

ON TOTAL VERTEX IRREGULARITY STRENGTH OF SOME CLASSES OF TADPOLE CHAIN GRAPHS

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ABSTRACT. A total k -labeling f that assigns $V \cup E$ into $\{1, 2, \dots, k\}$ on graph G is named vertex irregular if $wt_f(u) \neq wt_f(v)$ for dissimilar vertices u, v in G with the weights $wt_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. We call the minimum number k utilized in total labeling f as a total vertex irregularity strength of G , symbolized by $tv_s(G)$. In this research, we focus on tadpole chain graphs that are chain graphs which contain tadpole graphs in their blocks. We investigate tv_s of some classes of tadpole chain graphs, i.e., $T_r(4, n)$ and $T_r(5, n)$ with length r . Some formulas are derived as follows: $tv_s(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$ and $tv_s(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$.

Keywords: Vertex irregular total k -labeling, tv_s , tadpole graph, chain graph.

AMS Subject Classification: 83-02, 99A00

1. INTRODUCTION

We assume that $G(V, E)$ is a finite, undirected, and simple graph. A mapping f that assigns $V(G) \cup E(G)$ into a set of integers is named a total labeling. Further, the integers used in f are called as labels [14]. In addition, the total k -labeling f is called a total vertex irregular if the vertex weights $wt_f(u) \neq wt_f(v)$ for distinct vertices $u \neq v$ in G with $wt_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. Baca et al. [5] initiated the notion of total vertex irregularity strength of graph G , denoted by $tv_s(G)$, that is defined as the minimum number k in such a way that G has a vertex irregular total k -labeling.

Bača et al. [5] proposed the lower bound in the following: $\left\lceil \frac{p+\delta}{\Delta+1} \right\rceil \leq tv_s(G) \leq p + \Delta - 2\delta + 1$ for any graph $G(V, E)$ where $p = |V(G)|$, $\delta = \min\{d(v) | \forall v \in V(G)\}$, and $\Delta = \max\{d(v) | \forall v \in V(G)\}$, respectively. Whereas, Anholcer, et al. provided another bounds in [3]. Another way to get the lower bound of tv_s for any connected graph G was given by Nurdin et al. [11]. Let G be a connected graph where the number of vertices of

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§ Manuscript received: October, 25, 2019; accepted: April 20, 2020.

TWMS Journal of Applied and Engineering Mathematics, Vol.11, Special Issue © Işık University, Department of Mathematics, 2021; all rights reserved.

degrees i is n_i , for $i = \delta, \delta + 1, \delta + 2, \dots, \Delta$.

$$tvs(G) \geq \max \left\{ \left\lceil \frac{\delta(G) + n_{\delta(G)}}{\delta(G) + 1} \right\rceil, \left\lceil \frac{\delta(G) + n_{\delta(G)} + n_{\delta(G)+1}}{\delta(G) + 2} \right\rceil, \dots, \left\lceil \frac{\delta(G) + \sum_{i=\delta(G)}^{\Delta(G)} n_i}{\Delta(G) + 1} \right\rceil \right\}. \tag{1}$$

Nowdays, Many scholars have found total vertex irregularity strength of some graph classes such as in [5],[7], [10], [1], [2], [9], etc. Meanwhile, the results related to cactus chain graphs has been invented in [4] and [12]. Recently, Rosyida et al. published a result of tvs of $T_r(4, 1)$ tadpole chain graph [13]. To continue the result in [13], we investigate tvs of some tadpole chain graphs in this paper, i.e. $tvs(T_r(4, n))$ and $tvs(T_r(5, n))$.

2. MAIN RESULTS

In this part, we present formulas of tvs of $T_r(4, n)$ and $T_r(5, n)$. Let us consider the chain graph in Definition 2.1.

Definition 2.1. A graph which consists of a cycle graph C_m and a path graph P_n connected with a bridge is called as a tadpole graph, denoted by $T(m, n)$. Given a connected graph G . A bipartite graph (B, C) , where B is a set of blocks in graph G and C consists of cut vertices on each block in B , is named a block cut vertex of G . The edges of G join cut vertices with those blocks to which they belong. Further, G is called as a chain graph of length r if it contains r -blocks such that each pair of two blocks B_i and B_{i+1} has one common cut vertex for which the block cut vertex is a path ([8], [6]). Furthermore, tadpole chain graphs $T_r(4, n)$ and $T_r(5, n)$ are chain graphs which their blocks are $T(4, n)$ and $T(5, n)$, respectively.

2.1. Total vertex irregularity strength of $T_r(5, n)$. The chain graph $T_r(5, n)$ consists of:

- $\{y_1^n, y_2^n, \dots, y_r^n\}$, i.e. a set of vertices with degrees 1
- $\{y_1^{n-1}, y_2^{n-1}, \dots, y_r^{n-1}, y_1^{n-2}, y_2^{n-2}, \dots, y_r^{n-2}, \dots, y_1^2, y_2^2, \dots, y_r^2, y_1^1, y_2^1, \dots, y_r^1\} \cup \{u_1, u_2, \dots, u_{2r}, u_{2r+1}, u_{2r+2}\}$, i.e. a set of vertices of degrees 2;
- $\{x_1, x_2, \dots, x_r\}$ that is a set of vertices with degrees 3; and
- $\{v_1, v_2, \dots, v_{r-1}\}$ that is a set of vertices with degrees 4.

Theorem 2.1. If $T_r(5, 1)$ is a tadpole chain graph of length $r(r \geq 2)$, then

$$tvs(T_r(5, 1)) = r + 1.$$

Proof. Let y_1, y_2, \dots, y_r be vertices of $T_r(5, 1)$ with degrees 1.

TABLE 1. Labels of vertices and edges of $T_r(5, 1)$

Labels of vertices	Labels of edges
$f(u_1) = 1,$	$f(u_{2i-1}u_{2i}) = i, 1 \leq i \leq r,$
$f(u_{2i-1}) = i - 1, 2 \leq i \leq r,$	$f(u_1u_{2r+1}) = r,$
$f(u_{2i}) = r, 1 \leq i \leq r,$	$f(u_{2r}u_{2r+2}) = r + 1,$
$f(u_{2r+1}) = f(u_{2r+2}) = r + 1,$	$f(u_{2i}v_i) = r + 1, 1 \leq i \leq r - 1,$
$f(v_i) = r - 1, 1 \leq i \leq r - 1,$	$f(u_{2i+1}v_i) = r + 1, 1 \leq i \leq r - 1,$
$f(x_i) = r + 1, 1 \leq i \leq r,$	$f(v_i x_i) = r + 1, 1 \leq i \leq r - 1,$
$f(y_i) = 1, 1 \leq i \leq r.$	$f(v_i x_{i+1}) = r + 1, 1 \leq i \leq r - 1,$
	$f(u_{r+1}x_1) = r + 1, f(u_{r+2}x_r) = r + 1,$
	$f(x_i y_i) = i, 1 \leq i \leq r.$

According to Inequality (1), we obtain

$$tvs(T_r(5, 1)) \geq \max \left\{ \left\lceil \frac{r+1}{2} \right\rceil, \left\lceil \frac{3r+3}{3} \right\rceil, \left\lceil \frac{4r+3}{4} \right\rceil, \left\lceil \frac{5r+2}{5} \right\rceil \right\} = \left\lceil \frac{3r+3}{3} \right\rceil = r+1. \quad (2)$$

To proof that $tvs(T_r(5, 1)) \leq r+1$, we provide a total k -labeling $f : V \cup E \rightarrow \{1, 2, \dots, k\}$ with $k = r + 1$ and define labels of vertices and edges as in Table 1.

Under labeling f , we obtain that each vertex has the weight below:

$$\begin{aligned} wt_f(y_i) &= i + 1, & 1 \leq i \leq r, \\ wt_f(u_1) &= r + 2, \\ wt_f(u_i) &= i + r + 1, & 1 \leq i \leq 2r, \\ wt_f(u_{2r+1}) &= 3r + 2, \\ wt_f(u_{r+2}) &= 3(r + 1), \\ wt_f(v_i) &= i + 4r + 3, & 1 \leq i \leq r - 1, \\ wt_f(x_i) &= i + 3(r + 1), & 1 \leq i \leq r. \end{aligned}$$

The minimum label of vertices and edges is 1 and the maximum label is $r + 1$. Also, it is shown that the vertex weights are all diverse. Hence, we get upper bound $tvs(T_r(5, 1)) \leq r + 1$. Combining with Lower bound (2), it is proved that $tvs(T_r(5, 1)) = r + 1$. \square

Theorem 2.2. *If $T_r(5, n)$ are tadpole chain graphs with length $r (r \geq 2)$, $n = 4 \pmod 3$, and $n \geq 4$, then $tvs(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$.*

Proof. Based on (1), we acquire the lower bound

$$tvs(T_r(5, n)) \geq \max \left\{ \left\lceil \frac{r+1}{2} \right\rceil, \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \left\lceil \frac{(n+3)r+3}{4} \right\rceil, \left\lceil \frac{(n+4)r+2}{5} \right\rceil \right\} = \left\lceil \frac{(n+2)r+3}{3} \right\rceil. \quad (3)$$

We prove the upper bound through 3 cases.

Case 1. For $n = 4$.

TABLE 2. Labels of vertices and edges of $T_r(5, 4)$

Labels of vertices	Labels of edges
$f(u_{2i-1}) = i + r - 2, 1 \leq i \leq r,$	$f(u_{2i-1}u_{2i}) = i + r + 1, 1 \leq i \leq r,$
$f(u_{2i}) = (2r + 1) - 2 = 2r - 1, 1 \leq i \leq r,$	$f(u_1u_{2r+1}) = 2r + 1,$
$f(u_{2r+1}) = 2r + 1 - 1 = 2r,$	$f(u_{2r}u_{2r+2}) = 2r + 1,$
$f(u_{2r+2}) = 2r + 1,$	
$f(v_i) = 1, 1 \leq i \leq r - 1,$	$f(u_{2i}v_i) = i + r + 1, 1 \leq i \leq r - 1,$
$f(x_i) = 2r + 1, 1 \leq i \leq r,$	$f(u_{2i+1}v_i) = 2r + 1, 1 \leq i \leq r - 1,$
$f(y_i^1) = 2r - 1, 1 \leq i \leq r - 1,$	$f(v_i x_i) = 2r - 1, 1 \leq i \leq r - 1,$
$f(y_r^1) = 2r + 1,$	$f(v_i x_{i+1}) = 2r + 1, 1 \leq i \leq r - 1,$
$f(y_i^2) = r + 1, 1 \leq i \leq r,$	$f(u_{2r+1}x_1) = 2r + 1, f(u_{2r+2}x_r) = 2r + 1,$
$f(y_i^3) = r, 1 \leq i \leq r,$	$f(x_i y_i^1) = i + 2, 1 \leq i \leq r - 1,$
$f(y_i^4) = i, 1 \leq i \leq r.$	$f(x_r y_r^1) = r$
	$f(y_i^1 y_i^2) = r, 1 \leq i \leq r,$
	$f(y_i^2 y_i^3) = i; f(y_i^3 y_i^4) = 1, 1 \leq i \leq r,$

We assume that $f : V \cup E \rightarrow \{1, 2, \dots, k\}$ is a total k -labeling with $k = \left\lceil \frac{6r + 3}{3} \right\rceil = 2r + 1$. Vertex and edge labels are defined in Table 2.

Under labeling f , we derive the weights of vertices below:

$$\begin{aligned}
 wt_f(y_i^4) &= i + 1, & 1 \leq i \leq r, \\
 wt_f(y_i^3) &= i + r + 1, & 1 \leq i \leq r, \\
 wt_f(y_i^2) &= i + 2r + 1, & 1 \leq i \leq r, \\
 wt_f(y_i^1) &= i + 3r + 1, & 1 \leq i \leq r, \\
 wt_f(u_i) &= i + 4r + 1, & 1 \leq i \leq 2r, \\
 wt_f(u_{2r+1}) &= 3(2r + 1) - 1 = 6r + 2, \\
 wt_f(u_{r+2}) &= 3(2r + 1), \\
 wt_f(v_i) &= i + 2(2r + 1) + 3r + 1 = i + 7r + 3, & 1 \leq i \leq r - 1, \\
 wt_f(x_i) &= i + 2(2r + 1) + 2r + 1 = i + 6r + 3, & 1 \leq i \leq r.
 \end{aligned}$$

We observe that each vertex has a distinct weight. Also, vertex and edge labels used are less than or equal to $2r + 1$. Hence, $tvs(T_r(5, 4)) \leq \left\lceil \frac{6r+3}{3} \right\rceil = 2r + 1$. Combining with the lower bound, we have $tvs(T_r(5, 4)) = 2r + 1$.

Case 2. For $n = 7$.

We construct a function $f : V \cup E \rightarrow \{1, 2, \dots, k\}$ which is a total k -labeling with $k = \left\lceil \frac{9r + 3}{3} \right\rceil = 3r + 1$. We establish labels of vertices and edges as in Table 3.

TABLE 3. Labels of vertices and edges of $T_r(5, 7)$

Labels of vertices	Labels of edges
$f(u_{2i-1}) = i + 2r - 2, 1 \leq i \leq r,$	$f(u_{2i-1}u_{2i}) = i + 2r + 1, 1 \leq i \leq r,$
$f(u_{2i}) = (3r + 1) - 2 = 3r - 1, 1 \leq i \leq r,$	$f(u_1u_{2r+1}) = 3r + 1,$
$f(u_{2r+1}) = 3r + 1 - 1 = 3r,$	$f(u_{2r}u_{2r+2}) = 3r + 1,$
$f(u_{2r+2}) = 3r + 1,$	$f(u_{2i}v_i) = i + 2r + 1, 1 \leq i \leq r - 1,$
$f(v_i) = 1, 1 \leq i \leq r - 1,$	$f(u_{2i+1}v_i) = 3r + 1, 1 \leq i \leq r - 1,$
$f(x_i) = 3r + 1, 1 \leq i \leq r,$	$f(v_i x_i) = 2r - 1, 1 \leq i \leq r - 1,$
$f(y_i^1) = 2r - 1, 1 \leq i \leq r - 1,$	$f(v_i x_{i+1}) = 3r + 1, 1 \leq i \leq r - 1,$
$f(y_r^1) = 3r + 1,$	
$f(y_i^2) = 2r + 1, 1 \leq i \leq r,$	$f(u_{2r+1}x_1) = 3r + 1,$
$f(y_i^3) = 3r + 1, 1 \leq i \leq r,$	$f(u_{2r+2}x_r) = 3r + 1,$
$f(y_i^4) = 2r + 1, 1 \leq i \leq r,$	$f(x_i y_i^1) = i + r + 2, 1 \leq i \leq r - 1,$
$f(y_i^5) = 2r, 1 \leq i \leq r,$	$f(x_r y_r^1) = r; f(y_i^1 y_i^2) = 3r, 1 \leq i \leq r,$
$f(y_i^6) = r, 1 \leq i \leq r,$	$f(y_i^2 y_i^3) = i,$
$f(y_i^7) = 1, 1 \leq i \leq r.$	$f(y_i^3 y_i^4) = r,$
	$f(y_i^4 y_i^5) = i, f(y_i^5 y_i^6) = 1,$
	$f(y_i^6 y_i^7) = i, 1 \leq i \leq r$

By means of labeling f , we have the weights of vertices as follows:

$$\begin{aligned} wt_f(u_i) &= i + 7r + 1, & 1 \leq i \leq 2r, \\ wt_f(u_{2r+1}) &= 3(3r + 1) - 1, \\ wt_f(u_{r+2}) &= 3(3r + 1), \\ wt_f(v_i) &= i + 2(3r + 1) + 4r + 1 = i + 10r + 3, & 1 \leq i \leq r - 1, \\ wt_f(x_i) &= i + 2(3r + 1) + 3r + 1 = i + 9r + 3, & 1 \leq i \leq r, \\ wt_f(y_i^l) &= i + (7 - l)r + 1, & 1 \leq i \leq r; 1 \leq l \leq 7. \end{aligned}$$

The vertex weights are all distinct and the labels used are at most $\left\lceil \frac{9r+3}{3} \right\rceil = 3r + 1$. Thus, $tvs(T_r(5, 7)) = 3r + 1$.

Case 3. For $n = 3j + 10$ with $j \geq 0$.

We construct a total k -labeling $f : V \cup E \rightarrow \{1, 2, \dots, k\}$ with $k = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$. We

create vertex labels in the following:

$$\begin{aligned} f(u_{2i-1}) &= i + \left(\frac{n-1}{3}\right)r - 2, & 1 \leq i \leq r, \\ f(u_{2i}) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil - 2, & 1 \leq i \leq r, \\ f(u_{2r+1}) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil - 1, \\ f(u_{2r+2}) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \\ f(v_i) &= 1, & 1 \leq i \leq r - 1, \\ f(x_i) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil, & 1 \leq i \leq r - 1, \\ f(x_r) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil - \left(\left(\frac{n-7}{3}\right)r - 1\right), \\ f(y_i^1) &= \left(\frac{n-1}{3}\right)r - 2, & 1 \leq i \leq r - 1, \\ f(y_r^1) &= \left(\frac{n+2}{3}\right)r + 1, \\ f(y_i^2) &= f(y_i^3) = \dots = f(y_i^{2+2j-1}) = \left(\frac{n+2}{3}\right)r + 1, & 1 \leq i \leq r; j \geq 1, \\ f(y_i^{2+2j}) &= \left(\frac{n+2}{3}\right)r, & 1 \leq i \leq r; j \geq 0, \\ f(y_i^{2+2j+1}) &= \left(\frac{n+2}{3}\right)r + 1, & 1 \leq i \leq r; j \geq 0, \\ f(y_i^{2+2j+2}) &= \left(\frac{n-1}{3}\right)r + 1, & 1 \leq i \leq r; j \geq 0, \\ f(y_i^{2+2j+3}) &= \left(\frac{n+2}{3}\right)r + 1, & 1 \leq i \leq r; j \geq 0, \\ f(y_i^{2+2j+4}) &= \left(\frac{n-1}{3}\right)r + 1, & 1 \leq i \leq r; j \geq 0, \\ f(y_i^{2+2j+5}) &= \left(\frac{n-1}{3}\right)r, & 1 \leq i \leq r; j \geq 0, \\ f(y_i^{2+2j+l}) &= \left(\frac{n-1}{3} - (l-5)\right)r, & 1 \leq i \leq r; 6 \leq l \leq \left(\frac{n+2}{3} + 2\right); j \geq 0, \\ f(y_i^{2+2j+l}) &= r, & 1 \leq i \leq r; l = \frac{n+2}{3} + 3; j \geq 0, \\ f(y_i^n) &= i, & 1 \leq i \leq r; n \text{ is even}, \\ f(y_i^n) &= 1, & 1 \leq i \leq r; n \text{ is odd}. \end{aligned}$$

Meanwhile, each edge label is constructed below:

$$\begin{aligned}
 f(u_{2i-1}u_{2i}) &= i + \left(\frac{n-1}{3}\right)r + 1, & 1 \leq i \leq r, \\
 f(u_1u_{2r+1}) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil, f(u_{2r}u_{2r+2}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \\
 f(u_{2i}v_i) &= i + \left(\frac{n-1}{3}\right)r + 1, & 1 \leq i \leq r-1, \\
 f(u_{2i+1}v_i) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil, & 1 \leq i \leq r-1, \\
 f(v_i x_i) &= 2r - 1, & 1 \leq i \leq r-1, \\
 f(v_i x_{i+1}) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil, & 1 \leq i \leq r-1, \\
 f(u_{2r+1}x_1) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil; \\
 f(u_{2r+2}x_r) &= \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \\
 f(x_i y_i^1) &= i + \left(\frac{n-4}{3}\right)r + 2, & 1 \leq i \leq r-1, \\
 f(x_r y_r^1) &= \left(\frac{n-4}{3}\right)r - 1, \\
 f(y_i^{2l-1} y_i^{2l}) &= \left(\frac{n+2}{3} - (l-1)\right)r + 1, & 1 \leq i \leq r; 1 \leq l \leq j+1; j \geq 0, \\
 f(y_i^{2l} y_i^{2l+1}) &= i + \left(\frac{n-4}{3} - (l+1)\right)r - 1, & 1 \leq i \leq r; 1 \leq l \leq j; j \geq 1, \\
 f(y_i^{2j+2} y_i^{2j+3}) &= i; f(y_i^{2j+3} y_i^{2j+4}) = 3r, & 1 \leq i \leq r; j \geq 0, \\
 f(y_i^{2j+4} y_i^{2j+5}) &= i; f(y_i^{2j+5} y_i^{2j+6}) = r, & 1 \leq i \leq r; j \geq 0, \\
 f(y_i^{2j+6} y_i^{2j+7}) &= i; f(y_i^{2j+7} y_i^{2j+8}) = 1, & 1 \leq i \leq r; j \geq 0, \\
 f(y_i^{2j+8} y_i^{2j+9}) &= i; f(y_i^{2j+9} y_i^{2j+10}) = 1, & 1 \leq i \leq r; j \geq 0, \\
 \dots, \\
 f(y_i^{n-1} y_i^n) &= 1, & 1 \leq i \leq r; n \text{ is even,} \\
 f(y_i^{n-1} y_i^n) &= i, & 1 \leq i \leq r; n \text{ is odd.}
 \end{aligned}$$

Under labeling f , we derive the vertex weights in the following:

$$\begin{aligned}
 wt_f(u_i) &= i + nr + 1, & 1 \leq i \leq 2r, \\
 wt_f(u_{2r+1}) &= 3 \left\lceil \frac{(n+2)r+3}{3} \right\rceil - 1, \\
 wt_f(u_{r+2}) &= 3 \left\lceil \frac{(n+2)r+3}{3} \right\rceil, \\
 wt_f(v_i) &= i + 2 \left\lceil \frac{(n+2)r+3}{3} \right\rceil + \left(\frac{n+5}{3}\right)r + 1, & 1 \leq i \leq r-1, \\
 wt_f(x_i) &= i + 2 \left\lceil \frac{(n+2)r+3}{3} \right\rceil + \left(\frac{n+2}{3}\right)r + 1, & 1 \leq i \leq r, \\
 wt_f(y_i^l) &= i + (n-l)r + 1, & 1 \leq i \leq r; 1 \leq l \leq n.
 \end{aligned}$$

It is obvious that each vertex has a different weight and the labels used are not more than $\left\lceil \frac{(n+2)r+3}{3} \right\rceil$. Therefore, $tvs(T_r(5, n)) \leq \left\lceil \frac{(n+2)r+3}{3} \right\rceil$. Combining with Lower bound

(3), we get $tvs(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$.

□

Figure 1 describes the pattern of vertex and edge labels to get $tvs(T_5(5, 4)) = 11$.

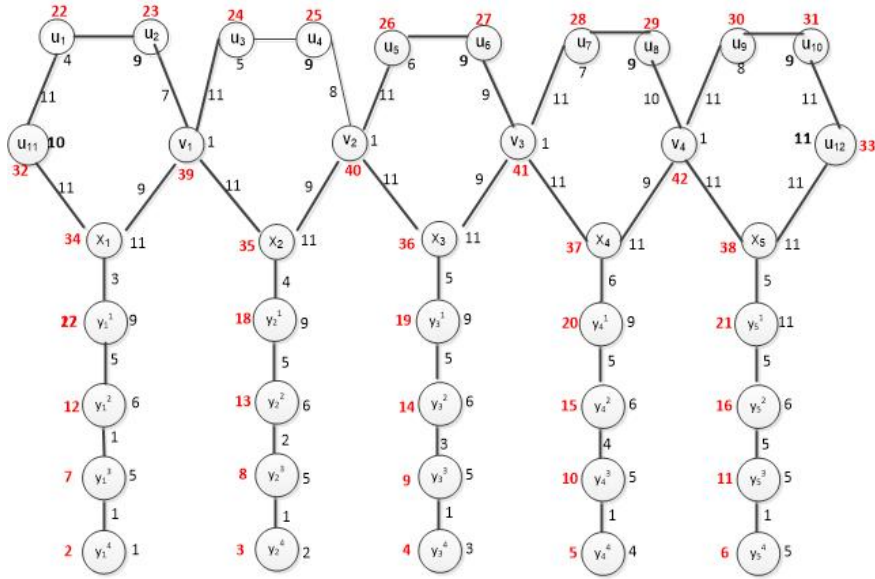


FIGURE 1. The pattern of vertex irregular total 11-labeling of $T_5(5, 4)$

2.2. **Total vertex irregularity strength of $T_r(4, n)$.** The graph $T_r(4, n)$ contains:

- a set of vertices with degrees 1, i.e. $\{y_1^n, y_2^n, \dots, y_r^n\}$;
- a set of vertices with degrees 2, i.e. $\{y_1^{n-1}, y_2^{n-1}, \dots, y_r^{n-1}, y_1^{n-2}, y_2^{n-2}, \dots, y_r^{n-2}, \dots, y_1^2, y_2^2, \dots, y_r^2, y_1^1, y_2^1, \dots, y_r^1\} \cup \{u_1, u_2, \dots, u_r, u_{r+1}, u_{r+2}\}$;
- a set of vertices with degrees 3, i.e. $\{x_1, x_2, \dots, x_r\}$; and
- a set of vertices with degrees 14, i.e. $\{v_1, v_2, \dots, v_{r-1}\}$.

Lemma 2.1. *If $T_r(4, 2)$ is a tadpole chain graph with length r ($r \geq 2$), then*

$$tvs(T_r(4, 2)) = r + 1.$$

Proof. Based on (1), we get the lower bound as follows:

$$tvs(T_r(4, 2)) \geq \max \left\{ \left\lceil \frac{r+1}{2} \right\rceil, \left\lceil \frac{3r+3}{3} \right\rceil, \left\lceil \frac{4r+3}{4} \right\rceil, \left\lceil \frac{5r+2}{5} \right\rceil \right\} = \left\lceil \frac{3r+3}{3} \right\rceil = r + 1. \quad (4)$$

To establish that $tvs(T_r(4, 2)) \leq r + 1$, we define a function f from $V \cup E$ into a set of integers $\{1, 2, \dots, k\}$ which is a total k -labeling with $k = \left\lceil \frac{3r+3}{3} \right\rceil = r + 1$.

We construct labels of vertices and edges as in Table 4.

We have the weights of vertices under the labeling f as follows:

$$\begin{aligned} wt_f(y_i^2) &= i + 1, & 1 \leq i \leq r, \\ wt_f(y_i^1) &= i + r + 1, & 1 \leq i \leq r, \\ wt_f(u_i) &= i + 2r + 1, & 1 \leq i \leq r + 2, \\ wt_f(v_i) &= i + 3(r + 1) + r, & 1 \leq i \leq r - 1, \\ wt_f(x_i) &= i + 3(r + 1), & 1 \leq i \leq r. \end{aligned}$$

TABLE 4. Labels of vertices and edges on $T_r(4, 2)$

Labels of vertices	Labels of edges
$f(u_i) = r - 1, 1 \leq i \leq r,$	$f(u_{r+1}u_1) = r + 1,$
$f(u_{r+1}) = r,$	$f(u_{r+2}u_r) = r + 1,$
$f(u_{r+2}) = r + 1,$	$f(u_i v_i) = i + 1, 1 \leq i \leq r - 1,$
$\Gamma f(v_i) = r - 1, 1 \leq i \leq r - 1,$	$f(u_{i+1}v_i) = r + 1, 1 \leq i \leq r - 1,$
$f(x_i) = r + 1, 1 \leq i \leq r,$	$f(v_i x_i) = r + 1; f(v_i x_{i+1}) = r + 1, 1 \leq i \leq r - 1$
$f(y_i^1) = r, 1 \leq i \leq r,$	$f(u_{r+1}x_1) = r + 1; f(u_{r+2}x_r) = r + 1,$
$f(y_i^2) = i, 1 \leq i \leq r,$	$f(x_i y_i^1) = i; f(y_i^1 y_i^2) = 1, 1 \leq i \leq r.$

We observe that each vertex has distinct label. Also, vertex and edge labels are less than or equal to $\left\lceil \frac{3r+3}{3} \right\rceil = r + 1$. Hence, we get upper bound $tvs(T_r(4, 2)) \leq r + 1$. Thus, $tvs(T_r(4, 2)) = r + 1$. \square

Theorem 2.3. *If $T_r(4, n)$ are tadpole chain graphs of length r where $r \geq 2, n = 5 \pmod{3}$, and $n \geq 5$, then $tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$.*

Proof. According to (1), we obtain the lower bound as follows:

$$tvs(T_r(4, n)) \geq \max \left\{ \left\lceil \frac{r+1}{2} \right\rceil, \left\lceil \frac{(n+1)r+3}{3} \right\rceil, \left\lceil \frac{(n+2)r+3}{4} \right\rceil, \left\lceil \frac{(n+3)r+2}{5} \right\rceil \right\} = \left\lceil \frac{(n+1)r+3}{3} \right\rceil. \quad (5)$$

We create a total k -labeling f from $V \cup E$ into $\{1, 2, \dots, k\}$ with $k = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$. Further, we prove the upper bound by means of 2 cases.

Case 1. For $n = 5$ and $n = 8$.

Firstly, we define Labels of vertices as follows:

$$f(u_i) = \begin{cases} 2r - 1, & 1 \leq i \leq r, n = 5, \\ 3r - 1, & 1 \leq i \leq r, n = 8. \end{cases}$$

$$f(u_{r+1}) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil - 1, \quad n = 5, 8$$

$$f(u_{r+2}) = \left\lceil \frac{3r+3}{3} \right\rceil, \quad n = 5, 8$$

$$f(v_i) = 1, \quad 1 \leq i \leq r - 1, n = 5, 8$$

$$f(x_i) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil, \quad 1 \leq i \leq r, n = 5, 8,$$

$$f(y_i^1) = \begin{cases} 2r - 1, & 1 \leq i \leq r - 1, n = 5, \\ 2r - 2, & 1 \leq i \leq r - 1, n = 8. \end{cases}$$

$$f(y_r^1) = \begin{cases} 2r, & n = 5, \\ 3r, & n = 8. \end{cases}$$

$$f(y_i^2) = \begin{cases} r, & 1 \leq i \leq r, n = 5, \\ 3r, & 1 \leq i \leq r, n = 8. \end{cases}$$

$$\begin{aligned}
 f(y_i^3) &= \begin{cases} 2r, & 1 \leq i \leq r, n = 5, \\ 3r + 1, & 1 \leq i \leq r, n = 8. \end{cases} \\
 f(y_i^4) &= \begin{cases} r, & 1 \leq i \leq r, n = 5, \\ 2r + 1, & 1 \leq i \leq r, n = 8. \end{cases} \\
 f(y_i^5) &= \begin{cases} 1, & 1 \leq i \leq r, n = 5, \\ 3r, & 1 \leq i \leq r, n = 8. \end{cases} \\
 f(y_i^6) &= 2r; f(y_i^7) = r; f(y_i^8) = i, \quad 1 \leq i \leq r, n = 8.
 \end{aligned}$$

Secondly, we construct labels of edges as follows:

$$\begin{aligned}
 f(u_{r+1}u_1) &= \left\lceil \frac{(n+1)r+3}{3} \right\rceil; f(u_{r+2}u_r) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil, \quad n = 5, 8 \\
 f(u_{i+1}v_i) &= \left\lceil \frac{(n+1)r+3}{3} \right\rceil, 1 \leq i \leq r - 1, \quad n = 5, 8, \\
 f(v_i x_i) &= 2r - 1, 1 \leq i \leq r - 1, \quad n = 5, 8 \\
 f(v_i x_{i+1}) &= \left\lceil \frac{(n+1)r+3}{3} \right\rceil, 1 \leq i \leq r - 1, \quad n = 5, 8, \\
 f(u_{r+1}x_1) &= \left\lceil \frac{(n+1)r+3}{3} \right\rceil; f(u_{r+2}x_r) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil, \\
 f(u_i v_i) &= \begin{cases} i + r + 1, & 1 \leq i \leq r - 1, n = 5, \\ i + 2r + 1, & 1 \leq i \leq r - 1, n = 8. \end{cases} \\
 f(x_i y_i^1) &= \begin{cases} i + 2, & 1 \leq i \leq r - 1, n = 5, \\ i + 2r + 2, & 1 \leq i \leq r - 1, n = 8. \end{cases} \\
 f(x_r y_r^1) &= \begin{cases} r, & n = 5, \\ 2r, & n = 8. \end{cases} \\
 f(y_i^1 y_i^2) &= \begin{cases} 2r + 1, & 1 \leq i \leq r, n = 5, \\ 3r + 2, & 1 \leq i \leq r, n = 8. \end{cases} \\
 f(y_i^2 y_i^3) &= i; f(y_i^4 y_i^5) = i, \quad 1 \leq i \leq r, n = 5, 8, \\
 f(y_i^3 y_i^4) &= \begin{cases} 1, & 1 \leq i \leq r, n = 5, \\ 2r, & 1 \leq i \leq r, n = 8. \end{cases} \\
 f(y_i^5 y_i^6) &= 1; f(y_i^6 y_i^7) = i; f(y_i^7 y_i^8) = 1, \quad 1 \leq i \leq r, n = 8.
 \end{aligned}$$

We have the vertex weights below:

$$\begin{aligned}
 wt_f(u_i) &= \begin{cases} i + 2 \left\lceil \frac{(n+1)r+3}{3} \right\rceil + (r - 1), & 1 \leq i \leq r + 2, n = 5, \\ i + 2 \left\lceil \frac{(n+1)r+3}{3} \right\rceil + (2r - 1), & 1 \leq i \leq r + 2, n = 8. \end{cases} \\
 wt_f(v_i) &= \begin{cases} i + 2 \left\lceil \frac{(n+1)r+3}{3} \right\rceil + (3r + 1), & 1 \leq i \leq r - 1, n = 5, \\ i + 2 \left\lceil \frac{(n+1)r+3}{3} \right\rceil + (4r + 1), & 1 \leq i \leq r - 1, n = 8. \end{cases} \\
 wt_f(x_i) &= \begin{cases} i + 2 \left\lceil \frac{(n+1)r+3}{3} \right\rceil + (2r + 1), & 1 \leq i \leq r, n = 5, \\ i + 2 \left\lceil \frac{(n+1)r+3}{3} \right\rceil + (3r + 1), & 1 \leq i \leq r, n = 8. \end{cases}
 \end{aligned}$$

$$wt_f(y_i^l) = i + (n - l)r + 1, \quad 1 \leq i \leq r, 1 \leq l \leq n, n = 5, 8.$$

It is obvious that each vertex has a different weight. Also, the labels are not more than $\left\lceil \frac{(n+1)r+3}{3} \right\rceil$. Hence, we establish the upper bound $tvs(T_r(4, n)) \leq \left\lceil \frac{(n+1)r+3}{3} \right\rceil$. According to Lower bound (5), we have $tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$.

Case 2. For $n \geq 11$.

Analog to the proof of Theorem 2.2 in Case 3, we get $tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$. □

Figure 2 shows the pattern of vertex and edge labels to obtain $tvs(T_5(4, 5)) = 11$.

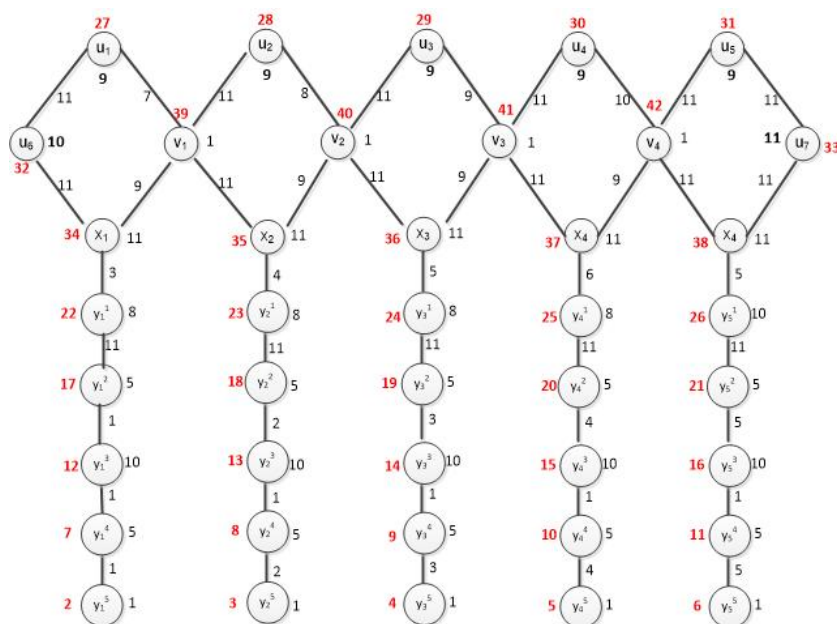


FIGURE 2. The pattern of vertex irregular total 11-labeling of $T_5(4, 5)$

3. CONCLUSIONS

We have verified tvs of $T_r(4, n)$ and $T_r(5, n)$ in this paper. The patterns to get tvs of the graphs were presented in the theorems. We have proved the upper bounds and got $tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$ and $tvs(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$. In further research, we are going to investigate tvs of general tadpole chain graph $T_r(m, n)$ for $m \geq 6$. Also, we will verify the formulas using algorithmic approaches.

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