S_2 = -8M_2(G) + 4\sum_{u_i\in V(G)} deg_G(u_i)^2 + 12\ t(G) = -8M_2(G) + 4M_1(G) + 12\ t(G)$$

Therefore, $M_2(U_2(G)) = 3|E(G)|M_1(G) - 2F(G) + (|V(G)| - 4)M_2(G) - 8M_2(G) + 4M_1(G) + 12\ t(G)$

$$= (3m+4)M_1(G) + (n-12)M_2(G) - 2F(G) + 12\ t(G),$$

where $F(G)$ is the Forgotten topological index and $t(G)$ is the number of triangles in the graph G.

Theorem 2.3. *Let G be a graph with $n \geq 2$. Then, $M_1(CU_2(G)) = (n-1)\, M_1(G) + 4m^2$.*

Proof.

$$M_1(CU_2(G)) = \sum_{(u_iu_j)\sim(u_iu_\ell)} \big[deg_{CU_2(G)}(u_iu_j) + deg_{CU_2(G)}(u_iu_\ell)\big]$$

$$= \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_i=u_j \\ u_iu_\ell\in E(G)}} \big[deg_{CU_2(G)}(u_iu_i) + deg_{CU_2(G)}(u_iu_\ell)\big]$$

$$+ \sum_{\substack{u_i\neq u_j \\ u_ju_\ell\in E(G)}} \big[deg_{CU_2(G)}(u_iu_j) + deg_{CU_2(G)}(u_iu_\ell)\big] \Bigg\}$$

$$= \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_i=u_j \\ u_iu_\ell\in E(G)}} [deg_G(u_i) + deg_G(u_i) + deg_G(u_\ell)]$$

$$+ \sum_{\substack{u_i\neq u_j \\ u_ju\ell\in E(G)}} [deg_G(u_i) + deg_G(u_j) + deg_G(u_i) + deg_G(u_\ell)] \Bigg\}$$

$$= (n-1)\, M_1(G) \; + \; 4m^2$$

Theorem 2.4. *Let G be a graph with $n \geq 2$. Then,*
$M_2(CU_2(G)) = 3mM_1(G) + (n-2)M_2(G) - F(G)$.

Proof.

$$M_2(CU_2(G)) = \sum_{(u_iu_j)\,\sim\,(u_iu_\ell)} \big[deg_{CU_2(G)}(u_iu_j)deg_{CU_2(G)}(u_iu_\ell)\big]$$

$$= \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_i=u_j \\ u_iu_\ell\in E(G)}} \big[deg_{CU_2(G)}(u_iu_i)deg_{CU_2(G)}(u_iu_\ell)\big]$$

$$+ \sum_{\substack{u_i\neq u_j \\ u_ju_\ell\in E(G)}} \big[deg_{CU_2(G)}(u_iu_j)deg_{CU_2(G)}(u_iu_\ell)\big] \Bigg\}$$

$$= \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_i=u_j \\ u_iu_\ell\in E(G)}} \Bigg[deg_G(u_i)\Big(deg_G(u_i) + deg_G(u_\ell)\Big)\Bigg]$$

$$+ \sum_{\substack{u_i\neq u_j \\ u_ju_\ell\in E(G)}} \Bigg[\Big(deg_G(u_i) + deg_G(u_j)\Big)\Big(deg_G(u_i) + deg_G(u_\ell)\Big)\Bigg] \Bigg\}$$

$$= \sum_{u_i\in V(G)} \Bigg\{ \sum_{\substack{u_i=u_j \\ u_iu_\ell\in E(G)}} \big[deg_G(u_i)^2 + deg_G(u_i)deg_G(u_\ell)\big]$$

$$+ \sum_{\substack{u_i\neq u_j \\ u_ju_\ell\in E(G)}} \big[deg_G(u_i)^2 + deg_G(u_i)deg_G(u_\ell) + deg_G(u_i)deg_G(u_j) + deg_G(u_j)deg_G(u_\ell)\big] \Bigg\}$$

$$= \sum_{u_i \in V(G)} \Bigg\{ \sum_{u_r u_s \in E(G)} \big[deg_G(u_i)^2 + deg_G(u_r) deg_G(u_s)\big]$$

$$+ \sum_{\substack{u_j u_\ell \in E(G) \\ i \neq j,\ell}} [deg_G(u_i) deg_G(u_j) + deg_G(u_i) deg_G(u_\ell)] \Bigg\}$$

$$= \sum_{u_i \in V(G)} \left\{ |E(G)| deg_G(u_i)^2 + M_2(G) + \sum_{\substack{u_j u_\ell \in E(G) \\ i \neq j,\ell}} \left[deg_G(u_i) \Big(deg_G(u_j) + deg_G(u_\ell) \Big) \right] \right\}$$

$$= |E(G)| M_1(G) + |V(G)| M_2(G) + \sum_{u_i \in V(G)} \{ deg_G(u_i) M_1(G - u_i) \}$$

$$= |E(G)| M_1(G) + |V(G)| M_2(G) + \sum_{u_i \in V(G)} deg_G(u_i) \left\{ M_1(G) - \sum_{u_r \in N(u_i)} \Big(deg_G(u_i) + deg_G(u_r) \Big) \right\}$$

$$= 3m M_1(G) + (n-2) M_2(G) - F(G).$$

3. Hyper-Zagreb index of Double vertex Graphs

The hyper-Zagreb index of G is defined as $HM(G) = \sum_{uv \in E(G)} [deg_G(u) + deg_G(v)]^2$. In this section we calculate $HM(U_2(G))$ and $HM(CU_2(G))$.

Remark 3.1. *We know that,* $HM(G) = \sum_{uv \in E(G)} [deg_G(u) + deg_G(v)]^2 = F(G) + 2M_2(G)$.

Proof.
$$HM(G) = \sum_{uv \in E(G)} (deg_G(u) + deg_G(v))^2$$
$$= \sum_{uv \in E(G)} (deg_G(u)^2 + deg_G(v)^2 + 2 deg_G(u) deg_G(v))$$
$$= \sum_{uv \in E(G)} (deg_G(u)^2 + deg_G(v)^2) + 2 \sum_{uv \in E(G)} deg_G(u) deg_G(v)$$
$$= F(G) + 2M_2(G).$$

Theorem 3.1. *Let G be a graph with $n \geq 2$. Then*
$F(U_2(G)) = (6m+12) M_1(G) + (n-10) F(G) - 12 M_2(G) - 8m.$

Proof.
$$F(U_2(G)) = \sum_{u_i u_j \sim u_i u_\ell} (deg^2_{U_2(G)}(u_i u_j) + deg^2_{U_2(G)}(u_i u_\ell))$$

$$= \sum_{u_i} \sum_{u_j \sim u_\ell} \Big(deg^2_{U_2(G)}(u_i u_j) + deg^2_{U_2(G)}(u_i u_\ell) \Big)$$

$$= \sum_{u_i} \Bigg\{ \sum_{u_j u_\ell \in R_1} \big[(deg_G(u_i) + deg_G(u_j))^2 + (deg_G(u_i) + deg_G(u_\ell))^2 \big]$$

$$+ \sum_{u_j u_\ell \in R_2} \big[(deg_G(u_i) + deg_G(u_j) - 2)^2 + (deg_G(u_i) + deg_G(u_\ell))^2 \big]$$

$$+ \sum_{u_j u_\ell \in R_3} \big[(deg_G(u_i) + deg_G(u_j) - 2)^2 + (deg_G(u_i) + deg_G(u_\ell) - 2)^2 \big] \Bigg\}$$

$$= \sum_{u_i} \big\{ 2|E(G - u_i)| deg^2_G(u_i) + F(G - u_i) + 2 d_i M_1(G - u_i) \big\}$$

$$+ \sum_{u_i} \Bigg\{ \sum_{u_j u\ell \in R_2} (-4 deg_G(u_i) - 4 deg_G(u_j) + 4)$$

$$+ \sum_{u_j u\ell \in R_3} (-8 deg_G(u_i) - 4 deg_G(u_j) - 4 deg_G(u_\ell) + 8) \Bigg\}$$

$= 6mM_1(G) + (n-6)F(G) - 4M_2(G) + \sum_{u_i} \sum_{u_j u_\ell \in R_2} 4 + \sum_{u_i} \sum_{u_j u_\ell \in R_3} 8$

$+ \sum_{u_i} \sum_{u_r \in N(u_i)} ((deg_G(u_r) - 1)(-4deg_G(u_i))) - 4 \sum_{u_i} \sum_{u_r \in N(u_i)} (deg_G(u_i)(deg_G(u_i) - 1))$

$= 6mM_1(G) + (n-6)F(G) - 4M_2(G) + \sum_{u_i} 4(deg_G(u_i)(deg_G(u_i) - 1) - 2t(u_i)) + \sum_{u_i} 8t(u_i) - 8M_2(G) + 4M_1(G) - 4 \sum_{u_i} \sum_{u_r \in N(u_i)} (deg_G^2(u_i) - deg_G(u_i))$

$= 6mM_1(G) + (n-6)F(G) - 4M_2(G) + 4M_1(G) - 8m - 24(G) + 24t(G) - 8M_2(G) + 4M_1(G) - 4F(G) + 4M_1(G)$

$= (6m + 12)M_1(G) + (n - 10)F(G) - 12M_2(G) - 8m.$

Theorem 3.2. *Let G be a graph with $n \geq 2$. Then $F(CU_2(G)) = (n-5)F(G) + 2HM(G) + 6mM_1(G) - 4M_2(G)$.*

Proof. We know that $F(CU_2(G)) = \sum_{u_i u_j \sim u_i u_\ell} (deg^2_{CU_2(G)}(u_i u_j) + deg^2_{CU_2(G)}(u_i u_\ell))$

$$= \sum_{u_i \in V(G)} \left\{ \sum_{u_i u_i \sim u_i u_\ell} \left(deg^2_{CU_2(G)}(u_i u_i) + deg^2_{CU_2(G)}(u_i u_\ell) \right) + \sum_{u_i u_j \sim u_i u_\ell} \left(deg^2_{CU_2(G)}(u_i u_j) + deg^2_{CU_2(G)}(u_i u_\ell) \right) \right\}$$

$$= \sum_{u_i \in V(G)} \left\{ \sum_{u_i \sim u_\ell} \left(deg^2_G(u_i) + (deg_G(u_i) + deg_G(u_\ell))^2 \right) + \sum_{u_j \sim u_\ell} \left((deg_G(u_i) + deg_G(u_j))^2 + (deg_G(u_i) + deg_G(u_\ell))^2 \right) \right\}$$

$$= \sum u_i \in V(G) \left\{ deg^3_G(u_i) + \sum_{u_i \sim u\ell} (deg_G(u_i) + deg_G(u_\ell))^2 + \sum_{u_j \sim u_\ell} \left(2deg^2_G(u_i) + deg^2_G(u_j) + deg^2_G(u_\ell) + 2deg_G(u_i)(deg_G(u_j) + deg_G(u_\ell)) \right) \right\}$$

$$= \sum u_i \in V(G) \left\{ F(G) + 2HM(G) + \sum_{u_j \in V(G)} \left(2|E(G-u_i)| deg^2_G(u_i) + F(G-u_i) + 2deg_G(u_i) M_1(G-u_i) \right) \right\}$$

$= (n-5)F(G) + 2HM(G) + 6mM_1(G) - 4M_2(G).$

Corollary 3.1. *The hyper-Zagreb index of the double vertex of graph $U_2(G)$ of G is $HM(U_2(G)) = (12m + 20)M_1(G) + (2n - 36)M_2(G) + (n - 14)F(G) - 8m + 24t(G)$.*

Proof. By Remark 3.1, $HM(G) = F(G) + 2M_2(G)$ and now the proof follows from Theorems 2.2 and 3.1.

Corollary 3.2. *The hyper-Zagreb index of the complete double vertex of graph $CU_2(G)$ of G is $HM(CU_2(G)) = 12mM_1(G) + (2n - 8)M_2(G) + 2HM(G) + (n - 7)F(G)$.*

Proof. Now the proof follows from Remark 3.1 and Theorems 2.4 and 3.2.

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Dr. S. Sampath Kumar graduated from Annamalai University, Annamalainagar, India in 2007. He received his MSc. degree and Ph.D. in Mathematics from Annamalai University 2008 and 2013, respectively. He is currently a faculty of Department of Mathematics in SSN College of Engineering, Chennai, India since 2014. His research interests focus mainly on graph theory.

Dr. R. Sundareswaran received his Master's degree in 1999, Ph. D degree in 2011 from Madurai Kamaraj University, Madurai, India. He did his Ph.D. degree in the major research project entitled "Domination Integrity in graphs" sponsored by Department of Science and Technology, New Delhi, India. His area of interest including vulnerability parameters of graphs, domination and colouring. He is currently working as Assistant Professor in the Department of Mathematics, SSN College of Engineering, Chennai, India. He published more than 20 research article in international referred journals. He is a reviewer of American Mathematical society.

Dr. Sundarakannan Mahilmaran was awarded his Ph.D. degree in Mathematics at National Center for Advanced Research in Discrete Mathematics (n-CARDMATH), Kalasalingam University, India in 2011. He has published over seven research papers in international journals and a book chapter in the book titled "Handbook of Graph Theory, Combinatorial Optimization, and Algorithms" by Chapman and Hall/CRC Press. He is working as an assistant professor, Department of Mathematics, SSN College of Engineering, Chennai, India.