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GENERALIZED FIXED POINT RESULTS WITH MULTI-VALUED MAPPINGS

P. KONAR[1], A. K. JANA[2], R. B. DAS[2], S. K. BHANDARI[2], R. R. DEVI[1] , §

ABSTRACT. In this article we deduce fixed point results for multi-valued contraction mappings. We primarily established two fixed results. One of them is the generalization of Nadler's contraction and the other result is the generalization of Mizoguchi-Takahashi's contraction. Some corollaries have been obtained from the main results and our results generalize some of the existing results. Illustrative examples are also constructed to support our main results.

Keywords: Fixed point, Hausdorff metric space, Multivalued mapping, Common fixed point.

AMS Subject Classification: 41A50, 47H10, 54H25

1. INTRODUCTION

Metric fixed point theory is one of the important tool for the existence of fixed point and allied problems for self mappings under different mathematical conditions. The method provides solutions for fractional differential equation, functional and matrix equations, integral equations etc. In this line of research, Banach [1] proved the Banach contraction mapping principle in 1922 and has been generalized in numerous research article [2], [3], [5], [4], [7]. Some preliminaries and basic works in this field are as follows.

Let (X, d) be a metric space . We denote by $CB(X)[\neq \{\phi\}]$ the family of closed and bounded subsets of X. Define $D(x, A) := inf\{d(x, a) : \forall\, a \in A\}$, where $A, B \in CB(X)$, and $x \in X$ and $H(A, B) := max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}$.
$H(\cdot, \cdot)$ is known as the pompeiu-Hausdorff distance on $CB(X)$.

[1] Department of Mathematics, Amity University, Kadampukur, 24PGS(S), Kolkata, West Bengal, 700135, India.
e-mail: pulakkonar@gmail.com; ORCID: https://orcid.org/0000-0003-0971-511X.
e-mail: rashmiirekha1995@gmail.com; ORCID: https://orcid.org/0000-0001-8563-4931.
[2] Department of Mathematics, Bajkul Milani Mahavidyalaya, P.O- Kismat Bajkul Dist, Purba Medinipur, West Bengal - 721655, India.
e-mail: ashimjana67@gmail.com; ORCID: https://orcid.org/0000-0002-0056-4960.
e-mail: radha.23j@gmail.com; ORCID: https://orcid.org/0000-0003-1979-3971.
e-mail: skbhit@yahoo.co.in; ORCID: https://orcid.org/0000-0002-5762-9315.
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Definition 1.1. *An elements $x \in X$ is a fixed point for a multi-valued mapping $T : X \to CB(X)$, if such that $x \in T(x)$.*

If (X, d) is a complete metric space then $(CB(X), H)$ is a complete Hausdorff metric space. (Lemma 8.1.4, of [13]).
Nadler [10] extended the Banach contraction mapping principle [1] to set-valued mappings in the year 1969. In 1989, Mizoguchi and Takahashi [9] extended the Nadler's theorem. Some of the existing literatures in this line are [6], [8], [11], [12], [14]. We have calculated the generalized form of Nadler's fixed point theorem and the gerelazied form of Mizoguchi - Takahasi fixed point theorem.

Example 1.1. *Every single valued mapping can be interpreted as a multi-valued mapping. Let $f : X \to Y$ be a single valued mapping. Consider $T : X \to 2^Y$ by $Tx = \{f(x)\}$. It may be noted that T is multi-valued mapping iff for each $x \in X$, $Tx \subseteq Y$. Unless otherwise we always assume Tx is non-empty for each $x \in X$.*

Definition 1.2. *Let (X, d) be a metric space. A map $T : X \to CB(X)$ is said to be multi valued contraction such that $H(Tx, Ty) \leq \lambda d(x, y)$, for all $x, y \in X$, where $0 \leq \lambda < 1$.*

Nadler [10] extended the Banach contraction mapping principle [1] to set-valued mappings in the year 1969. We have calculated the generalized form of the theorem of Nadler and Mizoguchi et. al.

Lemma 1.1. [10] *Let (X, d) be a metric space and $A, B \in CB(X)$. Then for each $a \in A$ and $\epsilon > 0$, there exists an $b \in B$ such that $d(a, b) \leq H(A, B) + \epsilon$.*

Theorem 1.1. *(Nadler [10]) Let (X, d) be a complete metric space and let $T : X \to CB(X)$ such that $H(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$, $0 \leq \alpha < 1$.
Then T has a fixed point.*

Theorem 1.2. *(Mizoguchi and Takahashi [9].) Let (X, d) be a complete metric space and let $T : X \to CB(X)$ such that $H(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$ for all $x, y \in X$ and $\alpha : [0, \infty) \to [0, 1)$ satisfying $\lim_{s \to t^+} \sup \alpha(s) < 1$ for all $t \in [0, \infty)$.
Then T has a fixed point.*

2. Main Results

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a mapping such that*

$$\begin{aligned} H(Tx, Ty) \leq & \alpha_1 d(x, y) + \alpha_2 D(x, Tx) + \alpha_3 D(y, Ty) + \alpha_4 [D(x, Tx) + D(y, Ty)] \\ & + \alpha_5 [D(x, Ty) + D(y, Tx)] + \alpha_6 [D(x, Tx) + D(y, Tx)] \\ & + \alpha_7 [D(y, Ty) + D(x, Ty)] \end{aligned}$$

*for all $x, y \in X$, where $\alpha_i \geq 0$ $(i = 1, 2, ..., 7)$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + 3\alpha_7 < 1$.
Then T has a fixed point.*

Proof. Let $x_0 \in X$, $x_1 \in Tx_0$ and we consider $r = \frac{\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7}{1 - (\alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_7)}$.
If $r = 0$ then the above theorem is trivially hold.

Assume that $r > 0$.
Then by lemma 1.1, we have

$$\begin{cases} \exists x_2 \in Tx_1; \quad d(x_1,x_2) \le H(Tx_0,Tx_1) + r, \\ \exists x_3 \in Tx_2; \quad d(x_2,x_3) \le H(Tx_1,Tx_2) + r^2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \exists x_{n+1} \in Tx_n; \quad d(x_n,x_{n+1}) \le H(Tx_{n-1},Tx_n) + r^n, \end{cases}$$

Hence , we have

$$\begin{aligned} d(x_n,x_{n+1}) \le & H(Tx_{n-1},Tx_n) + r^n \\ \le & \alpha_1 d(x_{n-1},x_n) + \alpha_2 D(x_{n-1},Tx_{n-1}) + \alpha_3 D(x_n,Tx_n) \\ & + \alpha_4[D(x_{n-1},Tx_{n-1}) + D(x_n,Tx_n)] + \alpha_5[D(x_{n-1},Tx_n) + D(x_n,Tx_{n-1})] \\ & + \alpha_6[D(x_{n-1},Tx_{n-1}) + D(x_n,Tx_{n-1})] + \alpha_7[D(x_n,Tx_n) + D(x_{n-1},Tx_n)] + r^n, \\ \le & \alpha_1 d(x_{n-1},x_n) + \alpha_2 d(x_{n-1},x_n) + \alpha_3 d(x_n,x_{n+1}) \\ & + \alpha_4[d(x_{n-1},x_n) + d(x_n,x_{n+1})] + \alpha_5[d(x_{n-1},x_{n+1}) + d(x_n,x_n)] \\ & + \alpha_6[d(x_{n-1},x_n) + d(x_n,x_n)] + \alpha_7[d(x_n,x_{n+1}) + d(x_{n-1},x_{n+1})] + r^n, \\ \le & (\alpha_1+\alpha_2+\alpha_4+\alpha_6) d(x_{n-1},x_n) + (\alpha_3+\alpha_4+\alpha_7) d(x_n,x_{n+1}) \\ & + \alpha_5[d(x_{n-1},x_n) + d(x_n,x_{n+1})] + \alpha_7[d(x_{n-1},x_n) + d(x_n,x_{n+1})] + r^n, \\ & \qquad\qquad\qquad\qquad \text{[By triangle inequality]} \\ = & (\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7) d(x_{n-1},x_n) + (\alpha_3+\alpha_4+\alpha_5+2\alpha_7) d(x_n,x_{n+1}) + r^n, \end{aligned}$$

which implies,
$\{1-(\alpha_3+\alpha_4+\alpha_5+2\alpha_7)\} d(x_n,x_{n+1}) \le (\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7) d(x_{n-1},x_n) + r^n$,
that is, $d(x_n,x_{n+1}) \le r\, d(x_{n-1},x_n) + \frac{r^n}{1-(\alpha_3+\alpha_4+\alpha_5+2\alpha_7)}$, for all $n \in N$.
Continuing the process, we have

$$d(x_n,x_{n+1}) \le r^n d(x_0,x_1) + \frac{n\, r^n}{1-(\alpha_3+\alpha_4+\alpha_5+2\alpha_7)}, \quad \text{for all}\, n \in N.$$

Now ,

$$r = \frac{\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7}{1-(\alpha_3+\alpha_4+\alpha_5+2\alpha_7)} < 1$$

So, $\sum\limits_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$.
Hence, $\{x_n\}$ is a Cauchy sequence in X.
By completeness of X, there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.
Now,

$$\begin{aligned} D(x^*,Tx^*) & \le d(x^*,x_{n+1}) + D(x_{n+1},Tx^*) \\ & \le d(x^*,x_{n+1}) + H(Tx_n,Tx^*) \\ & \le d(x^*,x_{n+1}) + \alpha_1 d(x_n,x^*) + \alpha_2 D(x_n,Tx_n) + \alpha_3 D(x^*,Tx^*) \\ & \quad + \alpha_4[D(x_n,Tx_n) + D(x^*,Tx^*)] + \alpha_5[D(x_n,Tx^*) + D(x^*,Tx_n)] \\ & \quad + \alpha_6[D(x_n,Tx_n) + D(x^*,Tx_n)] + \alpha_7[D(x^*,Tx^*) + D(x_n,Tx^*)], \quad \text{for all}\, n \in N \\ & \le d(x^*,x_{n+1}) + \alpha_1 d(x_n,x^*) + \alpha_2 d(x_n,x_{n+1}) + \alpha_3 D(x^*,Tx^*) \\ & \quad + \alpha_4[d(x_n,x_{n+1}) + D(x^*,Tx^*)] + \alpha_5[D(x_n,Tx^*) + d(x^*,x_{n+1})] \\ & \quad + \alpha_6[d(x_n,x_{n+1}) + d(x^*,x_{n+1})] + \alpha_7[D(x^*,Tx^*) + D(x_n,Tx^*)], \quad \text{for all}\, n \in N \end{aligned}$$

Taking a limit $n \to \infty$,we get

$$D(x^*, Tx^*) \leq (\alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_7) D(x^*, Tx^*)$$

Hence,

$$D(x^*, Tx^*) = 0. \quad (\text{since, } \alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_7 < 1)$$

It follows that $x^* \in Tx^*$.
Therefore, $\{x^*\}$ is a fixed point of T. □

Example 2.1. *Let* $X = [0, 1]$*. Define* $d : X \times X \longrightarrow X$ *by* $d(x, y) = |x - y|$*, for all* $x, y \in X$*. Then* (X, d) *is a complete metric space . Now consider the mapping* $T : X \to CB(X)$ *define by* $Tx = [0, \frac{x}{10}]$*, where* $x \in [0, 1]$*.*
Let us assume
$\alpha_1 = \frac{1}{9}, \alpha_2 = \frac{1}{6}, \alpha_3 = \frac{1}{72}, \alpha_4 = \frac{1}{36}, \alpha_5 = \frac{1}{18}, \alpha_6 = \frac{2}{9}, \alpha_7 = \frac{1}{54}$*, so that*

$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + 3\alpha_7 < 1$$

is satisfied.

Now, we have to consider the following two cases:

Case I:
If $x, y \in [0, 1]$*. The contractive condition of theorem 2.1 is trivially hold for the case when* $x = y = 0$*.*

Case II:
Suppose without any loss of generality, we can take $x < y$ *and* $x, y \neq 0$*.*

Then,
$d(x, y) = |x - y|$, $D(x, Tx) = \frac{9x}{10}$, $D(y, Ty) = \frac{9y}{10}$, $D(x, Ty) = |x - \frac{y}{10}|$ *and* $D(y, Tx) = |y - \frac{x}{10}|$.

$$\begin{aligned} L.H.S = H(Tx, Ty) &= Max\Big\{ \sup_{a \in Tx} D(a, Ty), \sup_{b \in Ty} D(b, Tx) \Big\} \\ &= Max\Big\{ \sup_{a \in Tx} [\inf\{d(a, p) : \forall p \in Ty\}], \sup_{b \in Ty} [\inf\{d(b, q) : \forall q \in Tx\}] \Big\} \\ &= Max\Big\{ \sup_{a \in [0, \frac{x}{10}]} [\inf\{d(a, p) : \forall p \in [0, \frac{y}{10}]\}], \sup_{b \in [0, \frac{y}{10}]} [\inf\{d(b, q) : \forall q \in [0, \frac{x}{10}]\}] \Big\} \\ &= Max\{0, |\frac{x}{10} - \frac{y}{10}|\} = |\frac{x}{10} - \frac{y}{10}| \end{aligned}$$

$$\begin{aligned} R.H.S &= \alpha_1 d(x, y) + \alpha_2 D(x, Tx) + \alpha_3 D(y, Ty) + \alpha_4 [D(x, Tx) + D(y, Ty)] \\ &\quad + \alpha_5 [D(x, Ty) + D(y, Tx)] + \alpha_6 [D(x, Tx) + D(y, Tx)] + \alpha_7 [D(y, Ty) + D(x, Ty)], \\ &= \frac{1}{9}|x - y| + \frac{1}{6}\frac{9x}{10} + \frac{1}{72}\frac{9y}{10} + \frac{1}{36}[\frac{9x}{10} + \frac{9y}{10}] + \frac{1}{18}[|x - \frac{y}{10}| + |y - \frac{x}{10}|] + \frac{2}{9}[\frac{9x}{10} + |y - \frac{x}{10}|] \\ &\quad + \frac{1}{54}[\frac{9y}{10} + |x - \frac{y}{10}|], \\ &= \frac{1}{9}|x - y| + \frac{9x}{10}(\frac{1}{6} + \frac{1}{36} + \frac{2}{9}) + \frac{9y}{10}(\frac{1}{72} + \frac{1}{36} + \frac{1}{54}) + |x + \frac{y}{10}|(\frac{1}{18} + \frac{1}{54}) + |y - \frac{x}{10}|(\frac{1}{18} + \frac{2}{9}) \\ &= \frac{1}{9}|x - y| + \frac{3x}{8} + \frac{13y}{240} + \frac{2}{27}|x - \frac{y}{10}| + \frac{5}{18}|y - \frac{x}{10}| \end{aligned}$$

Therefore, $L.H.S \leq R.H.S.$ *for all* $x, y (x < y) \in [0, 1]$ *and all the conditions of theorem*

2.1 are satisfied. Hence, we have $T0 = 0$, that is, $\{0\}$ is a fixed point of T.

Corollary 2.1. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ such that $H(Tx, Ty) \le \alpha d(x, y) + \beta[D(x, Tx) + D(y, Tx)] + \gamma[D(y, Ty) + D(x, Ty)]$ $\forall x, y \in X$, where $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + 3\gamma < 1$.
Then T has a fixed point.*

Proof. By the substitutions of $\alpha_1 = \alpha, \alpha_6 = \beta, \alpha_7 = \gamma$ in the theorem 2.1, we can obtain the proof of the corollary where $\alpha_i = 0\,(i = 2, 3, 4, 5)$. □

Corollary 2.2. *(Nadler [10]) Let (X, d) be a complete metric space and let $T : X \to CB(X)$ such that $H(Tx, Ty) \le \alpha d(x, y)$ for all $x, y \in X$, $0 \le \alpha < 1$.
Then T has a fixed point.*

Proof. We can obtain the proof by putting $\alpha_1 = \alpha$ and $\alpha_i = 0\,(i = 2, 3, ..., 7)$ in the theorem 2.1. □

Corollary 2.3. *([11], [12]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ such that $H(Tx, Ty) \le \beta[D(x, Tx) + D(y, Ty)]$ for all $x, y \in X$ and $\beta \in [0, \frac{1}{2})$.
Then T has a fixed point.*

Proof. The proof follows by putting $\alpha_4 = \beta$ and $\alpha_i = 0\,(i = 1, 2, 3, 5, 6, 7)$ in the theorem 2.1. □

Corollary 2.4. *([5]) Let (X, d) be a complete metric space and let $T : X \to X$ such that $d(Tx, Ty) \le \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + \gamma[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$, where $\alpha, \beta, \gamma \ge 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.*

Proof. If we put $\alpha_1 = \alpha, \alpha_4 = \beta, \alpha_5 = \gamma$ and $\alpha_i = 0\,(i = 2, 3, 6, 7)$ in the theorem 2.1. □

Corollary 2.5. *([4]) Let (X, d) be a complete metric space and let $T : X \to CB(X)$ such that $H(Tx, Ty) \le \alpha d(x, y) + \beta[D(x, Tx) + D(y, Ty)] + \gamma[D(x, Ty) + D(y, Tx)]$ $\forall x, y \in X$, where $\alpha, \beta, \gamma \ge 0$ and $\alpha + 2\beta + 2\gamma < 1$.
Then T has a fixed point.*

Proof. By the substitutions of $\alpha_1 = \alpha, \alpha_4 = \beta, \alpha_5 = \gamma$ in the theorem 2.1, we can obtain the proof of the corollary where $\alpha_i = 0\,(i = 2, 3, 6, 7)$. □

Corollary 2.6. *([4]) Let (X, d) be a complete metric space and let $T : X \to CB(X)$ such that $H(Tx, Ty) \le \gamma[D(x, Ty) + D(y, Tx)]$ for all $x, y \in X$, where $\gamma \in [0, \frac{1}{2})$.
Then T has a fixed point.*

Proof. By the substitutions of $\alpha_5 = \gamma$ in the theorem 2.1, we can obtain the proof of the corollary where $\alpha_i = 0\,(i = 1, 2, 3, 4, 6, 7)$. □

Theorem 2.2. *Let (X, d) be complete metric space and $T_1, T_2 : X \to CB(X)$ be a two multi-valued mappings, such that*

$$\begin{aligned} H(T_1x, T_2y) \le\, & \alpha_1'(d(x, y))d(x, y) + \alpha_2(d(x, y))D(x, T_1x) + \alpha_3(d(x, y))D(y, T_2y) \\ & + \alpha_4(d(x, y))[D(x, T_1x) + D(y, T_2y)] + \alpha_5(d(x, y))[D(x, T_2y) + D(y, T_1x)] \\ & + \alpha_6(d(x, y))[D(x, T_1x) + D(y, T_1x)] + \alpha_7(d(x, y))[D(y, T_2y) + D(x, T_2y)] \end{aligned}$$

for all $x, y \in X$, where $\alpha_i : [0, \infty) \to [0, 1)\,(i = 1, 2, ..., 7)$ such that

$$\alpha_1' : [0, \infty) \to [0, 1) \quad by\ \alpha_1'(t) = \frac{\alpha_1(t) + 1 - \alpha_3(t) - \alpha_2(t) - 2\alpha_4(t) - 2\alpha_5(t) - \alpha_6(t) - 3\alpha_7(t)}{2}$$

and

$$\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + 2\alpha_4(t) + 2\alpha_5(t) + \alpha_6(t) + 3\alpha_7(t) < 1$$

and

$$\lim_{n\to\infty} sup \frac{\alpha_1(t) + \alpha_2(t) + \alpha_4(t) + \alpha_5(t) + \alpha_6(t) + \alpha_7(t)}{1 - [\alpha_3(t) + \alpha_4(t) + \alpha_5(t) + 2\alpha_7(t)]} < 1 \text{ for all } t \in [0, \infty) \quad (1)$$

Then T_1 and T_2 have common fixed point.

Proof. By assumption $\alpha_1^{'} : [0,\infty) \to [0,1)$ by $\alpha_1^{'}(t) = \frac{\alpha_1(t)+1-\alpha_3(t)-\alpha_2(t)-2\alpha_4(t)-2\alpha_5(t)-\alpha_6(t)-3\alpha_7(t)}{2}$ for $t \in [0,\infty)$. Then we have the followings:

$$\alpha_1(t) < \alpha_1^{'}(t), \text{ for all } t \in [0,\infty) \quad (2)$$

$$\lim_{n\to\infty} sup \frac{\alpha_1(t) + \alpha_2(t) + \alpha_4(t) + \alpha_5(t) + \alpha_6(t) + \alpha_7(t)}{1 - [\alpha_3(t) + \alpha_4(t) + \alpha_5(t) + 2\alpha_7(t)]} < 1, \quad \text{for all } t \in [0, \infty) \quad (3)$$

For $x, y \in X$ and $p \in T_1x$ there exists $q \in T_2y$ such that

$$\begin{aligned} d(p,q) \leq{}& \alpha_1^{'}(d(x,y))d(x,y) + \alpha_2(d(x,y))D(x,T_1x) + \alpha_3(d(x,y))D(y,T_2y) \\ &+ \alpha_4(d(x,y))[D(x,T_1x) + D(y,T_2y)] + \alpha_5(d(x,y))[D(x,T_2y) + D(y,T_1x)] \\ &+ \alpha_6(d(x,y))[D(x,T_1x) + D(y,T_1x)] + \alpha_7(d(x,y))[D(y,T_2y) + D(x,T_2y)] \quad (4) \end{aligned}$$

Putting $p = y$ in (4), we obtain

$$\text{For } x, y \in X \text{ and } y \in T_1x \quad \text{there exists } q \in T_2y \quad (5)$$

such that

$$\begin{aligned} d(y,q) \leq{}& \alpha_1^{'}(d(x,y))d(x,y) + \alpha_2(d(x,y))D(x,T_1x) + \alpha_3(d(x,y))D(y,T_2y) \\ &+ \alpha_4(d(x,y))[D(x,T_1x) + D(y,T_2y)] + \alpha_5(d(x,y))[D(x,T_2y) + D(y,T_1x)] \\ &+ \alpha_6(d(x,y))[D(x,T_1x) + D(y,T_1x)] + \alpha_7(d(x,y))[D(y,T_2y) + D(x,T_2y)] \end{aligned}$$

We define sequence $\{x_{2n}\}$ such that $x_1 \in T_1x_0$ and $x_{2n+1} \in T_1x_{2n}$ *i.e.*, $x_{2n+1} = T_1x_{2n}$. Similarly we can have $x_2 \in T_2x_1$ and $x_{2n+2} \in T_2x_{2n+1}$ *i.e.*, $x_{2n+2} = T_2x_{2n+1}$.
Then we get,

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) \le\ & \alpha_1^{'}(d(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1}) + \alpha_2(d(x_{2n},x_{2n+1}))D(x_{2n},T_1x_{2n}) \\
& + \alpha_3(d(x_{2n},x_{2n+1}))D(x_{2n+1},T_2x_{2n+1}) \\
& + \alpha_4(d(x_{2n},x_{2n+1}))[D(x_{2n},T_1x_{2n}) + D(x_{2n+1},T_2x_{2n+1})] \\
& + \alpha_5(d(x_{2n},x_{2n+1}))[D(x_{2n},T_2x_{2n+1}) + D(x_{2n+1},T_1x_{2n})] \\
& + \alpha_6(d(x_{2n},x_{2n+1}))[D(x_{2n},T_1x_{2n}) + D(x_{2n+1},T_1x_{2n})] \\
& + \alpha_7(d(x_{2n},x_{2n+1}))[D(x_{2n+1},T_2x_{2n+1}) + D(x_{2n},T_2x_{2n+1})], \\
\le\ & \alpha_1^{'}(d(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1}) + \alpha_2(d(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1}) \\
& + \alpha_3(d(x_{2n},x_{2n+1}))d(x_{2n+1},x_{2n+2}) + \alpha_4(d(x_{2n},x_{2n+1}))[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\
& + \alpha_5(d(x_{2n},x_{2n+1}))[d(x_{2n},x_{2n+2}) + d(x_{2n+1},x_{2n+1})] \\
& + \alpha_6(d(x_{2n},x_{2n+1}))[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+1})] \\
& + \alpha_7(d(x_{2n},x_{2n+1}))[d(x_{2n+1},x_{2n+2}) + d(x_{2n},x_{2n+2})] \\
\le\ & \alpha_1^{'}(d(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1}) + \alpha_2(d(x_{2n},x_{2n+1}))d(x_{2n},x_{2n+1}) \\
& + \alpha_3(d(x_{2n},x_{2n+1}))d(x_{2n+1},x_{2n+2}) + \alpha_4(d(x_{2n},x_{2n+1}))[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\
& + \alpha_5(d(x_{2n},x_{2n+1}))[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] + \alpha_6(d(x_{2n},x_{2n+1}))[d(x_{2n},x_{2n+1})] \\
& + \alpha_7(d(x_{2n},x_{2n+1}))[d(x_{2n+1},x_{2n+2}) + d(x_{2n},x_{2n+2})]
\end{aligned}$$

[by triangle inequality]

which implies,

$$d(x_{2n+1},x_{2n+2}) \le \frac{P}{Q} d(x_{2n},x_{2n+1})$$

for all $n \in N$

where, for all $n \in N$,

$$\frac{P}{Q} = \frac{\alpha_1^{'}(t) + \alpha_2(t) + \alpha_4(t) + \alpha_5(t) + \alpha_6(t) + \alpha_7(t)}{1 - [\alpha_3(t) + \alpha_4(t) + \alpha_5(t) + 2\alpha_7(t)]},\ t = d(x_{2n},x_{2n+1}) \tag{6}$$

Therefore,

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) =\ & \frac{R}{S} d(x_{2n},x_{2n+1}) \\
<\ & d(x_{2n},x_{2n+1}) \quad \text{(using (4))}
\end{aligned}$$

where

$$\frac{R}{S} = \frac{\alpha_1(t) + 1 - \alpha_3(t) + \alpha_2(t) + \alpha_6(t) - \alpha_7(t)}{2[1 - \{\alpha_3(t) + \alpha_4(t) + \alpha_5(t) + 2\alpha_7(t)\}]} t,\ t = d(x_{2n},x_{2n+1})$$

which implies,

$$d(x_{2n+1},x_{2n+2}) < d(x_{2n},x_{2n+1}).$$

Therefore $\{d(x_{2n},x_{2n+1})\}$ is a non-increasing sequence in X.
Hence $\{d(x_{2n},x_{2n+1})\}$ converges to some non-negative integer r.
Now, by (1), we get

$$\lim_{s\to r^+} sup\frac{\alpha_1^{'}(s)+\alpha_2(s)+\alpha_4(s)+\alpha_5(s)+\alpha_6(s)+\alpha_7(s)}{1-[\alpha_3(s)+\alpha_4(s)+\alpha_5(s)+2\alpha_7(s)]} < 1.$$

So, we have

$$\frac{\alpha_1^{'}(r)+\alpha_2(r)+\alpha_4(r)+\alpha_5(r)+\alpha_6(r)+\alpha_7(r)}{1-[\alpha_3(r)+\alpha_4(r)+\alpha_5(r)+2\alpha_7(r)]} < 1.$$

Then there exists $k\in[0,1]$ and $\epsilon>0$ such that

$$\frac{\alpha_1^{'}(s)+\alpha_2(s)+\alpha_4(s)+\alpha_5(s)+\alpha_6(s)+\alpha_7(s)}{1-[\alpha_3(s)+\alpha_4(s)+\alpha_5(s)+2\alpha_7(s)]} < k, \quad \text{for all } s\in[r,r+\epsilon].$$

We can take $v\in N$ such that $r\le d(x_{2n},x_{2n+1})\le r+\epsilon$ for all $n\in N$ with $n\ge v$. It follows that, for all $n\in N$ with $n\ge v$,

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &\le \frac{P}{Q}d(x_{2n},x_{2n+1}), \quad \text{(Using (6))}\\ &\le kd(x_{2n},x_{2n+1}). \end{aligned}$$

where $k=\dfrac{P}{Q}$ and

$$\frac{P}{Q}=\frac{\alpha_1^{'}(t)+\alpha_2(t)+\alpha_4(t)+\alpha_5(t)+\alpha_6(t)+\alpha_7(t)}{1-[\alpha_3(t)+\alpha_4(t)+\alpha_5(t)+2\alpha_7(t)]}, \ t=d(x_{2n},x_{2n+1})$$

This implies that

$\sum\limits_{n=1}^{\infty} d(x_{2n+1},x_{2n+2}) \le \sum\limits_{n=1}^{v} d(x_{2n},x_{2n+1})+\sum\limits_{n=1}^{\infty} k^{2n}d(x_{2v},x_{2v+1}) < \infty$

Hence $\{x_{2n}\}$ is cauchy sequence in X.
Since (X,d) is complete metric space, then $\{x_{2n}\}$ converges to some point $x^*\in X$.
Now, we have

$$\begin{aligned} D(x^*,T_1x^*) &\le d(x^*,x_{2n+1})+D(x_{2n+1},T_1x^*),\\ &\le d(x^*,x_{2n+1})+H(T_1x_{2n},T_1x^*),\\ &\le d(x^*,x_{2n+1})+\alpha_1^{'}(d(x_{2n},x^*))d(x_{2n},x^*)+\alpha_2(d(x_{2n},x^*))D(x_{2n},T_1x_{2n})\\ &\quad+\alpha_3(d(x_{2n},x^*))D(x^*,T_1x^*)+\alpha_4(d(x_{2n},x^*))[D(x_{2n},T_1x_{2n})+D(x^*,T_1x^*)]\\ &\quad+\alpha_5(d(x_{2n},x^*))[D(x_{2n},T_1x^*)+D(x^*,T_1x_{2n})]\\ &\quad+\alpha_6(d(x_{2n},x^*))[D(x_{2n},T_1x_{2n})+D(x^*,T_1x_{2n})]\\ &\quad+\alpha_7(d(x_{2n},x^*))[D(x^*,T_1x^*)+D(x_{2n},T_1x^*)] \quad \text{for all } n\in N. \end{aligned}$$

$$\begin{aligned} d(x^*,T_1x^*) &\le d(x^*,x_{2n+1})+\alpha_1^{'}(d(x_{2n},x^*))d(x_{2n},x^*)+\alpha_2(d(x_{2n},x^*))d(x_{2n},x_{2n+1})\\ &\quad+\alpha_3(d(x_{2n},x^*))D(x^*,T_1x^*)+\alpha_4(d(x_{2n},x^*))[d(x_{2n},x_{2n+1})+D(x^*,T_1x^*)]\\ &\quad+\alpha_5(d(x_{2n},x^*))[D(x_{2n},T_1x^*)+d(x^*,x_{2n+1})]\\ &\quad+\alpha_6(d(x_{2n},x^*))[d(x_{2n},x_{2n+1})+d(x^*,x_{2n+1})]\\ &\quad+\alpha_7(d(x_{2n},x^*))[D(x^*,T_1x^*)+D(x_{2n},T_1x^*)] \quad \text{for all } n\in N. \end{aligned}$$

It follows that

$$\begin{aligned} D(x^*, T_1x^*) &\leq \lim_{n\to\infty} inf[\alpha_3(d(x_{2n}, x^*)) + \alpha_4(d(x_{2n}, x^*)) + \alpha_5(d(x_{2n}, x^*)) + 2\alpha_7(d(x_{2n}, x^*))]D(x^*, T_1x^*), \\ &= \lim_{s\to 0^+} inf[\alpha_3(s) + \alpha_4(s) + \alpha_5(s) + 2\alpha_7(s)]D(x^*, T_1x^*), \\ &\leq \lim_{s\to 0^+} sup\Big\{\frac{\alpha_1(s) + \alpha_2(s) + \alpha_4(s) + \alpha_5(s) + \alpha_6(s) + \alpha_7(s)}{1 - [\alpha_3(s) + \alpha_4(s) + \alpha_5(s) + 2\alpha_7(s)]}\Big\}D(x^*, T_1x^*). \end{aligned}$$

On the other hand, we have

$$\lim_{s\to 0^+} sup\Big\{\frac{\alpha_1(s)+\alpha_2(s)+\alpha_4(s)+\alpha_5(s)+\alpha_6(s)+\alpha_7(s)}{1-[\alpha_3(s)+\alpha_4(s)+\alpha_5(s)+2\alpha_7(s)]}\Big\} < 1.$$

Therefore $D(x^*, T_1x^*) = 0$
Since T_1x^* is closed, so, it follows that $x^* \in T_1x^*$.
Similarly if we can be established that $x^* \in T_2x^*$.
Thus $\{x^*\}$ is a common fixed point of T_1 and T_2.

□

Example 2.2. *Let $X = [0, 1]$. Define $d : X \times X \longrightarrow X$ by $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a complete metric space . Now consider the mappings $T : X \to CB(X)$ defined by $T_1x = [0, \frac{x}{10}]$ and $T_2y = [0, \frac{y}{5}]$, where $x, y \in [0, 1]$. Also consider the mappings $\alpha_i : [0, \infty) \longrightarrow [0, 1)(i = 1, 2, ..., 7.)$ defined by $\alpha_1(t) = \frac{t}{1+t}$, $\alpha_2(t) = \frac{t}{2(1+t)}$, $\alpha_3(t) = \frac{t}{1+3t}$, $\alpha_4(t) = \frac{1}{8(1+t^2)}$, $\alpha_5(t) = \frac{t^2}{8(1+t^2)}$, $\alpha_6(t) = \frac{1}{6}$, $\alpha_7(t) = \frac{1}{9}$, for all $t \in [0, \infty)$ such that*

$$\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + 2\alpha_4(t) + 2\alpha_5(t) + \alpha_6(t) + 3\alpha_7(t) < 1. \tag{7}$$

Therefore, using (6), we get $0 \leq t < 0.1206054$ and subsequently

$$\lim_{n\to\infty} sup\{\frac{\alpha_1(t) + \alpha_2(t) + \alpha_4(t) + \alpha_5(t) + \alpha_6(t) + \alpha_7(t)}{1 - [\alpha_3(t) + \alpha_4(t) + \alpha_5(t) + 2\alpha_7(t)]}\} = \lim_{n\to\infty} sup\{\frac{(143t + 35)(1 + 3t)}{(1 + t)(47 + 69t)}\} < 1.$$

Now, we have to consider the following two cases:

Case I:
If $x, y \in [0, 1]$. The contractive condition of theorem is trivially hold for the case when $x = y = 0$.

Case II:
*Suppose without any loss of generality, we can take $x < y$ and $x, y \neq 0$.
Then $d(x, y) = |x-y|$, $D(x, T_1x) = \frac{9x}{10}$, $D(y, T_2y) = \frac{4y}{5}$, $D(x, T_2y) = |x-\frac{y}{5}|$ and $D(y, T_1x) = |y - \frac{x}{10}|$.*

$$
\begin{aligned}
L.H.S = H(T_1x, T_2y) &= Max\Big\{ \sup_{a\in T_1x} D(a, T_2y), \sup_{b\in T_2y} D(b, T_1x)\Big\}, \\
&= Max\Big\{ \sup_{a\in T_1x} [inf\{d(a,p) : \forall p \in T_2y\}], \sup_{b\in T_2y} [inf\{d(b,q) : \forall q \in T_1x\}]\Big\}, \\
&= Max\Big\{ \sup_{a\in [0,\frac{x}{10}]} [inf\{d(a,p) : \forall p \in [0, \frac{y}{5}]\}], \sup_{b\in [0,\frac{y}{5}]} [inf\{d(b,q) : \forall q \in [0, \frac{x}{10}]\}]\Big\} \\
&= Max\{0, |\frac{x}{10} - \frac{y}{5}|\} \\
&= |\frac{x}{10} - \frac{y}{5}|, \quad for\ all\ x, y \in [0,1].
\end{aligned}
$$

$$
\begin{aligned}
R.H.S = {} & \alpha_1(d(x,y))d(x,y) + \alpha_2(d(x,y))D(x,T_1x) + \alpha_3(d(x,y))D(y,T_2y) \\
& + \alpha_4(d(x,y))[D(x,T_1x) + D(y,T_2y)] + \alpha_5(d(x,y))[D(x,T_2y) + D(y,T_1x)] \\
& + \alpha_6(d(x,y))[D(x,T_1x) + D(y,T_1x)] + \alpha_7(d(x,y))[D(y,T_2y) + D(x,T_2y)], \\
= {} & \frac{|x-y|^2}{1+|x-y|} + \frac{4y}{5}\Big(\frac{24|x-y|^3 + 80|x-y|^2 + 51|x-y| + 89}{72(1+3|x-y|)(1+|x-y|^2)}\Big) + \frac{9x}{10}\Big(\frac{2|x-y|^3 + 6|x-y|^2 + 3|x-y| -}{8(1+|x-y|)(1+|x-y|^2)} \\
= {} & |x - \frac{y}{5}|\Big(\frac{17|x-y|^2 + 8}{8(1+|x-y|^2)}\Big) \\
= {} & |y - \frac{x}{10}|\Big(\frac{3|x-y|^2 + 2}{8(1+|x-y|^2)}\Big)
\end{aligned}
$$

Therefore, L.H.S. $\leq$ *R.H.S. for all* $x, y (x < y) \in [0,1]$.
Hence all the conditions of our theorem 2.2 are satisfied. Here we have $T_1 0 = T_2 0 = 0$, *that is,* $\{0\}$ *is a common fixed point of* T_1 *and* T_2.

Corollary 2.7. *Let* (X,d) *be a complete metric space and let* $T_1, T_2 : X \to CB(X)$ *be two multi-valued mappings, such that* $H(T_1x, T_2y) \leq \alpha(d(x,y))[D(x,T_1x) + D(y,T_2y)]$ *for all* $x, y \in X$, *where* $\alpha : [0,\infty] \to [0,1)$ *such that* $\alpha(t) < \frac{1}{2}$ *and* $\lim_{s\to t^+} sup\ \alpha(t) < \frac{1}{2}$ *for all* $t \in [0,\infty)$. *Then* T_1 *and* T_2 *have a common fixed point.*

Proof. If we put $\alpha_4(t) = \alpha(t), \alpha_i(t) = 0, (i = 1,2,3,5,6,7)$ and for all $t \in [0,\infty)$ in the theorem 2.2. □

Corollary 2.8. *Let* (X,d) *be a complete metric space and let* $T_1, T_2 : X \to CB(X)$ *be two multi-valued mappings, such that* $H(T_1x, T_2y) \leq \alpha_1'(d(x,y))d(x,y) + \beta(d(x,y))[D(x,T_1x) + D(y,T_2y)]$ *for all* $x, y \in X$, *where* $\alpha, \beta : [0,\infty] \to [0,1)$ *such that* $\alpha(t) + 2\beta(t) < 1$ *and* $\lim_{s\to t^+} sup\ \frac{\alpha(t)+\beta(t)}{1-\beta(t)} < 1$ *for all* $t \in [0,\infty)$. *Then* T_1 *and* T_2 *have a common fixed point.*

Proof. If we put $\alpha_1(t) = \alpha(t), \alpha_4(t) = \beta(t), \alpha_i(t) = 0, (i = 2,3,5,6,7)$ and for all $t \in [0,\infty)$ in the theorem 2.2. □

Corollary 2.9. *Let* (X,d) *be a complete metric space and let* $T_1, T_2 : X \to CB(X)$ *be two multi-valued mappings, such that* $H(T_1x, T_2y) \leq \alpha_1'(d(x,y))d(x,y) + \beta(d(x,y))[D(x,T_1x) + D(y,T_2y)] + \gamma(d(x,y))[D(x,T_2y) + D(y,T_1x)]$ *for all* $x, y \in X$, *where* $\alpha, \beta, \gamma : [0,\infty] \to [0,1)$ *such that* $\alpha(t) + 2\beta(t) + 2\gamma(t) < 1$ *and* $\lim_{s\to t^+} sup\ \frac{\alpha(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))} < 1$ *for all* $t \in [0,\infty)$. *Then* T_1 *and* T_2 *have a common fixed point.*

Proof. If we put $\alpha_1(t) = \alpha(t), \alpha_4(t) = \beta(t), \alpha_5(t) = \gamma(t), \alpha_i(t) = 0, (i = 2, 3, 6, 7)$ and for all $t \in [0, \infty)$ in the theorem 2.2. □

3. Conclusions (mandatory)

In this article, we present two theorems which are generalized form of Nadler's theorem and Mizoguchi - Tahahasi's theorem. Also those are generalizing many existing result as the corollaries of our article. The explicit examples of the article help us to validate our theorems.

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Pulak Konar completed his M.Sc in 2007 from Guru Ghasidas Viswavidyalaya, C.G, India. He obtained his Ph.D in the year 2017 under the supervison of Prof.(Dr.) Binayak S. Choudhury from Indian Instiute of Engineering Science and Technology, Shibpur, W.B, India. Currently, he is working as an assistant professor in the Department of Mathematics in Amity University, Kolkata, India. His research interest is Nonlinear analysis, Functional analysis and Topology.

Ashim Kumar Jana obtained his M.Sc degree from Vidyasagar University in the year 2011. He also completed his B.Ed. degree on 2013 from Vidyasagar University. He is a guest teacher at Bajkul Milani Mahavidyalaya. He likes to teach and learn Algebra, Analysis and Differential equation.

Radha Binod Das obtained his M.Sc. degree from Vidyasagar University in the year 2013. He completed his B.Ed. degree on 2015 from Vidyasagar University. He works as a guest teacher at Bajkul Milani Mahavidyalaya. His teaching interest is Algebra, analysis and he is currently working on Functional Analysis.

Samir Kumar Bhandari obtained his Ph.D in 2012 from IIEST, Shibpur under the supervison of Prof.(Dr.) Binayak S. Choudhury. His research interest is Functional Analysis. Presently, he works at Bajkul Milani Mahavidyalaya as an assistant professor.

Rashmi Rekha Devi graduated from Guwahati University, Assam, India. She is currently a student of outgoing batch of M.Sc(Applied Mathematics) in the year 2019 in Amity University, Kolkata. She has done internship program in Calcutta Mathematical Society, Kolkata, India.