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CONNECTED MAJORITY DOMINATION VERTEX CRITICAL GRAPHS

J. JOSELINE MANORA[1], T. MUTHUKANI VAIRAVEL[2], §

Abstract. In this article, how the removal of a single vertex from a graph G can change the Connected Majority Domination number is surveyed for any graph G. A graph is Connected Domination Critical if the removal of any vertex decreases or increases its Connected Majority Domination Number. This paper gives examples and properties of CMD vertex critical graphs. There are two types namely CVR and UVR with respect to CMD sets of a graph. Also the vertex classification. $V^0_{CM}(G), V^-_{CM}(G)$ and V^+_{CM} are studied, characterisation theorems of these vertex classification are determined.

Keywords: Majority vertex critical, Connected Majority vertex critical, CVR_{CM}, UVR_{CM}.
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1. Introduction

Let G be a finite, simple, connected and undirected graph with vertex set $V(G)$ and edge set $E(G)$. A subset S of $V(G)$ is a dominating set [3] for G if every vertex of G either belongs to S or is adjacent to a vertex of S. The minimum cardinality of a minimal dominating set for G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set S is said to be a connected dominating set [10] if the subgraph $\langle S\rangle$ induced by S is connected in G. The minimum cardinality of a minimal connected dominating set for G is called the connected domination number of G and is denoted by $\gamma_c(G)$ [10].

Change of Majority Domination number in the case of removal of a single vertex is defined and studied. Here CVR means the change in vertex removal of a graph G and UVR means unchanged vertex removal of a graph G. The graphs in CVR were characterised by Bauer et. Al and H. B Walikar and B. D Acharya.

A subset S of $V(G)$ is a Majority Dominating set [9] if at least half of the vertices of $V(G)$ are either belong to S or adjacent to the elements of S i.e., $|N[S]| \geq \lceil \frac{V(G)}{2} \rceil$. The minimum cardinality of a minimal majority dominating set for G is called Majority Domination number of G and is denoted by $\gamma_M(G)$.

Let G be a graph with $p = |V(G)|$ and let $u \in V(G)$. Then u is said to be Majority Dominating (M.D) vertex if $d(u) \geq \lceil \frac{p}{2} \rceil - 1$. A full degree vertex is a M.D vertex but a M.D. vertex is not a full degree vertex

[1] Department of Mathematics, Tranquebar Bishop Manickam Lutheran College, Porayar, Tamil Nadu, India.
e-mail: joseline_manora@yahoo.co.in; ORCID: https://orcid.org/0000-0002-5265-7413.
[2] Department of Mathematics, Sir Issac Newton College, Nagapattinam, Tamil Nadu-India.
e-mail: muthukanivairavel@gmail.com@net.com; ORCID: https://orcid.org/0000-0001-5627-6987.
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For any graph G, CVR and UVR with respect to Domination numbers [12] are defined by

$$CVR : \gamma(G - v) \neq \gamma(G), for\ all\ v \in V(G)$$
$$UVR : \gamma(G - v) = \gamma(G), for\ all\ v \in V(G).$$

Similarly, for any graph G, CVR and UVR with respect to Majority Domination [8] numbers are defined by

$$CVR_M : \gamma_M(G - v) \neq \gamma_M(G), for\ all\ v \in V(G).$$
$$UVR_M : \gamma_M(G - v) = \gamma_M(G), for\ all\ v \in V(G).$$

2. Vertex Critical on CMD Sets and its Classifications

Definition 2.1. [6] *A subset S of $V(G)$ is a Connected Majority Dominating set CMDS if i S is a majority dominating set and ii the subgraph $\langle S \rangle$ induced by S is connected in G. The minimum cardinality of minimal Connected Majority dominating set for G is called the Connected Majority Domination number of G is denoted by $\gamma_{CM}(G)$.*

Definition 2.2. *For any graph G, CVR and UVR with respect to Connected Majority Domination number are defined by*

$$CVR_{CM} : \gamma_{CM}(G - v) \neq \gamma_{CM}(G), for\ all v \in V.$$

$$UVR_{CM} : \gamma_{CM}(G - v) = \gamma_{CM}(G), for\ all v \in V.$$

Example 2.1. *Let $G = P_7$ be a path of $p = 7$ vertices. Consider $V(G) = \{v_1, \cdots, v_7\}$. Let $S = \{v_2, v_3\}$ be a γ_{CM}-set of G and $\gamma_{CM}(G) = 2$. Consider the graph $(G–v_1)$ and its γ_{CM}-set is $S_1 = \{v_3\}$ and $\gamma_{CM}(G - v_1) = 1$. Therefore, $\gamma_{CM}(G - v_1) < \gamma_{CM}(G)$, for all $v \in V(G)$. Hence, $G = P_7 \in CVR_{CM}$.*

Example 2.2. *Let $G = F_9$ be a fan with $V(G) = \{v_1, \cdots, v_9\}$ and v_1 be the central vertex. Let γ_{CM}-set of G be $S = \{v_1\}$ and $\gamma_{CM}(G) = 1$. Consider the graph $(G–v_1), \gamma_{CM}$-set is $S_1 = \{v_2, v_3\}$ and $\gamma_{CM}(G–v_1) = 2$. Therefore, $\gamma_{CM}(G–v_1) > \gamma_{CM}(G)$. Hence $F_9 \in CVR_{CM}$*

Example 2.3. *Let $G = K_5$ with $V(G) = \{v_1, \cdots, v_5\}$ and each vertex is a full degree vertex. Let $S = \{v_1\}$ be a γ_{CM}-set of G and $\gamma_{CM}(G) = 1$. Consider the graph $(G–v_1)$ and its γ_{CM}-set is $S_1 = \{v_2\}$ and $\gamma_{CM}(G - v_1) = 1$. Therefore, $\gamma_{CM}(G - v_1) = \gamma_{CM}(G)$, for all $v \in V(G)$. Hence, $K_5 \in UVR_{CM}$.*

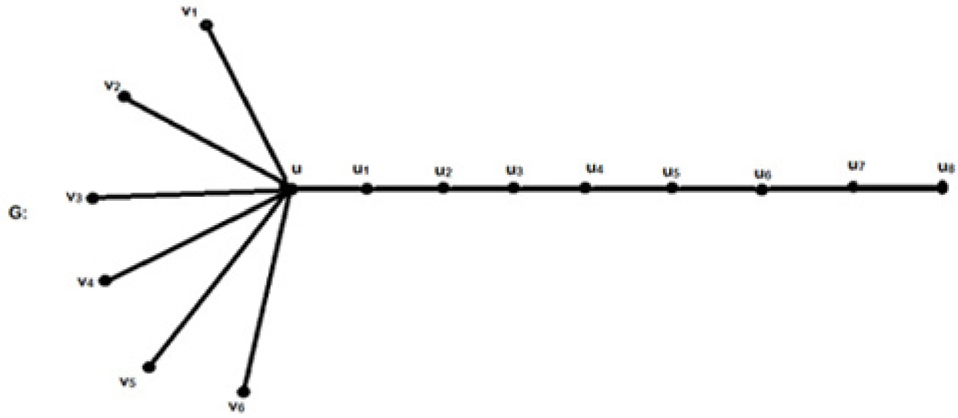
Definition 2.3. *For any graph G, the vertex set $V(G)$ can be partitioned with respect to Connected Majority Domination into three sets, namely $V^0_{CM}(G), V^-_{CM}(G)$ and $V^+_{CM}(G)$ are defined by*

$$V^0_{CM}(G) = \{v \in V(G) : \gamma_{CM}(G - v) = \gamma_{CM}(G)\}.$$

$$V^-_{CM}(G) = \{v \in V(G) : \gamma_{CM}(G - v) < \gamma_{CM}(G)\}.$$

$$V^+_{CM}(G) = \{v \in V(G) : \gamma_{CM}(G - v) > \gamma_{CM}(G)\}.$$

Example 2.4. *Consider the following structure of a graph G with $p = 15$ vertices.*



Let $S = \{u\}$ and $\gamma_{CM}(G) = 1$. Consider the graph $(G - u)$ and its γ_{CM}-set is $S_1 = \{u_2, u_3, u_4, u_5, u_6\}$ and $\gamma_{CM}(G - u) = 5$. Therefore, $\gamma_{CM}(G - u) > \gamma_{CM}(G)$. This Implies that $u \in V^+_{CM}(G)$ and $u_i \in V^0_{CM}(G)$, for all $u_i \in V(G)$.

3. Vertex Classification For Some Families of Graphs

Proposition 3.1. *If the graph G is complete, then $G \in UVR_{CM}$.*

Proposition 3.2. *Let W_p be the wheel on p vertices. Then the central vertex $u \in V^+_{CM}(G)$ and all other vertices $u_i \in V^0_{CM}(G)$.*

Proof: Let $G = W_p$ be a wheel. Let the centre vertex be u and the outer vertices be $\{u_1, \cdots u_{p-1}\}$. Here γ_{CM}-set of G is $S = \{u\}$ and $\gamma_{CM}(G) = 1$.

Case 1. *Let γ_{CM}-set of $(G-u)$ be $S_1 = \{u_1, \cdots, u_{(\lceil \frac{p}{2} \rceil - 2)}\}$. Thus, $\gamma_{CM}(G - u) = \lceil \frac{p}{2} \rceil - 2$, for a central vertex $u \in V(G)$. Hence, $\gamma_{CM}(G - u) > \gamma_{CM}(G)$. This implies that $u \in V^+_{CM}(G)$.*

Case 2. *Let $(G-u_1), u_1$ be an outer vertex of G. Then the γ_{CM}-set of $(G-u_1)$ is $S_2 = \{u\}$. Therefore $\gamma_{CM}(G - u_1) = 1$. Hence, $\gamma_{CM}(G - u_i) = \gamma_{CM}(G)$, for all $u_i \in (V - S)$. This implies that $u_i \in V^0_{CM}(G)$.* □

Proposition 3.3. *Let F_p be a fan on p vertices. Then the central vertex $u \in V^+_{CM}(G)$ and all other vertices $u_i \in V^0_{CM}(G)$.*

Proposition 3.4. *Let $G = K_{m,n}, m \leq n$ be a complete bipartite graph with $u_i \in V_1(G)$ and $v_i \in V_2(G)$. Then $u_i, v_i \in V^0_{CM}(G)$ for all u_i and v_i.*

Proposition 3.5. *Let C_p be a cycle with p vertices. Then*

(i) *$v_i \in V^0_{CM}(G)$, for all $v_i \in V(G)$ if p is even and*
(ii) *$v_i \in V^-_{CM}(G)$, for all $v_i \in V(G)$ if p is odd.*

Proof: Let $G = C_p$ be a cycle with $p \geq 3$ vertices. By the result [6], $\gamma_{CM}(G) = \lceil \frac{p}{2} \rceil - 2$. Then $(G - v)$ becomes a path, for any $v \in V(G)$.

Case 1. *Let p be even. Then $\gamma_{CM}(G - v_i) = \lceil \frac{p}{2} \rceil - 2$. This implies that $\gamma_{CM}(G - v_i) = \gamma_{CM}(G)$. Hence, $v_i \in V^0_{CM}(G)$, for all $v_i \in V(G)$.*

Case 2. *Let p be odd. Then $\gamma_{CM}(G - v_i) = \lceil \frac{p}{2} \rceil - 3$. This implies that $\gamma_{CM}(G - v_i) < \gamma_{CM}(G)$. Hence, $v_i \in V^-_{CM}(G)$, for all $v_i \in V(G)$.* □

4. Vertex Classification For Trees

Proposition 4.1. *Let $K_{1,p-1}$ be a star with p vertices. Then all pendants belong to $V^0_{CM}(G)$.*

Proposition 4.2. *Let $S(K_{1,p-1})$ be the subdivision of $K_{1,p-1}$ by dividing each edge exactly once. Then all Supports and all pendants belong to $V^0_{CM}(G)$.*

Proposition 4.3. *Let P_p be a path on p vertices. Then $v_i \in V^-_{CM}(G)$, if p is odd and $v_i \in V^0_{CM}(G)$, if p is even, for all $v_i \in V(G)$.*

Proposition 4.4. *Let $D_{r,s}, r \leq s$ be a double star with $p = r + s + 2$ vertices. Then the pendants and two M.D vertices belong to $V^0_{CM}(G)$.*

Proof: Let $G = D_{r,s}$ double star with $V(G) = \{u, v, u_1, \cdots, u_r, v_1, \cdots, v_s\}$, where the pendants $(u_1, \cdots, u_r)$ at u and the pedants $(v_1, \cdots v_s)$ at v.

Case 1. *Let $r = s$. Then the u and v both are M.D vertices, implies that $S = \{v\}$ is the CMD set of G. Therefore, $\gamma_{CM}(G) = 1$.*

Subcase 1 (a)**.** *Consider the graph $(G - v)$. Then $S_1 = \{u\}$ is the CMD set of $(G - v)$. Therefore $\gamma_{CM}(G - v) = 1$. Hence, $\gamma_{CM}(G) = \gamma_{CM}(G - v)$ and vice versa. This implies that $u, v \in V^0_{CM}(G)$.*

Subcase 1 (b)**.** *Consider the graph $(G - u_i)$ where u_i is any pendant vertex. Then $S_2 = \{v\}$ is the CMD set of $(G - u_i)$. Therefore, $\gamma_{CM}(G - u_i) = 1$. This implies that $u_i, v_i \in V^0_{CM}(G)$.*

Case 2. *Let $r < s$.*

Then $S = \{v\}$ is the CMD set of G. Therefore, $\gamma_{CM}(G) = 1$.

Subcase 2 (a)**.** *Consider the graph $(G - v)$. Then there is no CMD set of $(G - v)$. Therefore, $\gamma_{CM}(G - v)$ does not exist.*

Subcase 2 (b)**.** *Consider the graph $(G - u_i)$ where u_i is a pendant vertex. Then $\gamma_{CM}(G - u_i) = |S| = 1$. Hence, $\gamma_{CM}(G - u_i) = \gamma_{CM}(G)$. This implies that $u_i, v_i \in V^0_{CM}(G)$.* □

Proposition 4.5. *Let $C_{r,t}$ be a Comet with a M.D vertex u, w^-_r pendants attached to u and v^-_t path-at u. Then $u \in V^+_{CM}(G)$ and all w_r and v_t belong to $V^0_{CM}(G)$.*

Proposition 4.6. *Let G be a Caterpillar with p vertices.Then $u_i \in V^0_{CM}(G)$, if p is even and $u_i \in V^-_{CM}(G)$, if p is odd, for all $u_i \in V(G)$.*

Case 1. *Let p be even. By result [6],* $\gamma_{CM}(G) = \begin{cases} \frac{p}{e+1} - \lfloor \frac{k}{2} \rfloor & if\ k\ is\ odd \\ \frac{p}{e+1} - \frac{k}{2} & if\ k\ is\ even \end{cases}$. *Therefore, $S = \{u_2, u_3, \cdots u\}$.*

Subcase (a)**.** *Let $u \in S$. Then for the graph $G - u$, $S_1 = \{u_2, u_3, \cdots\}$ with $|S_1| = |S|$. Therefore, $\gamma_{CM}(G - u) = \gamma_{CM}(G)$. This implies that $u \in V^0_{CM}(G)$.*

Subcase (b)**.** *Let $u \in N(S)$ and $|N[S]| = \lceil \frac{p}{2} \rceil$. Then $|N[S]| = \lceil \frac{p-1}{2} \rceil$. $\gamma_{CM}(G - u) = |S| = \gamma_{CM}(G)$. This implies that $u \in V^0_{CM}(G)$.*

Subcase (c)**.** *Let $u \in V - N[S]$. Then also $\gamma_{CM}(G - u) = |S| = \gamma_{CM}(G)$. This implies that $u \in V^0_{CM}(G)$.*

Case 2. *Let p be odd.*

Subcase (a)**.** *Let* $u \in S$*. Then for the graph* $G - u, S_1 = \{u_2, u_3, \cdots\}$ *imply* $|S_1| = |S|$*. Therefore,* $\gamma_{CM}(G-u) < \gamma_{CM}(G)$*. This implies that* $u \in V^{-}_{CM}(G)$*.*

Subcase (b)**.** *Let* $u \in N(S)$ *and* $|N[S]| = \lceil \frac{p}{2} \rceil$*. Then also* $\gamma_{CM}(G-u) < \gamma_{CM}(G)$*. This implies that* $u \in V^{-}_{CM}(G)$*.*

Subcase (c)**.** *Let* $u \in V - N(S)$*. Then also* $\gamma_{CM}(G-u) < \gamma_{CM}(G)$*. This implies that* $u \in V^{-}_{CM}(G)$*.*

Proposition 4.7. *Let* T *be a binary tree with* p *vertices. Then every vertex* $u \in V^{0}_{CM}(G)$*.*

Proof: Let $V(T) = \{x, u_1, u_2, \cdots u_r, v_1, v_2, \cdots v_t\}$, where $d(x) = 2, d(u_i) = 3, i = 1 - r$ and $d(v_i) = 1, i = 1, \cdots t$. Then $p = 1 + r + t$. Let S be a $\gamma_{CM}(G)$-set of G.

Case 1. *When* p *is even. Since* $|S| = \gamma_{CM}(G), |N[S]| \geq \lceil \frac{p}{2} \rceil$*. If* $u \in S$*, then* $|N[S]| < \lceil \frac{p}{2} \rceil$ *and we could find another set* S_1 *with the same cardinality of* S*. It implies that* $\gamma_{CM}(G-u) = |S_1| = |S| = \gamma_{CM}(G)$*. This implies that* $u \in V^{0}_{CM}(G)$*. If* $u \in N(S)$*, then apply the same argument, we get* $u \in V^{0}_{CM}(G)$*.*

Case 2. *When* p *is odd. Since* $|S| = \gamma_{CM}(G), |N[S]| \geq \lceil \frac{p}{2} \rceil$*. If* $u \in S$ *and* $|N[S]| = \lceil \frac{p}{2} \rceil$*, then in* $G - u, |N_{(G')}[S]| = \lceil \frac{p}{2} \rceil - 1 = \left\lceil \frac{(p-1)}{2} \right\rceil$*, but* $\langle S \rangle$ *may be disconnected. So, we could find another set* S_1 *with the same cardinality of* S*. It implies that* $\gamma_{CM}(G-u) = |S_1| = |S| = \gamma_{CM}(G)$*. Therefore,* $u \in V^{0}_{CM}(G)$*. If* $u \in N(S)$*, then* $|N_{(G')}[S]| = \lceil \frac{p}{2} \rceil - 1 = \left\lceil \frac{(p-1)}{2} \right\rceil$ *and* $\langle S \rangle$ *is again connected. Therefore,* S *is again* γ_{CM}*-set for* $G - u = G'$*, implies* $u \in V^{0}_{CM}(G)$*.*
When either p *is even or odd. If* $u \notin N[S]$*, then* S *is again a* γ_{CM}*-set for* $G - u = G'$*, implies* $u \in V^{0}_{CM}(G)$*. In all the cases,* $u \in V^{0}_{CM}(G)$*, for every* $u \in V(G)$*.* □

5. Graphs on CVR_{CM} and UVR_{CM}

Proposition 5.1. *If the vertices of a connected a graph* G *are all M.D vertices, then the graph belongs to* UVR_{CM}*. For example i) a complete graph* K_p*, ii) a complete pipartite graph* $K_{m,n}$*.*

Proposition 5.2. *No tree belongs to* CVR_{CM}*.*

Proof: Since every tree has at least two pendants, all pendants $u_i \in V^{0}_{CM}(G)$. Therefore $\gamma_{CM}(G - u_i) = \gamma_{CM}(G)$. Hence, no tree belongs to CVR_{CM}. □

Proposition 5.3. *A tree* $D_{(r,s)}$ *and a binary tree belongs to* UVR_{CM}*.*

Proof: Let $G = D_{(r,s)}$ be a double star with $p = r+s+2$ and $V(G) = \{u, v, u_1, \cdots, u_r, v_1, \cdots, v_s\}$, where u and v are central vertices and the pendants $(u_1, \cdots, u_r)$ and $(v_1, \cdots, v_s)$ are attached in a vertex u and v respectively. By Proposition (4.4), we obtain $u, v \in V^{0}_{CM}(G), u_i, v_j \in V^{0}_{CM}(G)$, if $r = s$ where $i = 1, \cdots, r$ and $j = 1, \cdots, s$. When $r < s$ all pendants $u_i \in V^{0}_{CM}(G), i = 1, \cdots, r$. In all cases, $\gamma_{CM}(G - x) = \gamma_{CM}(G)$, for any vertex $x \in V(G)$. It implies $G = D_{r,s} \in UVR_{CM}$. □

Proposition 5.4. *In any graph* G *with an isolate, there exists a* γ_{CM}*-set of* G *not containing that isolate.*

Proof: Let v be an isolate of G and let S be a γ_{CM}-set of G. Suppose S is a γ_{CM}-set of G containing v. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and therefore, S is not a CMD set for G. Suppose $|N[S]| = \lceil \frac{p}{2} \rceil$ the induced subgraph $\langle S \rangle$ of S is not connected. Hence, there exists a γ_{CM}-set of G without an isolate v. □

Corollary 5.1. *If a graph G consists of exactly one component with $\lceil \frac{p}{2} \rceil$ vertices and other components may or may not be isolates, then the γ_{CM}-set contains no isolates.*
Eg.: i) $G = K_{1,7} \cup 8K_1$ ii) $G = D_{3,3} \cup 2K_3 \cup 2K_1$ with $p = 16$.

Proposition 5.5. *Let G be a disconnected graph with exactly two components. Then $u \in V^0_{CM}(G)$, for all and $u \in V(G)$, and $G \in UVR_{CM}$.*

Proof:

(i) When $|N[G_1]| = \lceil \frac{p}{2} \rceil = |N[G_2]|$ implies that $u \in V^0_{CM}(G)$, for all $u \in V(G)$.
(ii) When $|N[G_1]| > \lceil \frac{p}{2} \rceil$ and $|N[G_2]| < \lceil \frac{p}{2} \rceil$ and vice versa.

Let $S \subseteq V(G_1)$ be a γ_{CM}-set of G. Since $|N[G_1]| > \lceil \frac{p}{2} \rceil$ and $|N_{G_1}[S]| \geq \lceil \frac{p}{2} \rceil$. For $u \in G_1, |N[G_1 - u]| \geq \lceil \frac{p}{2} \rceil$ and $|N[G_2]| < \lceil \frac{p}{2} \rceil$. This implies that S is the γ_{CM}-set of $G - u$. It implies that $u \in V^0_{CM}(G)$, for all $u_i \in V(G_1)$.

Suppose $v \in V(G_2)$. Since $|N_{G_2}[S]| < \lceil \frac{p}{2} \rceil, S \subseteq V(G_1)$ is the same $\gamma_C M$-set of $G - v$. Hence $\gamma_{CM}(G - v) = \gamma_{CM}(G)$. It implies that $v \in V^0_{CM}(G)$, for all $v_i \in V(G_2)$. Therefore, all $u_i \in V^0_{CM}(G)$ and this graph $G \in UVR_{CM}$. □

Proposition 5.6. *Let T be a tree with even number of vertices. Then $\gamma_{CM}(T - u) \geq \gamma_{CM}(T)$ for every $u \in (V - S)$, where S is a γ_{CM}-set of G.*

Proof: Suppose $\gamma_{CM}(T - u) < \gamma_{CM}(T)$. Let $S = \{w_1, \cdots, w_s\}$ be a γ_{CM}-set of $(T - u)$. Then S cannot be a γ_{CM}-set of T. Therefore $|N[S]| \geq \lceil \frac{(p-1)}{2} \rceil$ and $|N[S]| < \lceil \frac{p}{2} \rceil$ i.e., $\lceil \frac{(p-1)}{2} \rceil \leq |N[S]| < \lceil \frac{p}{2} \rceil$. Given p is even, say, $p = 2r$. Then $r \leq |N[S]| < r$, a contradiction. Hence, $\gamma_{CM}(T - u) \geq \gamma_{CM}(T)$ for all $u \in (V - S)$. □

Corollary 5.2. *If a tree T has exactly one vertex u such that $d(u) = \lceil \frac{p}{2} \rceil - 1$ with p even number of vertices, then $\gamma_{CM}(T - u) > \gamma_{CM}(T)$, for $u \in S$. For example T is a comet $(C_{6,4})$*

Proposition 5.7. *Let T be a tree with odd vertices. Let S be a γ_{CM}-set of T. Then $\gamma_{CM}(T - u) \leq \gamma_{CM}(T)$, for all $u \in (V - S)$.*

Proof: Let S be a γ_{CM}-set of T. Then $\langle S \rangle$ is connected and $|N[S]| \geq \lceil \frac{p}{2} \rceil$ in T. In $T - u, |N[S]| \geq \lceil \frac{p}{2} \rceil - 1 = \frac{(p-1)}{2}$ if $u \in N(S)$ and $|N[S]| \geq \lceil \frac{p}{2} \rceil$ if $u \notin N(S)$. Therefore, in both cases, S is a CMD set of $T - u$. Therefore, $\gamma_{CM}(T - u) \leq |S| = \gamma_{CM}(T)$. □

Proposition 5.8. *Let S be a γ_{CM}-set of a tree T with p is odd such that there exists a vertex $u \notin N[S]$.Then $\gamma_{CM}(T - u) \leq \gamma_{CM}(T)$.*

Proof: Since S is a γ_{CM}-set of T, $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\langle S \rangle$ is connected in T. Let $T' = (T - u)$. When $|N_T[S]| = \lceil \frac{p}{2} \rceil$ and p is odd, since $u \notin N[S], |N_T[S]| = \frac{(p-1)}{2}$ and $\langle S \rangle$ is connected in T'. It implies S is again a γ_{CM}-set of T'. Hence, $\gamma_{CM}(T') = |S| = \gamma_{CM}(T)$. When $|N_T[S]| > \lceil \frac{p}{2} \rceil$ and p is odd. Since $u \notin N[S], |N_{T'}[S]| > \frac{p-1}{2}$ and S is a CMD set of T' but not a minimum set. Therefore, there exists a set $S_1 \subseteq S$ such that $|N_{T'}[S_1]| > \frac{p-1}{2}$ and $\langle S \rangle$ is connected in T'. It implies $\gamma_{CM}(T') = |S_1| < |S| = \gamma_{CM}(T)$. Hence, if $u \notin N[S]$, then $\gamma_{CM}(T - u) \geq \gamma_{CM}(T)$. □

Corollary 5.3. *Let T be a tree with even number of vertices. Let S be a γ_{CM}-set of T such that there exists a vertex $u \notin N[S]$. Then $\gamma_{CM}(T - u) = \gamma_{CM}(T)$.*

Proposition 5.9. *If there exists a γ_{CM}-set S of a tree T and there exists a γ_{CM}-set S_1 of $T - u$ with $|N[S_1] \ \geq \lceil \frac{p}{2} \rceil$, such that $u \notin N[S]$. Then $\gamma_{CM}(T - u) = \gamma_{CM}(T)$.*

Proof: Let S and S_1 be the γ_{CM}-set T and T' respectively. Since $|N[S_1]| \geq \lceil \frac{p}{2} \rceil, S_1$ is also a CMD set of T. Therefore, $\gamma_{CM}(T) \leq |S_1| = \gamma_{CM}(T-u)$. Since there exists a γ_{CM}-set S of T and a vertex $u \notin N[S], S$ is again a γ_{CM}-set of $T-u$. Therefore, $\gamma_{CM}(T-u) \leq |S| = \gamma_{CM}(T). Hence, \gamma_{CM}(T-u) = \gamma_{CM}(T)$. □

Proposition 5.10. *Let G be any graph with p vertices and u be a pendant vertex. Then $\gamma_{CM}(G-u) \leq \gamma_{CM}(G)$.*

Proof: Let $u \in V(G)$ be a pendant and $|V(G)| = p$. Let S be a γ_{CM}-set of G (1)

Case 1. *Let p be even. Since u is pendant, $u \notin N[S]$. Then by Corollary (5.3), we get $\gamma_{CM}(G-u) = \gamma_{CM}(G)$. Let $u \in N(S)$. By (1), since $|N[S] > \frac{p}{2}, |N_{(G')}[S] \geq \left\lceil \frac{(p-1)}{2} \right\rceil = \frac{p}{2}$ and $\langle S \rangle$ is connected in G'. Hence, S is again a γ_{CM}-set for G'. It implies $\gamma_{CM}(G') = \gamma_{CM}(G)$.*

Case 2. *Let p be odd. Let S be a γ_{CM}-set of G. Then $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\langle S \rangle$ is connected in G Since p is odd, $\lceil \frac{p}{2} \rceil = \frac{p}{2} + 1$. Therefore, in $(G-u), (p-1)$ is even and there exists a γ_{CM}-set S_1 of $(G-u)$ such that $|N[S_1]| = \frac{(p-1)}{2} = \frac{p}{2}, p$ is odd with $|S_1| < |S|$ implies $\gamma_{CM}(G-u) < \gamma_{CM}(G)$.* □

6. Characterization of $V^{+}_{CM}(G)$

Proposition 6.1. *Let G be a graph with p vertices and S be a γ_{CM}-set of G. A vertex $u \in V^{+}_{CM}(G)$ if and only if u satisfies the following conditions If p is odd and $\langle G' \rangle$ is connected with $d_{G'}(u_i) < \lceil \frac{p}{2} \rceil - 2$, for every $u_i \in V(G')$ then $u \in S$.*

Proof: Let $u \in V^{+}_{CM}(G)$. Then

$$\gamma'_{CM}(G) > \gamma_{CM}(G) \tag{1}$$

Let S be a CMD set of G, implies that $\langle S \rangle$ is connected and $|N[S]| \geq \lceil \frac{p}{2} \rceil$.
Let $p = 2n+1 = odd$.To prove that $u \in S$. Suppose $u \notin S$. Then $u \in N(S)$ or $u \notin N(S)$.

Case 1. *When $u \in N(S)$ such that $|N[S]| = \lceil \frac{p}{2} \rceil$. Then in $G' = G-u, |N_{(G')}[S]| = \lceil \frac{p}{2} \rceil - 1 = \frac{(p-1)}{2}$. Since, $\langle G' \rangle$ is connected, $\langle S \rangle$ is connected in G'. This implies that S is CMD set of G'. Therefore, $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$, which is a contradiction to (1).*

Case 2. *When $u \in N(S)$ such that $|N_G[S]| > \lceil \frac{p}{2} \rceil$ and $p = 2n+1$. Therefore $p-1 = 2n$. If $u \in N(S)$ then $|N_{(G')}[S]| = \lceil \frac{p}{2} \rceil > \frac{(p-1)}{2}$. Since, $\langle G' \rangle$ is connected, $\langle S \rangle$ is connected. This implies S is again a CMD set for G'. Therefore, $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$, which is a contradiction to (1).*

Case 3. *When $u \notin N(S)$ and $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Then $u \in ((V(G) - N(S))$.
Since, $\langle S \rangle$ is connected in $G' = G-u$ and $|N_{(G')}[S]| \geq \lceil \frac{p}{2} \rceil = \frac{(p-1)}{2}$. This implies that S is again a CMD set of G'. Therefore, $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$, which is a contradiction to (1).*

From the above all cases, we find $u \in S$. □

Proposition 6.2. *Let S be a CMD set of a connected graph G with p vertices. Let $u \in V^{+}_{CM}(G)$ and $G' = G-u$. If p is even and $\langle G' \rangle$ is connected with $d_{G'}(u_i) < d(u)$, for every $u_i \in V(G')$ then $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$ and $u \in S$ such that $|N[S]| > \frac{p}{2}$.*

Proof: Let $u \in V^+_{CM}(G)$. Then $\gamma_{CM}(G') > \gamma_{CM}(G)$ (1)

Let $p = 2n =$ even and $\langle G' \rangle$ is connected. To prove (i) $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$, (ii) $u \in S$ such that $|N[S]| > \frac{p}{2}$.

Case 1. *Suppose $u \notin N[S]$ such that $|N[S]| \geq \frac{p}{2}$ (or) $u \in N[S]$ such that $|N[S]| > \frac{p}{2}$.*

Subcase 1. *Suppose $u \notin N[S]$ such that $|N[S]| \geq \frac{p}{2}$. Then $u \in V(G) - N[S]$.*
When $|N[S]| = \frac{p}{2}$ and $p = 2n, p-1 = 2n-1$ i.e., $\frac{p}{2} = \left\lceil \frac{(p-1)}{2} \right\rceil$. Since, $\langle G' \rangle$ is connected, $\langle S \rangle$ is also connected in G'. Then $|N_{G'}[S]| \geq \frac{p}{2} = \left\lceil \frac{(p-1)}{2} \right\rceil$. It implies that S is a CMD set of G'. Therefore $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$, contradicts (1).

Subcase 2. *Suppose $u \in N[S]$ such that $|N[S]| > \frac{p}{2}$. Then in $G' = G - u, |N_{G'}[S]| \geq \frac{p}{2} = \left\lceil \frac{(p-1)}{2} \right\rceil$ and $\langle S \rangle$ is connected in G'. It implies that S is a CMD set of G', contradicts (1). Hence, $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$.*

Case 2. *Suppose $u \notin N(S)$ such that $|N[S]| > \lceil \frac{p}{2} \rceil$. Then $u \notin N(S)$ or $u \in N(S)$ such that $|N[S]| > \lceil \frac{p}{2} \rceil$ Or $u \in S$ such that $|N[S]| = \frac{p}{2}$.*

Subcase 1. *Suppose $u \in N(S)$ such that $|N[S]| > \lceil \frac{p}{2} \rceil$. Let $p = 2n, \frac{p}{2} = \left\lceil \frac{(p-1)}{2} \right\rceil$.*
In $G' = G - u, |N_{G'}[S]| \geq \frac{p}{2} = \left\lceil \frac{(p-1)}{2} \right\rceil$ and $\langle S \rangle$ is connected in G'. It implies that S is a CMD set of G', by similar arguments, we get a contradiction to (1).

Subcase 2. *Suppose $u \notin N(S)$ such that $|N[S]| > \frac{p}{2}$. Then in $G' = G - u$, $\langle S \rangle$ is connected and $|N_{G'}[S]| \geq \frac{p}{2} = \left\lceil \frac{(p-1)}{2} \right\rceil$. It implies that S is a CMD set of G', we get a contradiction to (1).*

Subcase 3. *Suppose $u \in S$ such that $|N[S]| = \frac{p}{2}$. This is contained in case (i). Hence, if $u \in V^+_{CM}(G)$ then $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$ and $u \in S$ such that $|N[S]| > \frac{p}{2}$.* □

Proposition 6.3. *Let S be a CMD set of a connected graph G with p vertices. Let $u \in V^+_{CM}(G)$ and $G' = G - u$. If $\langle G' \rangle$ is disconnected with a component g such that $|N_{G'}[g]| \geq \lceil \frac{p}{2} \rceil$ and $d_g(u_i) < d(u)$, for every $u_i \in V(g)$ then $u \in S$ if p odd and $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$ if p is even.*

Proof: Let $u \in V^+_{CM}(G)$. Then $\gamma_{CM}(G') > \gamma_{CM}(G)$ (1)

Let S be a CMD set of G and $G' = G - u$. To prove $u \in S$.
Given $\langle G' \rangle$ is disconnected with atleast one component g such that $|N_{(G')}[g]| \geq \lceil \frac{p}{2} \rceil$ and $d_g(u_i) < d(u)$, for every $u_i \in V(g)$ (2)

Suppose $p = 2n + 1 =$ odd $\notin S$. Then either $u \in N(S)$ or Suppose $u \notin N(S)$.

Case 1. *When $p = 2n + 1 =$* odd. *Then $p - 1 = 2n$.*

Subcase 1. *Let $u \in N(S)$, then $|N_{G'}[g]| \leq \lceil \frac{p}{2} \rceil$. Since $\langle G' \rangle$ is disconnected, $\langle S' \rangle$ is connected. Also, when p is odd, $\lceil \frac{p}{2} \rceil - 1 = \frac{p-1}{2}$. Therefore $|N_{G'}[g]| = \lceil \frac{p}{2} \rceil - 1 = \frac{p-1}{2}$. It implies that S is a CMD set of G'. Thus, $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$, contradiction to (1). Hence, $u \in S$.*

Subcase 2. *Let $u \notin N(S)$. Then $u \in ((V(G) - N(S))$ and $\langle G' \rangle$ is disconnected. Now CMD set of S of G is contained in any one of the component of G'. Therefore $|N_{G'}[S]| \geq$*

$\lceil \frac{p}{2} \rceil > \frac{(p-1)}{2}$ and $\langle S' \rangle$ is connected in G'. This implies that S is also a CMD set of G'. Therefore $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$ contradiction to (1). Hence, $u \in S$.

Case 2. *When $p = 2n =$ even. Then $p-1 = 2n-1$. To prove $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$. Suppose $u \notin N[S]$ such that $|N[S]| > \lceil \frac{p}{2} \rceil$ and $u \in N[S]$ such that $|N[S]| > \frac{p}{2}$.*

Subcase 1. *Suppose $u \in N[S]$ such that $|N[S]| > \frac{p}{2}$. Then $u \in (V(G) - N[S])$. Since $\langle G' \rangle$ is disconnected, $\langle S \rangle$ is connected in any one of the component of G' and $|N_{G'}[S]| \geq \lceil \frac{p}{2} \rceil$ and $\langle S \rangle$ is connected in G'. It implies that S is a CMD set of G'. Thus, $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$, contradiction to (1). Hence, $u \in S$ such that $|N[S]| = \frac{p}{2}$.*

Subcase 2. *Suppose $u \in N[S]$ such that $|N[S]| = \lceil \frac{p}{2} \rceil$. Then in $G' = G - u, |N_{G'}[S]| \geq \frac{p}{2} = \left\lceil \frac{p-1}{2} \right\rceil$. Since $\langle G' \rangle$ is disconnected, $\langle S \rangle$ is connected and it is contained in G'. It implies that S is a CMD set of G'. Thus, $\gamma'_{CM}(G) = |S| = \gamma_{CM}(G)$, contradiction to (1). Hence, $u \in S$ such that $|N[S]| = \frac{p}{2}$.* □

Proposition 6.4. *If $u \in V^{+}_{CM}$, then no subset of $V(G) - N[u]$ with the same cardinality of $\gamma_{CM}(G)$ majority dominates $(G - u)$ connectedly.*

Proof: Let $u \in \gamma^{+}_{CM}(G)$ Then $\gamma_{CM}(G - u) > \gamma_{CM}(G)$ (1)
and let S be the γ_{CM}- set of G. Suppose S_1 is the subset of $V(G) - N[u]$ with the same cardinality of S majority dominates $(G-u)$ and $\langle S_1 \rangle$ is also connected. Then S_1 is a CMD set of $(G - u) = G'$. Therefore $\gamma_{CM}(G') \leq |S_1| = |S| = \gamma_{CM}(G)$, which is a contradiction to (1). Hence the result. □

Theorem 6.1. *Let G be a graph with p vertices and S be a γ_{CM}-set of G and let $G' = G-u$. A vertex $u \in V^{+}_{CM}(G)$ if and only if u satisfies the following conditions:*

- A) *If p is odd and $\langle G' \rangle$ is connected with $d_{G'}(u_i) < d(u)$ for every $u_i \in V(G')$ then $u \in S$.*
- B) *If p is even and $\langle G' \rangle$ is connected with $d_{G'}(u_i) < d(u)$ for every $u_i \in V(G')$ then $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$ and $u \in S$ such that $|N[S]| \geq \frac{p}{2}$.*
- C) *If $\langle G' \rangle$ is disconnected with a component 'g' such that $|N_{G'}[g]| \geq \lceil \frac{p}{2} \rceil$ and $d_g(u_i) < d(u)$, for every $u_i \in V(g)$ then $u \in S$, if p is odd and $u \in N[S]$ such that $|N[S]| = \frac{p}{2}$, if p is even.*
- D) *No subset of $V(G)-N[u]$ with the same cardinality of $\gamma_{CM}(G)$ majority dominates G' connectedly.*

Proof: Let $u \in V^{+}_{CM}(G)$. Let S be a γ_{CM}-set of G. From the proof of Propositions 6.1, 6.2, 6.3 and 6.4, we get all the four conditions.
Conversely, assume that the conditions (A), (B), (C) & (D) are true. Let $G' = G - u$.
A) Assume that if p is odd and $\langle G' \rangle$ is connected with $d_{G'}(u_i) < d(u)$, for every $u_i \in V(G')$.
Let S be a $\gamma_{CM}(G)$-set of G. This implies that $|N[S]| \geq \lceil \frac{p}{2} \rceil$ and $\langle S \rangle$ is connected in G. By hypothesis (A), $\langle G' \rangle$ is connected and $d_{G'}(u_i) < \lceil \frac{p}{2} \rceil - 2$, for every $u_i \in V(G')$. Let $u \in S$. Then $|N_{G'}[G]| < \lceil \frac{p}{2} \rceil = \frac{p-1}{2}$. This implies that S is not a γ_{CM}-set of G'. Since, $d_{G'}(u_i) < \lceil \frac{p}{2} \rceil - 2$, there exists a set $S_1 \subseteq V(G')$ with $|S_1| > |S|$ such that $|N[S_1]| \geq \frac{p-1}{2}$. Since $\langle G' \rangle$ is connected, $\langle S_1 \rangle$ is connected. Therefore, S_1 is a $\gamma_{CM}(G)$-set of G'. This implies that $\gamma_{CM}(G') = |S_1| > |S| = \gamma_{CM}(G)$, for $G' = G - u$. Hence, $u \in V^{+}_{CM}(G)$.
B) Assume that if p is even and $\langle G' \rangle$ is connected with $d_{G'}(u_i) < d(u)$, for every $u_i \in V(G')$.

Case 1. *Let $u \in N[S]$ such that $N[S] = \frac{p}{2}$. Then $|N_{G'}[S]| < \frac{p}{2} = \left\lceil \frac{p-1}{2} \right\rceil$. It implies that S is not a γ_{CM}-set of G'. By hypothesis (B), since $d_{G'}(u_i) < d(u)$ for every $u_i \in V(G')$,* *there exists a set* $S_1 \subseteq V(G')$ *with* $|S_1| > |S|$ *such that* $|N[S_1]| \geq \left\lceil \frac{p-1}{2} \right\rceil$*. Since* $\langle G' \rangle$ *is connected,* $\langle S_1 \rangle$ *is also connected in* G'*. It implies that* S_1 *is a* γ_{CM}*-set of* G'*. Therefore,* $\gamma_{CM}(G') = |S_1| > |S| = \gamma_{CM}(G)$*, for* $G' = G - u$*. Hence,* $u \in V^+_{CM}(G)$*.*

Case 2. *Let* $u \in S$ *such that* $|N[S]| \geq \frac{p}{2}$*. Then* S *is connected and* $|N_{G'}[S]| < \left\lceil \frac{p-1}{2} \right\rceil$*. This implies that* S *is not a* γ_{CM}*-set of* G'*. By hypothesis (B), there exists a set* $S_1 \subseteq V(G')$ *with* $|S_1| > |S|$ *such that* $|N_{G'}[S_1]| \geq \left\lceil \frac{p-1}{2} \right\rceil$*. It implies that* S_1 *is a* γ_{CM}*-set of* G'*. Therefore,* $\gamma_{CM}(G') = |S_1| > |S| = \gamma_{CM}(G)$*, for* $G' = G - u$*. Hence,* $u \in V^+_{CM}(G)$*.*

C) Let S *be a* γ_{CM}*-set of* G*, imply that* $|N[S]| \geq \lceil \frac{p}{2} \rceil$*. Since,* $\langle G' \rangle$ *is not connected,* S *is not a* $\gamma_{CM}(G)$*-set of* G'*.*

Case 1. *Let* $u \in S$ *and* p *is odd. By hypothesis, since there is a component 'g' such that* $|N_{G'}[g]| \geq \lceil \frac{p}{2} \rceil$ *in* G'*. Since,* $d_g(u_i) < d(u)$*, for every* $u_i \in V(g)$*, there exists a set* $S_1 \subseteq V(g) \subseteq V(G')$ *with* $|S_1| > |S|$ *such that* $|N_{G'}[S_1]| \geq \lceil \frac{p}{2} \rceil$ *and* $\langle S_1 \rangle$ *is also connected in* G'*. It implies that* S_1 *is a* γ_{CM}*-set of* G' *in* g*. Therefore,* $\gamma_{CM}(G') = |S_1| > |S| = \gamma_{CM}(G)$*, for* $G' = G - u$*. Hence,* $u \in V^+_{CM}(G)$*.*

Case 2. *Let* $u \in N[S]$ *such that* $|N[S]| = \frac{p}{2}$*, if* p *is even. Then in* G'*,* $|N_{G'}[S]| < \left\lceil \frac{p-1}{2} \right\rceil$ *and since* $\langle G' \rangle$ *is connected,* S *is not a* γ_{CM}*-set of* G'*. Since* $d_g(u_i) < d(u)$*, for every* $u_i \in V(g)$*, apply the same argument, we get* $\gamma_{CM}(G') > \gamma_{CM}(G)$*. Hence,* $u \in V_C M^+(G)$*.*

D) Let $G' = G - u$*. Suppose* $S_1 \subseteq V(G) - N[u]$ *is a* γ_{CM}*-set of* G' *with* $|S_1| = |S|$*. Then* $\gamma_{CM}(G') = |S_1| = |S| = \gamma_{CM}(G)$*, imply that* $u \in V^0_{CM}(G)$*. It is a contradiction to assumption* $u \in V^+_{CM}(G)$*.* □

6.1. Characterization of $V^0_{CM}(G)$.

Theorem 6.2. *Let* G *be a graph with* p *vertices and* S *be a* γ_{CM}*-set of* G*. A vertex* $u \in V^0_{CM}(G)$ *if and only if the following conditions are true:*

(i) $u_i \notin N[S]$*, for all* $u_i \in V(G)$*.*
(ii) *A pendant and* $u \in N(S)$ *such that* $|N[S]| > \lceil \frac{p}{2} \rceil$*.*
(iii) *When* p *is odd* $u \in N(S)$ *such that* $|N[S]| \geq \lceil \frac{p}{2} \rceil$*.*
(iv) *When* p *is even* $u \in N(S)$ *such that* $|N[S]| > \lceil \frac{p}{2} \rceil$*.*

Proof:

(i) Let $u \in V^0_{CM}(G)$ and $G' = G - u$. Then $\gamma_{CM}(G') = \gamma_{CM}(G)$ (1)
and S is a γ_{CM}-set of G and G'.
This implies that $\langle S \rangle$ is connected and $|N[S]| \geq \lceil \frac{p}{2} \rceil$ in both G and G'. (2).
Then $\langle S \rangle$ is connected in G' and $|N_G[S]| \geq \lceil \frac{p}{2} \rceil$, for $u \notin N[S]$. Therefore, S is also a γ_{CM}-set of G', for all $u_i \in N[S]$. Hence, (i) holds.

(ii) Since $\langle S \rangle$ is connected in G' and $|N_G[S]| \geq \lceil \frac{p}{2} \rceil$, S is a γ_{CM}-set of G'. When $|N[S]| = \lceil \frac{p}{2} \rceil, |N_{G'}[S]| \leq \lceil \frac{p}{2} \rceil - 1$, if $u \in N(S)$. It implies that S is not a γ_{CM}-set of G'. When $|N[S]| > \lceil \frac{p}{2} \rceil, |N_{G'}[S]| \geq \lceil \frac{p}{2} \rceil$, if $u \in N(S)$ and $\langle S \rangle$ is connected in G'. This implies that u is a pendant and $u \in N(S)$. Therefore, (ii) holds.

(iii) Let p be odd. By (2), if $|N_G[S]| = \lceil \frac{p}{2} \rceil$ and $\langle S \rangle$ is connected in $G', |N_{G'}[S]| = \lceil \frac{p}{2} \rceil = \frac{p-1}{2}$, implies that S is a γ_{CM}-set of G' where $u \in N(S)$. If $|N_G[S]| > \lceil \frac{p}{2} \rceil$, then $|N_{G'}[S]| \geq \frac{p-1}{2}$ and by (2), $\langle S \rangle$ is connected in G', implies that $u \in N(S)$. Hence, $u \in N(S)$ such that $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Therefore (iii) holds.

(iv) Let p be even. Suppose $u \in N(S)$ such that $|N[S]| = \frac{p}{2}$. Then $|N_{G'}[S]| = \frac{p-1}{2}$ for $u \in N(S)$ and $\langle S \rangle$ is connected in G', implies that S is not a γ_{CM}-set of G' where $u \in N(S)$ which is a contradiction to (2). Hence, $u \in N(S)$ such that $|N[S]| > \frac{p}{2}$.

Suppose $u \notin N(S)$. Then $u \in S$ or $u \in V(G) - N(S)$. If $u \in S$ then $|N[S]| \geq \frac{p}{2}$ and $\langle S \rangle$ is not connected in G'. It implies that S is not a γ_{CM}-set of G', contradicts to (2).
If $u \in V(G) - N(S)$, then $u \in N[S]$. Then by (i), it is true. Hence, $u \in N(S)$ such that $|N[S]| > \frac{p}{2}$, if p is even.
Conversely, let S be a γ_{CM}-set of G. If all the vertices $u_i \in V(G), u_i \notin N[S]$, then $|N_{G'}[S]| \geq \lceil \frac{p}{2} \rceil$ and $\langle S \rangle$ is connected in G', implies that S is again a γ_{CM}-set for G'. Hence, $u \in V^0_{CM}(G)$.
When p is odd, $\lceil \frac{p}{2} \rceil = \frac{p-1}{2}$. Therefore, if $u \in N(S)$ such that $|N[S]| \geq \lceil \frac{p}{2} \rceil$, then $|N_{G'}[S]| \geq \frac{p-1}{2}$ and $\langle S \rangle$ is connected in G', implies that S is again a γ_{CM}-set for G'. Hence, $u \in V^0_{CM}(G)$.
When p is even, $\frac{p}{2} = \left\lceil \frac{p-1}{2} \right\rceil$. Therefore, if $u \in N(S)$ such that $|N[S]| > \lceil \frac{p}{2} \rceil$, then $|N_{G'}[S]| \geq \lceil \frac{p-1}{2} \rceil$ and $\langle S \rangle$ is connected in G', implies that S is again a γ_{CM}-set for G'. Hence, $u \in V^0_{CM}(G)$. □

6.2. Characterization of $V^-_{CM}(G)$.

Theorem 6.3. *Let S be a γ_{CM}-set of a graph G and $G' = G - u$. A vertex $u \in V^-_{CM}(G)$ if and only if it satisfies the following conditions:*

(i) $|V(G)| = p$ *is odd.*
(ii) $|N[S]| = \lceil \frac{p}{2} \rceil$.
(iii) $\gamma_{CM}(G') = \gamma_{CM}(G) - 1$.

Proof: Let S be a γ_{CM}-set of a graph G. Then $\langle S \rangle$ is connected and $|N[S]| \geq \lceil \frac{p}{2} \rceil$. Let $G' = G - u$. Let $u \in V^-_{CM}(G)$. Then $\gamma_{CM}(G') < \gamma_{CM}(G)$ (1).

(i) Suppose p is even. Then $\frac{p}{2} = \left\lceil \frac{p-1}{2} \right\rceil$. If $u \in N[S]$ and $|N[S]| > \frac{p}{2}$, then $|N_{G'}[S]| \geq \frac{p}{2} = \left\lceil \frac{p-1}{2} \right\rceil$. This implies that S is again a γ_{CM}-set of a graph G'. Therefore, $\gamma_{CM}(G') = \gamma_{CM}(G)$, contradiction to (1). If $u \in N[S]$ and $|N[S]| = \frac{p}{2}$, then $|N_{G'}[S]| < \left\lceil \frac{p-1}{2} \right\rceil = \frac{p}{2}$. Therefore, there exists a set $S_1 \subseteq V(G')$ with $|S_1| = |S|$ such that $|N_{G'}[S]| = \left\lceil \frac{p-1}{2} \right\rceil$. It implies that S_1 is a γ_{CM}-set of G'. Thus, $\gamma_{CM}(G') = |S_1| = |S| = \gamma_{CM}(G)$, but contradicting (1). Suppose $u \notin N[S]$, then S is again a γ_{CM}-set of G'. This implies that $\gamma_{CM}(G') = |S| = \gamma_{CM}(G)$. Hence, this leads to contradiction to (1). Therefore, $|V(G)| = p$ is odd.
(ii) Suppose $|N[S]| > \lceil \frac{p}{2} \rceil$. If $u \in N[S]$ and $u \notin N[S]$, then $|N_{G'}[S]| \geq \lceil \frac{p}{2} \rceil = \frac{p-1}{2}$. It implies that S is a γ_{CM}-set of G and G'. Therefore, $\gamma_{CM}(G') = \gamma_{CM}(G)$, which is a contradiction to (1). Hence, $|N[S]| = \lceil \frac{p}{2} \rceil$.
(iii) By (1), $\gamma_{CM}(G') < \gamma_{CM}(G)$ and p is odd. This implies S_1 is a γ_{CM}-set of G' and $|S_1| < |S|$. Since p is odd and $|N[S]| = \lceil \frac{p}{2} \rceil$, $|N_{G'}[S_1]| = \frac{p-1}{2}$. Since $\langle S \rangle$ is connected in G, all vertices of S are adjacent. Also $\langle S_1 \rangle$ is connected in G' and $|S_1| = |S| - 1$ such that $|N_{G'}[S_1]| = \frac{p-1}{2}$. Hence, $\gamma_{CM}(G') = \gamma_{CM}(G) - 1$.

Conversely, the conditions (i) – (iii) are true. Let S be a γ_{CM}-set of G and G'. When p is odd, $|N[S]| = \lceil \frac{p}{2} \rceil$ and $\gamma_{CM}(G') = \gamma_{CM}(G) - 1$. It implies $\gamma_{CM}(G') < \gamma_{CM}(G)$ for any vertex $u \in V(G)$. Then by definition, $u \in V^-_{CM}(G)$. □

7. Conclusion

In this research article, it has been discussed that the removal of any vertex of a graph G how affects the Connected Majority Domination number of G. Also the vertex critical some classifications are discussed. The characterization theorems are also determined for $V^+_{CM}(G), V^-_{CM}(G)$ and $V^0_{CM}(G)$.

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J. Joseline Manora is a research supervisor with an experience of 32 years of service in the P.G and Research Department of mathematics, Tranquebar Bishop Manickam-Lutheren College, Porayar, affiliated to Bharathidasan University, Thiruchirappalli, Tamilnadu, India. She has received Ph.D Mathematics from Madurai Kamaraj University Madurai. Her area of interest and specialisation includes Graph Theory and Mathematical Statistics.

T. Muthukani Vairavel is in the Department of Mathematics in Sir Issac Newton College, Nagapattinam, Tamilnadu, India. His area of interest is mainly in Graph Theory and Real Analysis.