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EDGE-VERTEX DOMINATION AND TOTAL EDGE DOMINATION IN TREES

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ABSTRACT. An edge $e \in E(G)$ dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. An edge-vertex dominating set of a graph G is a set D of edges of G such that every vertex of G is edge-vertex dominated by an edge of D. The edge-vertex domination number of a graph G is the minimum cardinality of an edge-vertex dominating set of G. A subset $D \subseteq E(G)$ is a total edge dominating set of G if every edge of G has a neighbor in D. The total edge domination number of G is the minimum cardinality of a total edge dominating set of G. We characterize all trees with total edge domination number equal to edge-vertex domination number.

Keywords: Edge-vertex domination, Total Edge Domination, Tree.
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1. INTRODUCTION

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The edge incident with a leaf is called an end edge. The path on n vertices we denote by P_n. Let T be a tree, and let v be a vertex of T. We say that v is adjacent to a path P_n if there is a neighbor of v, say x, such that one of the components of $T - vx$ is a path P_n containing x as a leaf. By a star we mean a connected graph in which exactly one vertex has degree greater than one called its center. Let uv be an edge of a graph G. By subdividing the edge uv we mean removing it, and adding a new vertex, say x, along with two new edges ux and xv. Subdivided star, SS_k is a graph obtained from a star, $K_{1,r}$ by subdividing each one of its edges.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A subset $D \subseteq E(G)$ is a total edge dominating set, abbreviated TEDS, of G if every edge of G has a neighbor in D. The total edge domination number of

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a graph G, denoted by $\gamma_t^{'}(G)$, is the minimum cardinality of a total edge dominating set of G. For a comprehensive survey of domination in graphs, see [1].

An edge $e \in E(G)$ dominates a vertex $v \in V(G)$ if e is incident with v or e is incident with a vertex adjacent to v. A subset $D \subseteq E(G)$ is an edge-vertex dominating set, abbreviated EVDS, of a graph G if every vertex of G is edge-vertex dominated by an edge of D. The edge-vertex domination number of a graph G, denoted by $\gamma_{ev}(G)$, is the minimum cardinality of an edge-vertex dominating set of G. Edge-vertex domination in graphs was introduced in [4], and further studied in [2, 3, 5].

Trees with equal total domination number equal to edge-vertex domination number plus one were characterized in [2]. We characterize all trees with total edge domination number equal to edge-vertex domination number.

2. Results

We begin this section by proving that for any graph G, edge-vertex domination number is less than or equal to total edge domination number. Since the one-vertex graph does not have a total edge dominating set or an edge-vertex dominating set, we consider graphs with at least two vertices.

Proposition 2.1. *For any graph G, $\gamma_{ev}(G) \leq \gamma_t^{'}(G)$.*

Proof. Let D be a $\gamma_t^{'}(G)$-set. For every edge $e \in E(G)$ there exist an edge $f \in D$ such that e and f are adjacent. Every vertex incident with every edge is dominated by an edge in D. Hence, D is an EVDS of the graph G. Thus $\gamma_{ev}(G) \leq \gamma_t^{'}(G)$. □

We now characterize all trees with equal edge-vertex domination number and total edge domination number. For the purpose we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a subdivided star $SS_k (k \geq 2)$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a vertex to a vertex of T_k not a leaf and adjacent to a support vertex.
- Operation $\mathcal{O}_3$: Attach a center of a subdivided star $SS_k (k \geq 2)$ to a vertex not a leaf of T_k.
- Operation $\mathcal{O}_4$: Attach a path P_5 by joining its support vertex to a vertex of T_k adjacent to P_5 through its support vertex.
- Operation $\mathcal{O}_5$: Attach a path P_5 by joining its support vertex to a vertex of T_k adjacent to a path P_2.

Now we prove that for every tree of the family $\mathcal{T}$, the total edge domination number is equal to the $ev-$domination number.

Theorem 2.1. *If $T \in \mathcal{T}$, then $\gamma_{ev}(T) = \gamma_t^{'}(T)$.*

Proof. We use the induction on the number k of operations performed to construct tree T. If $T = SS_k(k \geq 2)$, then obviously $\gamma_{ev}(T) = k = \gamma_t^{'}(T)$. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let y be the vertex joined to a support vertex x. Let z be a leaf adjacent to x other than y. Let D be a $\gamma_{ev}(T)$-set. To dominate the leaves y and z, the edge incident with x which is not xz and xy is in D. Obviously D is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \leq \gamma_{ev}(T)$. Let D' be a $\gamma_{ev}(T')$-set. It is obvious that D' is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. This implies that $\gamma_{ev}(T) = \gamma_{ev}(T')$. Let S' be a $\gamma_t^{'}(T')$-set. The edge which dominates z

also dominates y. Hence S' is a TEDS of the tree T. Thus $\gamma_t^{'}(T) \leq \gamma_t^{'}(T')$. Let S be a $\gamma_t^{'}(T)$-set. Obviously S is an TEDS of the tree T'. This implies that $\gamma_t^{'}(T) = \gamma_t^{'}(T')$. We now get $\gamma_{ev}(T) = \gamma_{ev}(T') = \gamma_t^{'}(T') = \gamma_t^{'}(T)$.

Assume that T is obtained from T' by operation $\mathcal{O}_2$. The vertex to which a vertex is attached we denote by x. Let y be the attached vertex. Let α be the support vertex adjacent to x. Let β the leaf adjacent to α. Let D be a $\gamma_{ev}(T)$-set. To dominate β, the edge $x\alpha \in D$. It is easy to observe that D is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \leq \gamma_{ev}(T)$. Let D' be a $\gamma_{ev}(T')$-set. To dominate β, the edge $x\alpha \in D$. The edge $x\alpha$ dominates y in the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. This implies that $\gamma_{ev}(T) = \gamma_{ev}(T')$. Let S be a $\gamma_t^{'}(T)$-set. To dominate the edge $x\alpha$, the edge incident with x other than $x\alpha$ and xy belongs to S. It is easy to observe that S is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T)$. Let S' be a $\gamma_t^{'}(T')$-set. To dominate the edge $\alpha\beta$, the edge $x\alpha \in S'$. To dominate $x\alpha$, the edge incident with x other than $x\alpha$ belongs to S'. This obvious that S' is a TEDS of the tree T. Thus $\gamma_t^{'}(T) \leq \gamma_t^{'}(T')$. This implies that $\gamma_t^{'}(T) = \gamma_t^{'}(T')$. Now we get $\gamma_{ev}(T) = \gamma_{ev}(T') = \gamma_t^{'}(T') = \gamma_t^{'}(T)$.

Assume that tree T is obtained from T' by operation $\mathcal{O}_3$. The vertex to which a subdivided star $SS_k(k \geq 2)$ is attached we denote by x. Let α be the center of the star. Let $u_{11}, u_{21}, \ldots, u_{k1}$ be the support vertices of the subdivided star. Let $u_{12}, u_{22}, \ldots, u_{k2}$ be the leaf adjacent to $u_{11}, u_{21}, \ldots, u_{k1}$ respectively. Let α be adjacent to x. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{u_{11}\alpha, u_{21}\alpha, \ldots, u_{k1}\alpha\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + k$. Let D be a $\gamma_{ev}(T)$-set. To dominate the vertices $u_{12}, u_{22}, \ldots, u_{k2}$, the edges $u_{11}\alpha, u_{21}\alpha, \ldots, u_{k1}\alpha$ belongs to D. It is easy to observe that $D \setminus \{u_{11}\alpha, u_{21}\alpha, \ldots, u_{k1}\alpha\}$ is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \leq \gamma_{ev}(T) - k$. This implies that $\gamma_{ev}(T) = \gamma_{ev}(T') + k$. Let D' be a $\gamma_t^{'}(T')$-set. The set $D' \cup \{u_{11}\alpha, u_{21}\alpha, \cdots, u_{k1}\alpha\}$ is a TEDS of the tree T. Thus $\gamma_t^{'}(T) \leq \gamma_t^{'}(T') + k$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges $u_{11}u_{12}, u_{21}u_{22}, \ldots, u_{k1}u_{k2}$ the edges $u_{11}\alpha, u_{21}\alpha, \ldots, u_{k1}\alpha$ belongs to D. It is obvious that $D \setminus \{u_{11}\alpha, u_{21}\alpha, \ldots, u_{k1}\alpha\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - k$. This implies that $\gamma_t^{'}(T) = \gamma_t^{'}(T') + k$. We now get $\gamma_{ev}(T) = \gamma_{ev}(T') + k = \gamma_t^{'}(T') + k = \gamma_t^{'}(T)$.

Assume that tree T is obtained from T' by operation $\mathcal{O}_4$. The vertex to which a support vertex of a path P_5 is attached we denote by x. Let $u_1u_2u_3u_4u_5$ be the attached path. Let u_2 be adjacent to x. Let $v_1v_2v_3v_4v_5$ be a path different from $u_1u_2u_3u_4u_5$ adjacent to x. Let x and v_2 be adjacent. Let D' be a $\gamma_{ev}(T')$-set. It is easy to observe that $D' \cup \{u_2u_3, u_3u_4\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Let S be a $\gamma_{ev}(T)$-set. To dominate the vertices u_5, u_1, v_5 and v_1 the edges $u_3u_4, xu_2, v_3v_4, xv_2 \in S$. It is obvious that $S \setminus \{u_3u_4, xu_2\}$ is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 2$. This implies that $\gamma_{ev}(T) = \gamma_{ev}(T') + 2$. Let S be a $\gamma_t^{'}(T)$-set. To dominate edges u_4u_5, u_1u_2, v_4v_5 and v_1v_2 the edges $u_2u_3, u_3u_4, v_2v_3, v_3v_4 \in S$. It is obvious that $S \setminus \{u_2u_3, u_3u_4\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2$. Let S' be a $\gamma_t^{'}(T')$-set. It is obvious that $S \cup \{u_2u_3, u_3u_4\}$ is a TEDS of the tree T. Thus $\gamma_t^{'}(T) \leq \gamma_t^{'}(T') + 2$. This implies that $\gamma_t^{'}(T) = \gamma_t^{'}(T') + 2$. We now get $\gamma_{ev}(T) = \gamma_{ev}(T') + 2 = \gamma_t^{'}(T') + 2 = \gamma_t^{'}(T)$.

Assume that tree T is obtained from T' by operation $\mathcal{O}_5$. The vertex to which a support vertex of P_5 is attached we denote by x. Let $u_1u_2u_3u_4u_5$ be the attached path. Let u_2 be adjacent to x. Let v_1v_2 be a path adjacent to x. Let x and v_1 be adjacent. Let D be a $\gamma_{ev}(T)$-set. To dominate u_5, u_1 and v_2 the edges xu_2, u_3u_4 and xv_1 belongs to D. It is easy to observe that $D \setminus \{xu_2, u_3u_4\}$ is an EVDS of the tree T'. Thus $\gamma_{ev}(T') \leq \gamma_{ev}(T) - 2$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{xu_2, u_3u_4\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. This implies that $\gamma_{ev}(T) = \gamma_{ev}(T') + 2$. Let S be a $\gamma_t^{'}(T)$-set. To dominate the edges v_1v_2, u_1u_2 and u_4u_5 the edges $xv_1, u_2u_3, u_3u_4 \in S$. To dominate the edge xv_1, the edge incident with x other than xu_2 is in the set S. It is obvious that $S \setminus \{u_2u_3, u_3u_4\}$ is a TEDS of the set T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2$. Let S' be a $\gamma_t^{'}(T')$-set. It is clear that $S' \cup \{u_2u_3, u_3u_4\}$ is a TEDS of the tree T. Thus $\gamma_t^{'}(T) \leq \gamma_t^{'}(T') + 2$. This imples that $\gamma_t^{'}(T) = \gamma_t^{'}(T') + 2$. We now get $\gamma_{ev}(T) = \gamma_{ev}(T') + 2 = \gamma_t^{'}(T') + 2 = \gamma_t^{'}(T)$. □

Now we prove that if the total edge domination number of a tree is equal to edge-vertex domination number, then the tree belongs to the family $\mathcal{T}$.

Theorem 2.2. *Let T be a tree. If $\gamma_{ev}(T) = \gamma_t^{'}(T)$, then $T \in \mathcal{T}$.*

Proof. Let diam(T)=2, then T is a star. We get $\gamma_{ev}(T) = 1 < 2 = \gamma_t^{'}(T)$. Now assume diam(T) ≥ 3. Thus the order of the tree T is at least four. We prove the result by induction on n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. Let D be any $\gamma_{ev}(T')$-set. It is obvious that D is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. Let D be a $\gamma_t^{'}(T)$-set. It is clear that D is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T)$. This implies that $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) = \gamma_{ev}(T) \leq \gamma_{ev}(T')$. On the other hand $\gamma_t^{'}(T') \geq \gamma_{ev}(T')$. Thus we get $\gamma_t^{'}(T') = \gamma_{ev}(T')$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity diam(T). Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If diam(T) ≥ 4, then let w be the parent of u. If diam(T) ≥ 5, then let d be the parent of w. If diam(T) ≥ 6, then let e be the parent of d. By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - x$. Let D' be a $\gamma_{ev}(T')$-set. It is obvious that D' is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T')$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edge vt, the edge $uv \in D$. To dominate the edge uv, the edge $uw \in D$. It is clear that D is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T)$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) = \gamma_{ev}(T) \leq \gamma_{ev}(T')$. This implies that $\gamma_{ev}(T') = \gamma_t^{'}(T')$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Assume that some child of u, other than v, say x, is at a distance two from a vertex of T_k. Let y be the leaf adjacent to x. If $u = r$ and then $T = SS_k (k \geq 2)$. Thus $\gamma_{ev}(T) = k = \gamma_t^{'}(T)$, we have $T \in \mathcal{T}$. Assume that $u \neq r$. Let $T' = T - T_u$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup A$ where A is the set of edges incident with u other than uw is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + |A|$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges incident with the leaves, the support edges belongs to D. Obviously $A \subseteq D$. It is clear that $D \setminus A$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - |A|$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - |A| = \gamma_{ev}(T) - |A| \leq \gamma_{ev}(T')$. This implies that $\gamma_t^{'}(T') = \gamma_{ev}(T')$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained by T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume $d_T(u) = 2$. Assume that some child of w, other than u, say x is at a distance three from a vertex of T_k. Let y be adjacent to x and z be adjacent to y. Let $T' = T - T_u$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edge vt and yz the edges $uv, xy \in D$. To dominate uv, xy, the edge $wu, wx \in D$. It is easy to see that $D \setminus \{wu, vu\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T') - 2 + 1 < \gamma_{ev}(T')$.

Assume that some child of w, other than u, say x is at a distance two from a vertex of T_k. It suffices to consider the case that w is adjacent to path $P_2 = xy$. Let $T' = T - T_w$.

Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{wx, uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges vt, xy, the edges $uv, wx \in D$. To dominate the above two edges $uw \in D$. It is clear that $D \setminus \{uv, uw, wx\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 3 = \gamma_{ev}(T) - 3 \leq \gamma_{ev}(T') + 2 - 3 < \gamma_{ev}(T')$.

Assume that some child of w, other than u, say x, is a leaf. Now fix $d_T(w) = 3$. Now assume that some child of d, other than w, say x is at a distance four from a vertex of T_k. It suffices to consider the case T_k is isomorphic to T_w or T_k is $P_4 = abcd$. First assume that T_k is $P_4 = abcd$. Let $T' = T - T_a$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to observe that $D' \cup \{bc\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges vt and cd the edges $uv, bc \in D$. To dominate uv and bc, the edges $wu, ab \in D$. It is easy to see that $D \setminus \{ab, bc\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T') - 2 + 1 < \gamma_{ev}(T')$.

Now assume that T_k is isomorphic to T_w. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{wu, uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges vt and wx, the edges $uv, wu \in D$. It is easy to observe that $D \setminus \{uv, wu\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T') - 2 + 2 = \gamma_{ev}(T')$. This implies that $\gamma_{ev}(T') = \gamma_t^{'}(T')$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{T}$.

Assume that some child of d, other than w, say x is at a distance three from a vertex of T_k. Let d be adjacent to more than one P_3 paths. Let $u_1u_2u_3$ and $v_1v_2v_3$ be two paths adjacent to d. Let $T' = T - T_{u_1}$. Let D' be a $\gamma_{ev}(T')$-set. It is clear that $D' \cup \{u_1u_2\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges u_2u_3 and v_2v_3 the edges $u_1u_2, v_1v_2 \in D$. To dominate the edges u_2u_3 and v_2v_3, the edges $u_1d, v_1d \in D$. It is easy to observe that $D \setminus \{u_1d, u_2u_3\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T') + 1 - 2 < \gamma_{ev}(T')$, a contradiction. Hence the vertex d is adjacent to exactly one path P_3, say $v_1v_2v_3$. Let $T' = T - T_d$. Let D' be a $\gamma_{ev}(T')$-set. It is clear that $D' \cup \{v_1v_2, wu, uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 3$. Let D be an $\gamma_t^{'}(T)$-set. To dominate the edges u_2u_3, vt and wx, the edges $u_1u_2, vu, uw \in D$. To dominate the edge u_1u_2, the edge $du_1 \in D$. It is obvious that $D \setminus \{u_1u_2, vu, vw, du_1\}$ is a TEDS of the tree T'. Thus $\gamma_t^{(}T') \leq \gamma_t^{'}(T) - 4$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 4 = \gamma_{ev}(T) - 4 \leq \gamma_{ev}(T') + 3 - 4 < \gamma_{ev}(T')$.

Assume that some child of d, other than w, say a, is at a distance two from a vertex of T_k. Let b be the vertex adjacent to a. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv, wd\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges vt and wx, the edges $uv, uw \in D$. It is easy to observe that $D \setminus \{uv, uw\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 2 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T')$. This implies that $\gamma_{ev}(T') = \gamma_t^{'}(T')$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree is obtained from T' by operation $\mathcal{O}_5$. Thus $T \in \mathcal{T}$.

Assume that some child of d, other than w, say a, is a leaf. Let $T' = T - T_d$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to observe that $D' \cup \{dw, uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Let D be a $\gamma_t^{'}(T)$-set. To dominate the edges vt, wx and da, the edges $uv, uw, wd \in D$. It is easy to see that $D \setminus \{uv, uw, wd\}$ is a TEDS of the tree T'. Thus $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 3$. We now get $\gamma_t^{'}(T') \leq \gamma_t^{'}(T) - 3 = \gamma_{ev}(T) - 3 \leq \gamma_{ev}(T') - 3 + 2 < \gamma_{ev}(T')$.

Now assume that $d_T(d) = 2$. Let $T' = T - T_d$. Let D' be a $\gamma_{ev}(T')$-set. It is obvious to see that $D' \cup \{wu, uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 2$. Let D be an $\gamma_t^{'}(T)$-set. To dominate the edges vt, wx and ed, the edges $dw, wu, uv \in D$. It is clear that $D \setminus \{dw, uw, uv\}$ is a TEDS of the tree T'. Thus $\gamma_t'(T') \leq \gamma_t'(T) - 3$. We now get $\gamma_t'(T') \leq \gamma_t'(T) - 3 = \gamma_{ev}(T) - 3 \leq \gamma_{ev}(T') + 2 - 3 < \gamma_{ev}(T')$.

Now assume that $d_T(w) = 2$. Let $d_T(d) \geq 2$. Let $T' = T - T_w$. Let D' be a $\gamma_{ev}(T')$-set. It is easy to see that $D' \cup \{uv\}$ is an EVDS of the tree T. Thus $\gamma_{ev}(T) \leq \gamma_{ev}(T') + 1$. Let D be a $\gamma_t'(T)$-set. To dominate vt, the edge $uv \in D$. To dominate uv, the edge $wu \in D$. It is easy to see that $D \setminus \{wu, uv\}$ is a TEDS of the tree T'. Thus $\gamma_t'(T') \leq \gamma_t'(T) - 2$. We now get $\gamma_t'(T') \leq \gamma_t'(T) - 2 = \gamma_{ev}(T) - 2 \leq \gamma_{ev}(T') + 1 - 2 < \gamma_{ev}(T')$. □

As an immediate consequence of Theorems 2.1 and 2.2, we have the following characterization of trees with total edge domination number equal to edge-vertex domination number.

Theorem 2.3. *Let T be a tree. Then $\gamma_{ev}(T) = \gamma_t'(T)$ if and only if $T \in \mathcal{T}$.*

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