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TOTAL VERTEX IRREGULARITY STRENGTH OF INTERVAL GRAPHS

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Abstract. A labeling of a graph is a mapping that maps some set of graph elements to a set of numbers (usually positive integers). For a simple graph $G = (V, E)$ with vertex set V and edge set E, a labeling $\phi : V \cup E \to \{1, 2, ..., k\}$ is called total k-labeling. The associated vertex weight of a vertex $x \in V(G)$ under a total k-labeling ϕ is defined as $wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy)$ where $N(x)$ is the set of neighbors of the vertex x. A total k-labeling is defined to be a vertex irregular total labeling of a graph G, if $wt(x) \neq wt(y)$ holds for every two different vertices x and y of G. The minimum k for which a graph G has a vertex irregular total k-labeling is called the total vertex irregularity strength of G, $tvs(G)$. In this paper, total vertex irregularity strength of interval graphs is studied. In particular, an efficient algorithm is designed to compute tvs of proper interval graphs and bounds of tvs are presented for interval graphs.

Keywords: Interval graphs, vertex irregular total labeling, total vertex irregularity strength, design of algorithms.
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1. Introduction

Labeling of graphs growing fast during the last three decades. Labeled graphs has wide range of applications including coding theory, x-ray crystallography, radar, astronomy, circuit design, channel assignments of FM radio stations and communication network addressing. A labeling of a graph is a function from the vertex set V or the edge set E or both to the set of natural numbers subject to certain conditions. If the domain is the vertex-set (edge-set) the labeling is called vertex-labeling (edge labeling). For a graph G a labeling $\phi : V \cup E \to \{1, 2, ..., k\}$ is said to be a vertex irregular total k-labeling of the graph G if for every two different vertices x and y of G, $wt(x) \neq wt(y)$ where the weight of a vertex x in the labeling ϕ is

$$wt(x) = \phi(x) + \sum_{y \in N(x)} \phi(xy)$$

where $N(x)$ is the set of neighbors of x. The minimum k for which the graph G has an vertex irregular total k-labeling is called the total vertex irregularity strength of the graph G and is denoted by $tvs(G)$. It is easy to verify that irregularity strength $s(G)$ of a graph G can be defined only for graphs containing at most one isolated vertex and no connected

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component of order 2. On the other hand, the total vertex irregularity strength $tvs(G)$ is defined for every graph G. This paper dedicated to the study of total vertex irregularity strength of a subclass of interval graphs, called proper interval graphs.

An undirected graph $G = (V, E)$ is an interval graph, if the vertex set V can be put into one to one correspondence with a set of intervals I on the real line R such that two vertices are adjacent in G, if and only if their corresponding intervals have non empty intersection. A graph G is a proper interval graph, if there is an interval representation of G in which no interval properly contains another.

The intervals and the vertices of an interval graph are the same things. Interval graphs are discussed extensively in [10]. Here, we assume that the input graph is given by an interval representation I which is the set of n sorted intervals labeled by $1, 2, \ldots, n$. Let $I = \{I_1, I_2, \ldots, I_n\}$ where $I_j = [a_j, b_j]$, $j = 1, 2, 3, \ldots, n$; be the interval representation of the given interval graph $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, a_j and b_j are respectively the left and the right end points of the interval I_j. Without any loss of generality, we assume that each interval contains both its end points and that no two intervals share a common end point. Also, we assume that the intervals in I are indexed by increasing right end points, that is, $b_1 < b_2 < \cdots < b_n$. This indexing known as IG ordering. Figure 1 shows an interval graph and its corresponding interval representation.

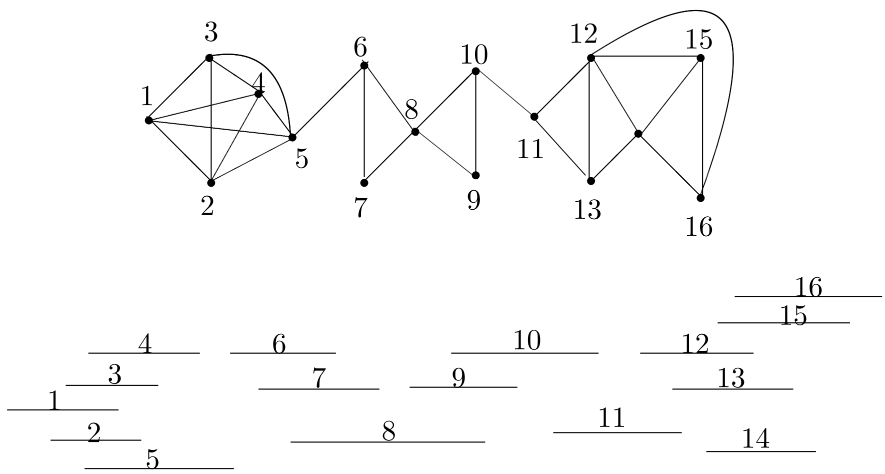

FIGURE 1. An interval graph and its interval representation.

An interval graph can be recognized in $O(n + m)$ time [6] which output an interval representation in the positive case. All graphs considered in this paper are connected, undirected and have no self loop or parallel edges.

The study of graphs labeling was initiated by Sadlacek in 1964 [14]. This study was continued by Stewart [16]. In 1986 Chartrand et al. [7] introduced irregular assignments and the irregularity strength of graphs. The motivation of Baca et al. [5] behind the definition of total irregularity strength of graphs came from [7]. After that, a lot of work done for both vertex and edge variants of total irregularity strength of graphs, please see [1–5] etc.. To date, the total vertex irregularity strengths are known only for some special classes of graphs. In [5], Baca et al. determined exact value of tvs for stars, cliques and prisms. Also, they fixed boundary value of tvs for tree, (p, q) graph etc.

If G is a (p, q) graph with minimum degree δ and maximum degree Δ, then Baca et al. proved that

$$\lceil \frac{p+\delta}{\Delta+1} \rceil \leq tvs(G) \leq p+\delta-2\Delta+1.$$

These results were then improved by Przybylo [12] for sparse graphs and for graphs with large minimum degree. In [4] Anholcer et al. established a new upper bound of the form

$$tvs(G) \leq 3\frac{p}{\delta}+1.$$

Among the others, Ahmad et al. [1] fixed the lower bound of *tvs* of any graph. Moreover, Ahmad et al. [2,3] found an exact value of the total vertex irregularity strength for Jahangir graphs, circulant graphs and wheel related graphs. In [17], Wijaya and Slamin determined the exact values of the total vertex irregularity strength of wheels, fans, suns and friendship graphs. Furthermore, Wijaya et al. [18] found the total vertex irregularity strength for complete bipartite graphs.

To date, the *tvs* of interval graphs is unknown. The main objective of this paper is to find the *tvs* of proper interval graphs, a subclass of interval graphs. Here, we present an efficient algorithm to determine the *tvs* of proper interval graphs. Moreover the boundary values of *tvs* of proper interval graphs and interval graphs are presented.

2. Preliminaries

A clique of a graph is a set of vertices, such that, there exists an edge between every pair of vertices in the set. A clique is a maximal clique, if no proper subset of it is a clique. An interval graph with n vertices can have at most n maximal cliques [9].

Lemma 2.1. (*Fulkerson and Gross [9]*). *A triangulated graph and so an interval graph with n vertices has at most n maximal cliques. The number of maximal cliques is n if and only if the graph has no edges.*

A connected interval graph with n vertices has at most $n-1$ maximal cliques, when the graph is a path. Gilmore and Hoffman [11] have shown that the maximal cliques of an interval graph G can be linearly ordered such that for every vertex $x \in V$ the maximal cliques containing x occur consecutively.

Theorem 2.1. (*Gilmore and Hoffman [11]*). *Let G be an undirected graph. The following statements are equivalent.*
(*i*) *G is an interval graph.*
(*ii*) *G contains no chord less cycle with four or more vertices and its complement $\bar{G}$ is a comparability graph.*
(*iii*) *The maximal cliques of G can be linearly ordered such that for every vertex x of G, the maximal cliques containing x occur consecutively.*

An interval graph with n vertices has at most n maximal cliques [9]. A connected interval graph with n vertices has at least 1 maximal clique and at most $n-1$ maximal cliques. If the graph is a path then the number of maximal cliques are $n-1$ whereas it becomes a complete graph if the number of maximal clique is 1. A clique with n vertices is denoted by C_n.

Lemma 2.2. *For a clique C_n,*

$$tvs(C_n) = 2.$$

Proof: Let C_n be a clique with n vertices $v_1, v_2, \ldots, v_n$. Then there exist an edge between every pair of vertices v_i and v_j. The edge between v_i and v_j is denoted by e_{ij}. Let us define a function $\phi : V \cup E \to \{1, 2\}$ as follows:

$\phi(v_1) = 1$, $\phi(e_{1j}) = 1$, $j = 2, 3, \ldots, n-1$ and $\phi(e_{1n}) = 1$.

$\phi(v_2) = 2$, $\phi(e_{2j}) = 1$, $j = 1, 3, 4, \ldots, n-1$, $\phi(e_{2n}) = 2$.

$\phi(v_3) = 2$, $\phi(e_{3j}) = 1$, $j = 1, 2, 4, \ldots, n-2$, $\phi(e_{3(n-1)}) = \phi(e_{3n}) = 2$.

$\cdots \; \cdots \; \cdots$

$\phi(v_n) = 2$, $\phi(e_{nj}) = 2$, $j = 1, 2, \ldots, n-1$.

Then weight of the vertices $v_1, v_2, \ldots, v_n$ are $n+1, n+2, \ldots, 2n$ respectively. Therefore ϕ is a vertex irregular total labeling function and C_n has vertex irregular total labeling . Clearly, 2 is the minimum integer such that $\phi : V \cup E \to \{1, 2\}$ is a vertex irregular total labeling function. Therefore $tvs(C_n) = 2$. □

The degree of all vertices of an interval graph G can be obtained in $O(n^2)$ time. Let D_r be the set of all vertices of G of degree r. Then the vertices of G can be partitioned into the sets D_δ, $D_{\delta+1}$, …, D_Δ, where δ and Δ are the minimum and maximum degree of G respectively.

Now, we prove a property of proper interval graph which plays an important role in our algorithm.

Lemma 2.3. *The number of leaf vertex of a connected proper interval graph is at most 2.*

Proof: Let G be a connected proper interval graph with n vertices $1, 2, \ldots, n$. Since no interval properly contains another interval, the intervals $2, 3, \ldots, n-1$ must intersect at least two intervals. This implies that the degree of each of the internal vertices $2, 3, \ldots, n-1$ must be greater than 1. Therefore, only the end vertices 1 and n are potential leaf vertex. □

From this lemma, it is observed that the cardinality of the set D_1 is 0 or 1 or 2, i.e., $|D_1| \leq 2$.

Lemma 2.4. *For any graph G with at least two vertices, $tvs(G) \geq 2$.*

Proof: Let G be graph with two vertices. Then degree of each vertex must be 1. With the integer 1, we can label only one vertex and weight of that vertex will be 2 (both vertex and edge gets the label 1). To label the other vertex, another integer is needed except 1. Therefore, at least two integers are required to label the vertices of G. □

Observe that, if it is possible to label the vertices and edges of a graph such that weight of the vertices are consecutive integers starting from least possible weight, then tvs is achieved. So, our aim is to label the vertices and edges of G in such a way that weight of the vertices are consecutive integers subject to the condition that the highest possible wight is least.

Lemma 2.5. *If $|D_k| = (k+1)p + 1$ $(k > 1)$ for some positive integer p, then at least $p+1$ integers are required to label the vertices of D_k to follow vertex irregular total labeling.*

Proof: Using the integer 1, we can label only one vertex of D_k. Each of the k edges incident on the vertex receive label 1 and the vertex also get label 1. Hence the weight of that vertex is $k+1$.

Using the integers 1 and 2, we can label $k+1$ vertices and edges incident on that vertices as $\underbrace{1, 1, \ldots, 1, 2}_{k+1 \text{ times}}, \underbrace{1, 1, \ldots, 1, 2, 2}_{k+1 \text{ times}}, \ldots, \underbrace{2, 2, \ldots, 2, 2}_{k+1 \text{ times}}$.

Hence by using the integers 1 and 2, total $(k+1)+1$ vertices can be labeled.

Again, using the integers 1, 2 and 3, we can label $k+1$ vertices of D_k and edges incident on that vertices as $\underbrace{2,2,\ldots,2,3}_{k+1 \text{ times}}, \underbrace{2,2,\ldots,2,3,3}_{k+1 \text{ times}}, \ldots, \underbrace{3,3,\ldots,3,3}_{k+1 \text{ times}}$.

Hence by the integers 1, 2 and 3, total $2(k+1)+1$ vertices can be labeled.

Proceeding with similar logic, we have $(k+1)p+1$ vertices of D_k can be labeled by using $p+1$ integers $1, 2, \ldots, p, p+1$. □

It should be noted that, if p be any real number other than integer then $\lceil p \rceil + 1$ integers are required to label $(k+1)p+1$ vertices of D_k. The vertices and adjacent edges are labeled in increasing order of degree. At first we label all vertices of degree one, if there be any. Then all the vertices of degree two (if exists) are labeled and so on. Observe that, using same number of integers we can label some vertices of higher degree.

Lemma 2.6. *Let $|D_k| = (k+1)p+1$ ($k > 1$ and p be an integer) and $p+1$ consecutive integers are used to label the vertices of D_k. Then $p+1$ vertices of D_{k+1} can be labeled using these $p+1$ consecutive integers.*

Proof: Using $p+1$ consecutive integers $\{1, 2, \ldots, p+1\}$, one can generate a maximum weight $(k+2)(p+1)$ of the vertices of D_{k+1}. In that case each of the $k+1$ edges and the only vertex get label $p+1$. Again by using these $p+1$ consecutive integers, a maximum weight $(k+1)(p+1)$ is possible of the vertices of D_k. Hence $(k+2)(p+1)-(k+1)(p+1) = p+1$ consecutive wights can be generated using the same $p+1$ consecutive integers. These $p+1$ weights $(k+1)(p+1)+1$, $(k+1)(p+1)+2$ …, $(k+1)(p+1)+(p+1)$ can be used to label $p+1$ vertices of D_{k+1}. □

Lemma 2.7. *Let $|D_k| = (k+1)p+1$ ($k > 1$ and p be a fraction) and $\lceil p \rceil + 1$ consecutive integers are used to label the vertices of D_k. Then $(k+2)\lceil p \rceil - (k+1)p + 1$ vertices of D_{k+1} can be labeled using these $\lceil p \rceil + 1$ consecutive integers.*

Proof: Similar logic as in the previous Lemma. □

From the above two lemmas it is observed that without increasing the value of tvs, we can label some extra vertices of higher degree. The number of extra vertices of D_{k+1} can be labeled by the integers used to label the vertices of D_k is denoted by e_{k+1}. The values of e_i, ($i = 3, 4, \ldots \Delta$) can be determined using previous two lemmas.

3. The Algorithm

The main idea of the proposed algorithm is follows. The algorithm proceeds by labeling the vertices in increasing order of degree of the given graph. At first, the sets $D_1, D_2, \ldots, D_\Delta$ are computed. Depending on the cardinal number of these sets, $tvs(G)$ is determined based on the Lemma 4.

A formal description of the algorithm is given below.

Algorithm tvs
Input: A set of $n(> 1)$ sorted intervals of a proper interval graph $G = (V, E)$.
Output: Total Vertex Irregularity Strength of G, i.e., $tvs(G)$.
Initialize $i = 2$ and $T = 0$.
Step 1: Compute the sets D_1, D_2, …, D_Δ.
Step 2: If $\mid D_1 \mid = 0$ then
$e_2 = 0$ and goto next step.
else if $\mid D_1 \mid = 1$ then
$T = T + 1$, $e_2 = 1$ and goto next step.

else $T = T + 2$, $e_2 = 3$ and goto next step.
endIf
Step 3: If $\mid D_i \mid = 0$
$i = i + 1$
If $i > \Delta$, goto the last Step 4
endIf
else $\mid D_i \mid = \mid D_i \mid - e_i$
If $\mid D_i \mid > 0$ express $\mid D_i \mid = (i+1)p + 1$
If $T > 1$, $T = T + \lceil p \rceil$, $i = i + 1$, repeat step 3
else $T = T + \lceil p \rceil + 1$, $i = i + 1$, repeat step 3
endIf
else $i = i + 1$, repeat step 3
endIf
endIf
Step 4: $tvs(G) = T$.
end Algorithm tvs.

The correctness of the algorithm follows from the previous Lemmas.

Theorem 3.1. *For a proper interval graph G with n (> 1) vertices ,*

$$2 \leq tvs(G) \leq \left\lceil \frac{n-6}{3} \right\rceil + 2.$$

Proof: Observe that, if G be a complete graph of order n then $tvs(G) = 2$ (Lemma 2). If G be not complete then $tvs(G) > 2$. Therefore, $tvs(G) \geq 2$.

Again, the tvs of a graph G will be maximum, if G consist of maximum number of vertices of less degree. Now, a connected proper interval graph with n vertices can have at most 2 leaf vertices and $n - 2$ vertices of degree 2. In this case, G becomes a path P_n of n vertices.

Therefore $\mid D_1 \mid = 2$ and $\mid D_2 \mid = n - 2$. Two integers 1 and 2 are necessary to label the vertices of D_1. Using these two integers, we can label 3 vertices of D_2, i.e., $e_2 = 3$.

Hence by the algorithm tvs, we have $tvs(G) = \lceil \frac{n-6}{3} \rceil + 2$. □

Note that, the algorithm tvs does not work for general interval graph, as the Lemma 2 is not true for general interval graph. The maximum number of leaf vertex of an interval graph is $n - 1$. So, the algorithm needs modification depending on the number of leaf vertex to compute tvs of a general interval graph.

Theorem 3.2. *For a connected interval graph G with n (> 1) vertices ,*

$$2 \leq tvs(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

Proof: The lower bound follows from the previous theorem.

Since maximum value of tvs can be attained when the graph has maximum number of vertices of less degree. Therefore, upper bound is attained if the graph has maximum number of leaf vertex. Now, a connected interval graph G with n vertices can have at most $n - 1$ vertices of degree 1 (leaf vertex). In that case, G becomes a star and $tvs(G) = \lceil \frac{n}{2} \rceil$.

Therefore maximum value of tvs of G is $\lceil \frac{n}{2} \rceil$.

Hence

$$tvs(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

□

4. Conclusion

To date, the total vertex irregularity strength is known only for very few classes of graphs. In this paper, an algorithm is designed to find the exact value of *tvs* of proper interval graphs. Based on this algorithm the boundary values of tvs presented for proper interval graphs and interval graphs. It seems, our approach can be generalized to find the *tvs* of other intersection graphs, viz. permutation graphs, trapezoid graphs and circular arc graphs. Future study can be done to find the value of tvs for these intersection graphs. □

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