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VERTEX COLORING EDGE WEIGHTINGS OF SOME SQUARE GRAPHS

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Abstract. A k-edge-weighting w of a graph G is an assignment of integer weight, $w(e) \in \{1, 2, \ldots, k\}$, to each edge e. A k-edge-weighting w induces a vertex coloring c by defining $c(u) = \sum_{u \sim e} w(e)$ for every $u \in V(G)$, where $u \sim e$ denote that u is an end-vertex of e. A k-edge-weighting w of a graph G is a vertex coloring of G if the induced coloring c is proper, i.e., $c(u) \neq c(v)$ for any edge $uv \in E(G)$. In this paper, vertex coloring edge weighting of square of Cartesian product of paths is considered.

Keywords: edge weighting, vertex coloring, Cartesian product
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1. Introduction

For graph-theoretical terminology and notation, we in general follow [1]. In this paper, we assume that the graphs G in discussion are finite, connected, undirected and simple with order $|V(G)| \geq 3$. For a vertex v of a graph $G = (V, E)$, $N_G(v)$ denotes the set of vertices which are adjacent to v in G. For $v \in V(G)$ and $e \in E(G)$, $v \sim e$ denote that v is an end-vertex of e. A *k-vertex coloring* c of G is an assignment of k integers, $1, 2, \ldots, k$, to the vertices of G, the color of a vertex v is denoted by $c(v)$. The coloring is *proper* if no two distinct adjacent vertices share the same color. A graph G is k-colorable if G has a proper k-vertex coloring. The chromatic number $\chi(G)$ is the minimum number r such that G is r-colorable.

A k-edge-weighting w of a graph G is an assignment of an integer weight $w(e) \in \{1, 2, \ldots, k\}$ to each edge e of G. An edge weighting induces a vertex coloring by defining $c(u) = \sum_{u \sim e} w(e)$ for every $u \in V(G)$. A k-edge-weighting of G is a vertex-coloring if for every edge $e = uv$, $c(u) \neq c(v)$ and then say G admitting a *vertex-coloring k-edge weighting.* The minimum k for which G has a *vertex-coloring k-edge weighting* is denoted by $sd(G)$, called the *sum distinguishing index* of G.

The *Cartesian product* $G \Box H$ of two graphs G and H has $V(G \Box H) = V(G) \times V(H)$, and two vertices (u_1, u_2) and (v_1, v_2) of $G \Box H$ are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(H)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G)$.

Given any graph G, its *square graph* G^2 has the same vertex set G, with two vertices adjacent in G^2 whenever they are at distance 1 or 2 in G.

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If a graph has an edge as a component, clearly it cannot have a vertex-coloring k-edge-weighting.

In [4], Karonski, Luczak and Thomason initiated the study of vertex-coloring k-edge-weighting and they brought forward a conjecture as following.

1-2-3-Conjecture. *If G is a connected graph of order 3 or more, then $sd(G) \leq 3$.*

Furthermore, they proved that the conjecture holds for 3-colorable graphs. For cubic graphs, by Brooks' theorem, if $G \neq K_4$, then $\chi(G) \leq 3$ and hence by the above result $sd(G) \leq 3$. Recently, Kalkowski et al. [3] showed that every connected graph of order 3 or more admits a vertex-coloring 5-edge-weighting.

In this paper, we consider vertex coloring edge weighting of square of Cartesian product of paths.

2. Results

Theorem 2.1. *If $m \geq 3$ and $n \geq 3$, then $sd((P_m \square P_n)^2) = 3$.*

Proof:
First, we consider.
Case 1. $m = n \geq 8$, $m \equiv 2 \mod 3$.
Subcase 1.1. $n \equiv 0 \mod 3$.
Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{2(n-1)}v_{2n}\} \bigcup \{v_{i1}v_{i2} : i \in \{5, 8, 11, \ldots, m-3\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{5, 8, 11, \ldots, m-3\}, j \in \{2, 5, 8, \ldots, n-4\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{i1}v_{(i+1)1} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{i2}v_{(i+1)2} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{(m-1)2}v_{m2}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-3\}, j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-3\}, j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{i(n)}v_{(i+1)n} : i \in \{3, 6, 9, \ldots, m-2\}\} \bigcup \{v_{i(n)}v_{(i+1)n} : i \in \{4, 7, 10, \ldots, m-4\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-3\}\}$. Next, we assign $w(e) = 3$ if $e \in \{v_{ij}v_{i(j+1)} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{4, 7, 10, \ldots, m-4\}, j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{5, 8, 11, \ldots, m-3\}\} \bigcup \{v_{1j}v_{2(j+1)} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-2\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Next, we consider.
Subcase 1.2. $n \equiv 1 \mod 3$.
Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)}\} : j \in \{3, 6, 9, \ldots, n-4\} \bigcup \{v_{i1}v_{i2} : i \in \{5, 8, 11, \ldots, m-3\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{5, 8, 11, \ldots, m-3\}, j \in \{2, 5, 8, \ldots, n-2\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j \in \{3, 6, 9, \ldots, n-1\}\} \bigcup \{v_{m(n-1)}v_{mn}\} \bigcup \{v_{i1}v_{(i+1)1} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{i2}v_{(i+1)2} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{(m-1)2}v_{m2}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-3\}, j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-3\}, j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{3, 6, 9, \ldots, m-2\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{4, 7, 10, \ldots, m-4\}\} \bigcup \{v_{in}v_{(i+1)n} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-1\}\}$. Next, we assign $w(e) = 3$ if $e \in \{v_{ij}v_{i(j+1)} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{4, 7, 10, \ldots, m-4\}, j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{4, 7, 10, \ldots, m-4\}\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{5, 8, 11, \ldots, m-3\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-3\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Subcase 1.3. $n \equiv 2 \mod 3$.

Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)}\} : j \in \{3, 6, 9, \ldots, n-2\} \bigcup \{v_{in-1}v_{in} : i \in \{4, 7, 10, \ldots, m-4\}\} \bigcup \{v_{i1}v_{i2} : i \in \{5, 8, 11, \ldots, m-3\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{5, 8, 11, \ldots, m-3\}, j \in \{2, 5, 8, \ldots, n-3\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{(m-1)(n-1)}v_{(m-1)n}\} \bigcup \{v_{i1}v_{(i+1)1} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{i2}v_{(i+1)2} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{(m-1)2}v_{m2}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-3\}, j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-3\}, j \in \{4, 7, 10, \ldots, n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-4\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{2, 5, 8, \ldots, m-3\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{3, 6, 9, \ldots, m-2\}\} \bigcup \{v_{(m-1)n}v_{mn}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{in}v_{(i+2)n} : i \in \{1, 4, 7, \ldots, m-4\}\}$. Next, we assign $w(e) = 3$ if $e \in \{v_{ij}v_{i(j+1)} : i \in \{3, 6, 9, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{3, 6, 9, \ldots, m-2\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{4, 7, 10, \ldots, m-4\}, j \in \{4, 7, 10, \ldots, n-4\}\} \bigcup \{v_{1j}v_{2j} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-4\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.

Case 2. $m = n \geq 9$, $m \equiv 0 \mod 3$.

Subcase 2.1. $n \equiv 0 \mod 3$.

Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)}\} : j \in \{3, 6, 9, \ldots, n-3\} \bigcup \{v_{2(n-1)}v_{2n)}\} \bigcup \{v_{i1}v_{i2} : i \in \{5, 8, 11, \ldots, m-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{5, 8, 11, \ldots, m-4\}, j \in \{2, 5, 8, \ldots, n-4\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{mj}v_{m(j+1)} : j \in \{2, 5, 8, \ldots, n-4\}\} \bigcup \{v_{mj}v_{m(j+1)} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{i1}v_{(i+1)1} : i \in \{2, 5, 8, \ldots, m-4\} \bigcup \{v_{i2}v_{(i+1)2} : i \in \{2, 5, 8, \ldots, m-4\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-4\}, j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-3\}, j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-4\}, j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-3\}, j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{2, 5, 8, \ldots, m-4\} \bigcup \{v_{(m-1)(n-1)}v_{m(n-1)}\} \bigcup \{v_{in}v_{(i+1)n} : i \in \{3, 6, 9, \ldots, m-3\} \bigcup \{v_{in}v_{(i+1)n} : i \in \{4, 7, 10, \ldots, m-2\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-3\}\}$.

Next, we assign $w(e) = 3$ if $e \in \{v_{ij}v_{i(j+1)} : i \in \{3, 6, 9, \ldots, m-3\}, j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{4, 7, 10, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{5, 8, 11, \ldots, m-4\} \bigcup \{v_{mj}v_{m(j+1)} : j \in \{3, 6, 9, \ldots, n-3\} \bigcup \{v_{1j}v_{2j} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{(m-1)j}v_{mj} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-2\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.

Subcase 2.2. $n \equiv 1 \mod 3$.

Let us define, $w(e) = 2$ if $e \in \{v_{1(n-1)}v_{1n}\} \bigcup \{v_{2j}v_{2(j+1)} : j \in \{3, 6, 9, \ldots, n-4\} \bigcup \{v_{i1}v_{i2} : i \in \{5, 8, 11, \ldots, m-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{5, 8, 11, \ldots, m-4\}, j \in \{2, 5, 8, \ldots, n-2\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{mj}v_{m(j+1)} : j \in \{2, 5, 8, \ldots, n-2\}\} \bigcup \{v_{mj}v_{m(j+1)} : j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{i1}v_{(i+1)1} : i \in \{2, 5, 8, \ldots, m-4\} \bigcup \{v_{i2}v_{(i+1)2} : i \in \{2, 5, 8, \ldots, m-4\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-4\}, j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-3\}, j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-4\}, j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-3\}, j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{3, 6, 9, \ldots, m-3\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{4, 7, 10, \ldots, m-2\} \bigcup \{v_{in}v_{(i+1)n} : i \in \{2, 5, 8, \ldots, m-4\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-1\}\}$.

Next, we assign $w(e) = 3$ if $e \in \{v_{ij}v_{i(j+1)} : i \in \{3, 6, 9, \ldots, m-3\}, j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{4, 7, 10, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{4, 7, 10, \ldots, m-2\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{5, 8, 11, \ldots, m-4\} \bigcup \{v_{mj}v_{m(j+1)} : j \in \{3, 6, 9, \ldots, n-4\} \bigcup \{v_{1j}v_{2j} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{(m-1)j}v_{mj} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{2, 5, 8, \ldots, m-4\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in$

$\{4,7,10,\ldots,n-3\}\}$; $w(e)=1$ otherwise. Consequently, for any edge $e=uv$, $c(u)\neq c(v)$.

Subcase 2.3. $n\equiv 2 \mod 3$.

Let us define, $w(e)=2$ if $e\in\{v_{2j}v_{2(j+1)}\} : j\in\{3,6,9,\ldots,n-2\} \bigcup \{v_{i(n-1)}v_{in} : i\in \{4,7,10,\ldots,m-2\}\} \bigcup \{v_{i1}v_{i2} : i\in\{5,8,11,\ldots,m-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in\{5,8,11,\ldots,m-4\}, j\in\{2,5,8,\ldots,n-3\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j\in\{3,6,9,\ldots,n-2\}\} \bigcup \{v_{(m-1)(n-1)} v_{(m-1)n}\} \bigcup \{v_{mj}v_{m(j+1)} : j\in\{2,5,8,\ldots,n-3\}\} \bigcup \{v_{mj}v_{m(j+1)} : j\in\{4,7,10,\ldots,n-4\}\} \bigcup \{v_{i1}v_{(i+1)1} : i\in\{2,5,8,\ldots,m-4\}\} \bigcup \{v_{i2}v_{(i+1)2} : i\in\{2,5,8,\ldots,m-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{2,5,8,\ldots,m-4\}, j\in\{3,6,9,\ldots,n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in \{3,6,9,\ldots,m-3\}, j\in\{3,6,9,\ldots,n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{2,5,8,\ldots,m-4\}, j\in\{4,7,10,\ldots,n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{3,6,9,\ldots,m-3\}, j\in\{4,7,10,\ldots,n-4\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i\in\{2,5,8,\ldots,m-4\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i\in\{3,8,11,\ldots,m-3\}\} \bigcup \{v_{ij}v_{(i+2)j} : i\in\{1,2,3,\ldots,m-2\}, j\in\{3,6,9,\ldots,n-2\}\} \bigcup \{v_{in}v_{(i+2)n} : i\in \{1,4,7,\ldots,m-2\}\}$.

Next, we assign $w(e)=3$ if $e\in\{v_{ij}v_{i(j+1)} : i\in\{3,6,9,\ldots,m-3\}, j\in\{3,6,9,\ldots,n-2\}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in\{4,7,10,\ldots,m-2\}, j\in\{4,7,10,\ldots,n-4\}\} \bigcup \{v_{i(n-1)}v_{in} : i\in \{3,6,9,\ldots,m-3\} \bigcup \{v_{mj}v_{m(j+1)} : j\in\{3,6,9,\ldots,n-2\} \bigcup \{v_{1j}v_{2j} : j\in\{1,2,3,\ldots,n-1\}\} \bigcup \{v_{(m-1)(n-1)}v_{m(n-1)} \bigcup \{v_{ij}v_{(i+2)j} : i\in\{1,2,3,\ldots,m-2\}, j\in\{4,7,10,\ldots,n-4\}\}$; $w(e)=1$ otherwise. Consequently, for any edge $e=uv$, $c(u)\neq c(v)$.

Case 3. $m=n\geq 7$, $m\equiv 1 \mod 3$.

Subcase 3.1. $n\equiv 0 \mod 3$.

Let us define, $w(e)=2$ if $e\in\{v_{2j}v_{2(j+1)}\} : j\in\{3,6,9,\ldots,n-3\} \bigcup \{v_{2(n-1)}v_{2n}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in\{5,8,11,\ldots,m-2\}\ j\in\{2,5,8,\ldots,n-4\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j\in \{3,6,9,\ldots,n-3\}\} \bigcup \{v_{(m-1)(n-1)}v_{(m-1)n}\} \bigcup \{v_{i1}v_{(i+1)1} : i\in\{2,5,8,\ldots,m-2\} \bigcup \{v_{i2}v_{(i+1)2} : i\in\{2,5,8,\ldots,m-2\} \bigcup \{v_{(m-1)1}v_{m1}, v_{(m-1)2}v_{m2}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in \{2,5,8,\ldots,m-2\}, j\in\{3,6,9,\ldots,n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{3,6,9,\ldots,m-4\}, j\in\{3,6,9,\ldots,n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{2,5,8,\ldots,m-2\}, j\in\{4,7,10,\ldots,n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{3,6,9,\ldots,m-4\}, j\in\{4,7,10,\ldots,n-2\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i\in \{2,5,8,\ldots,m-2\} \bigcup \{v_{in}v_{(i+1)n} : i\in\{3,6,9,\ldots,m-4\} \bigcup \{v_{in}v_{(i+1)n} : i\in\{4,7,10,\ldots,m-3\} \bigcup \{v_{i1}v_{(i+2)1} : i\in\{3,6,9,\ldots,m-4\}\} \bigcup \{v_{ij}v_{(i+2)j} : i\in\{1,2,3,\ldots,m-2\}, j\in \{3,6,9,\ldots,n-3\}\}$.

Next, we assign $w(e)=3$ if $e\in\{v_{i1}v_{i2} : i\in\{3,6,9,\ldots,m-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in \{3,6,9,\ldots,m-4\}, j\in\{3,6,9,\ldots,n-3\}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in\{4,7,10,\ldots,m-3\}, j\in\{4,7,10,\ldots,n-2\}\} \bigcup \{v_{i(n-1)}v_{in} : i\in\{5,8,11,\ldots,m-2\} \bigcup \{v_{1j}v_{2j} : j\in \{1,2,3,\ldots,n-1\}\} \bigcup \{v_{ij}v_{(i+2)j} : i\in\{1,2,3,\ldots,m-2\}, j\in\{4,7,10,\ldots,n-2\}\}$; $w(e)=1$ otherwise. Consequently, for any edge $e=uv$, $c(u)\neq c(v)$.

Subcase 3.2. $n\equiv 1 \mod 3$.

Let us define, $w(e)=2$ if $e\in\{v_{2j}v_{2(j+1)}\} : j\in\{3,6,9,\ldots,n-4\} \bigcup \{v_{1(n-1)}v_{1n}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in\{5,8,11,\ldots,m-2\}\}\ j\in\{2,5,8,\ldots,n-2\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j\in \{3,6,9,\ldots,n-1\}\} \bigcup \{v_{i1}v_{(i+1)1} : i\in\{2,5,8,\ldots,m-2\}\} \bigcup \{v_{i2}v_{(i+1)2} : i\in\{2,5,8,\ldots,m-2\}\} \bigcup \{v_{(m-1)1}v_{m1}, v_{(m-1)2}v_{m2} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{2,5,8,\ldots,m-2\}, j\in\{3,6,9,\ldots,n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{3,6,9,\ldots,m-4\}, j\in\{3,6,9,\ldots,n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in \{2,5,8,\ldots,m-2\}, j\in\{4,7,10,\ldots,n-3\}\} \bigcup \{v_{ij}v_{(i+1)j} : i\in\{3,6,9,\ldots,m-4\}, j\in \{4,7,10,\ldots,n-3\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i\in\{3,6,9,\ldots,m-4\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i\in \{4,7,10,\ldots,m-3\}\} \bigcup \{v_{in}v_{(i+1)n} : i\in\{2,5,8,\ldots,m-2\}\} \bigcup \{v_{i1}v_{(i+2)1} : i\in\{3,6,9,\ldots,m-4\}\} \bigcup \{v_{ij}v_{(i+2)j} : i\in\{1,2,3,\ldots,m-2\}, j\in\{3,6,9,\ldots,n-1\}\}$.

Next, we assign $w(e)=3$ if $e\in\{v_{i1}v_{i2} : i\in\{3,6,9,\ldots,m-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in \{3,6,9,\ldots,m-4\}, j\in\{3,6,9,\ldots,n-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i\in\{4,7,10,\ldots,m-3\}, j\in\{4,7,10,\ldots,n-3\}\} \bigcup \{v_{i(n-1)}v_{in} : i\in\{4,7,10,\ldots,m-3\}\} \bigcup \{v_{i(n-1)}v_{in} : i\in$

$\{5, 8, 11, \ldots, m-2\}\} \bigcup \{v_{m(n-1)}v_{mn}, v_{(m-1)(n-1)}v_{m(n-1)}\} \bigcup \{v_{1j}v_{2j} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{2, 5, 8, \ldots, m-2\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-3\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.

Subcase 3.3. $n \equiv 2 \mod 3$.

Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)}\} : j \in \{3, 6, 9, \ldots, n-2\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{4, 7, 10, \ldots, m-3\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{5, 8, 11, \ldots, m-2\}\ j \in \{2, 5, 8, \ldots, n-3\}\} \bigcup \{v_{(m-1)j}v_{(m-1)(j+1)} : j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{(m-1)(n-1)}v_{(m-1)n}, v_{m(n-1)}v_{mn}\} \bigcup \{v_{i1}v_{(i+1)1} : i \in \{2, 5, 8, \ldots, m-2\} \bigcup \{v_{i2}v_{(i+1)2} : i \in \{2, 5, 8, \ldots, m-2\} \bigcup \{v_{(m-1)1}v_{m1}, v_{(m-1)2}v_{m2}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-4\}, j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{2, 5, 8, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-4\}\} \bigcup \{v_{ij}v_{(i+1)j} : i \in \{3, 6, 9, \ldots, m-4\}, j \in \{4, 7, 10, \ldots, n-4\}\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{2, 5, 8, \ldots, m-2\} \bigcup \{v_{i(n-1)}v_{(i+1)(n-1)} : i \in \{3, 6, 9, \ldots, m-4\} \bigcup \{v_{i1}v_{(i+2)1} : i \in \{3, 6, 9, \ldots, m-4\}\} \bigcup \{v_{in}v_{(i+2)n} : i \in \{1, 4, 7, \ldots, m-3\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{3, 6, 9, \ldots, n-2\}\}$.

Next, we assign $w(e) = 3$ if $e \in \{v_{i1}v_{i2} : i \in \{3, 6, 9, \ldots, m-4\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{3, 6, 9, \ldots, m-4\}, j \in \{3, 6, 9, \ldots, n-2\}\} \bigcup \{v_{ij}v_{i(j+1)} : i \in \{4, 7, 10, \ldots, m-3\}, j \in \{4, 7, 10, \ldots, n-4\}\} \bigcup \{v_{i(n-1)}v_{in} : i \in \{3, 6, 9, \ldots, m-4\} \bigcup \{v_{1j}v_{2j} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{(m-1)(n-1)}v_{m(n-1)}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, \ldots, m-2\}, j \in \{4, 7, 10, \ldots, n-4\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$. Thus $sd((P_m \Box P_n)^2) = 3$.

Theorem 2.2. *If* $n \geq 6$, *then* $sd((P_6 \Box P_n)^2) = 3$.

Proof:

Case 1. $n \equiv 0 \mod 3$.

Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)}\} : j \in \{3, 6, 9, \ldots, n-3\} \bigcup \{v_{2(n-1)}v_{2n}\} \bigcup \{v_{5j}v_{5(j+1)} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{2, 5, 8, \ldots, n-4\}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{1, 2\} \bigcup \{v_{2j}v_{3j} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{3j}v_{4j} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{3j}v_{4j} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{2(n-1)}v_{3(n-1)}\} \bigcup \{v_{5(n-1)}v_{6(n-1)}\} \bigcup \{v_{3n}v_{4n}\} \bigcup \{v_{4n}v_{5n}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, 4\}, j \in \{3, 6, 9, \ldots, n-3\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{3j}v_{3(j+1)} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{4j}v_{4(j+1)} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{3, 6, 9, \ldots, n-3\}\} \bigcup \{v_{1j}v_{2j} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{5j}v_{6j} : j \in \{4, 7, 10, \ldots, n-2\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, 4\}, j \in \{4, 7, 10, \ldots, n-2\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.

Case 2. $n \equiv 1 \mod 3$.

Let us define, $w(e) = 2$ if $e \in \{v_{1(n-1)}v_{1n}\} \bigcup \{v_{2j}v_{2(j+1)} : j \in \{3, 6, 9, \ldots, n-4\} \bigcup \{v_{5j}v_{5(j+1)} : j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{2, 5, 8, \ldots, n-2\}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{2j}v_{3j} : j \in \{1, 2\} \bigcup \{v_{2j}v_{3j} : j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{3j}v_{4j} : j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{3j}v_{4j} : j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{3(n-1)}v_{4(n-1)}\} \bigcup \{v_{4(n-1)}v_{5(n-1)}\} \bigcup \{v_{2n}v_{3n}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, 4\}, j \in \{3, 6, 9, \ldots, n-1\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{3j}v_{3(j+1)} : j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{4j}v_{4(j+1)} : j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{4(n-1)}v_{4n}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{3, 6, 9, \ldots, n-4\}\} \bigcup \{v_{1j}v_{2j} : j \in \{1, 2, 3, \ldots, n-1\}\} \bigcup \{v_{2(n-1)}v_{3(n-1)}\} \bigcup \{v_{5j}v_{6j} : j \in \{4, 7, 10, \ldots, n-3\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1, 2, 3, 4\}, j \in \{4, 7, 10, \ldots, n-3\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.

Case 3. $n \equiv 2 \mod 3$.

Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)} : j \in \{3, 6, 9, \ldots, n-2\} \bigcup \{v_{4(n-1)}v_{4n} \bigcup$

$\{v_{5j}v_{5(j+1)} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{5(n-1)}v_{5n}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{2,5,8,\dots,n-3\}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{4,7,10,\dots,n-4\}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{3j}v_{4j} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4,7,10,\dots,n-4\}\} \bigcup \{v_{3j}v_{4j} : j \in \{4,7,10,\dots,n-4\}\} \bigcup \{v_{2(n-1)}v_{3(n-1)}, v_{3(n-1)}v_{4(n-1)}, v_{1n}v_{3n}, v_{4n}v_{6n}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3,4\},\ j \in \{3,6,9,\dots,n-2\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{3j}v_{3(j+1)} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{3(n-1)}v_{3n}\} \bigcup \{v_{4j}v_{4(j+1)} : j \in \{4,7,10,\dots,n-4\}\} \bigcup \{v_{6j}v_{6(j+1)} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{1j}v_{2j} : j \in \{1,2,3,\dots,n-1\}\} \bigcup \{v_{5j}v_{6j} : j \in \{4,7,10,\dots,n-4\}\} \bigcup \{v_{5(n-1)}v_{6(n-1)}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3,4\},\ j \in \{4,7,10,\dots,n-4\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$. Thus $sd((P_6 \square P_n)^2) = 3$.

Theorem 2.3. *If $n \geq 5$, then $sd((P_5 \square P_n)^2) = 3$.*

Proof:
First, we consider.
Case 1. $n = 5$.
Let us define, $w(e) = 2$ if $e \in \{v_{23}v_{24}, v_{43}v_{44}, v_{44}v_{45}, v_{54}v_{55}, v_{21}v_{31}, v_{22}v_{32}, v_{23}v_{33}, v_{24}v_{34}, v_{33}v_{43}, v_{34}v_{44}, v_{45}v_{55}, v_{13}v_{33}, v_{23}v_{43}, v_{33}v_{53}, v_{14}v_{34}, v_{24}v_{44}, v_{34}v_{54}, v_{15}v_{35}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{33}v_{34}, v_{34}v_{35}, v_{11}v_{21}, v_{12}v_{22}, v_{13}v_{23}, v_{14}v_{24}, v_{44}v_{54}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Case 2. $n \equiv 0 \mod 3$.
Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)} : j \in \{3,6,9,\dots,n-3\}\} \bigcup \{v_{2(n-1)}v_{2n}\} \bigcup \{v_{4j}v_{4(j+1)} : j \in \{3,6,9,\dots,n-3\}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2,n\}\} \bigcup \{v_{2j}v_{3j} : j \in \{3,6,9\dots,n-3\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4,7,10\dots,n-2\}\} \bigcup \{v_{2(n-1)}v_{3(n-1)}\} \bigcup \{v_{3n}v_{4n}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3\}\ j \in \{3,6,9\dots,n-3\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{3j}v_{3(j+1)} : j \in \{3,6,9,\dots,n-3\}\} \bigcup \{v_{1j}v_{2j} : j \in \{1,2,3,\dots,n\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3\}\ j \in \{4,7,10,\dots,n-2\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Case 3. $n \equiv 1 \mod 3$.
Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)} : j \in \{3,6,9,\dots,n-4\}\} \bigcup \{v_{1(n-1)}v_{1n}\} \bigcup \{v_{4j}v_{4(j+1)} : j \in \{3,6,9,\dots,n-1\}\} \bigcup \{v_{5(n-1)}v_{5n}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{3,6,9\dots,n-4\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4,7,10\dots,n-3\}\} \bigcup \{v_{3j}v_{4j} : j \in \{3,6,9\dots,n-4\}\} \bigcup \{v_{3j}v_{4j} : j \in \{4,7,10\dots,n-3\}\} \bigcup \{v_{3(n-1)}v_{4(n-1)}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3\}\ j \in \{3,6,9\dots,n-1\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{3j}v_{3(j+1)} : j \in \{3,6,9,\dots,n-4\}\} \bigcup \{v_{1j}v_{2j} : j \in \{1,2,3,\dots,n-1\}\} \bigcup \{v_{2j}v_{3j} : j \in \{n-1,n\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3\}\ j \in \{4,7,10,\dots,n-3\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Case 4. $n \equiv 2 \mod 3$.
Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{4j}v_{4(j+1)} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{4(n-1)}v_{4n}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2,n-1\}\} \bigcup \{v_{2j}v_{3j} : j \in \{3,6,9\dots,n-2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4,7,10\dots,n-4\}\} \bigcup \{v_{3j}v_{4j} : j \in \{3,6,9\dots,n-2\}\} \bigcup \{v_{3j}v_{4j} : j \in \{4,7,10\dots,n-4\}\} \bigcup \{v_{2(n-1)}v_{3(n-1)}\} \bigcup \{v_{3(n-1)}v_{4(n-1)}\} \bigcup \{v_{4n}v_{5n}\} \bigcup \{v_{1n}v_{3n} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3\}\ j \in \{3,6,9\dots,n-2\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{3j}v_{3(j+1)} : j \in \{3,6,9,\dots,n-2\}\} \bigcup \{v_{3(n-1)}v_{3n} \bigcup \{v_{1j}v_{2j} : j \in \{1,2,3,\dots,n-1\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2,3\}\ j \in \{4,7,10,\dots,n-4\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$. Thus $sd((P_5 \square P_n)^2) = 3$.

Theorem 2.4. *If $n \geq 4$, then $sd((P_4 \square P_n)^2) = 3$.*

Proof:
First we consider.

Case 1. $n = 4$.
Let us define, $w(e) = 2$ if $e \in \{v_{23}v_{24}, v_{43}v_{44}, v_{13}v_{33}, v_{23}v_{43}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2,3,4\}\} \bigcup \{v_{3j}v_{4j} : j \in \{1,2\}\}$; Next, define $w(e) = 3$ if $e \in \{v_{31}v_{32}\} \bigcup \{v_{1j}v_{2j} : j \in \{1,2,3,4\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Case 2. $n \equiv 2 \mod 3$.
Let us define, $w(e) = 2$ if $e \in \{v_{2j}v_{2(j+1)} : j \in \{3,6,9,\ldots,n-2\}\} \bigcup \{v_{3j}v_{3(j+1)} : j \in \{3,6,9,\ldots,n-2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2,n-1\}\} \bigcup \{v_{3j}v_{4j} : j \in \{1,2,n\}\} \bigcup \{v_{2j}v_{3j} : j \in \{3,6,9\ldots,n-2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4,7,10\ldots,n-4\}\} \bigcup \{v_{1n}v_{3n} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2\}\ j \in \{3,6,9\ldots,n-2\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{1(n-1)}v_{1n} \bigcup \{v_{4(n-1)}v_{4n} \bigcup \{v_{1j}v_{2j} : j \in \{1,2,3,\ldots,n-1\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2\}\ j \in \{4,7,10,\ldots,n-4\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Case 3. $n \equiv 0 \mod 3$.
Let us define, $w(e) = 2$ if $e \in \{v_{ij}v_{i(j+1)} : i \in \{2,3\}, j \in \{3,6,9,\ldots,n-3\}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2,n-1\}\} \bigcup \{v_{2j}v_{3j} : j \in \{3,6,9\ldots,n-3\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4,7,10\ldots,n-2\}\} \bigcup \{v_{3j}v_{4j} : j \in \{1,2,n\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2\}\ j \in \{3,6,9\ldots,n-3\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{1j}v_{2j} : j \in \{1,2,3,\ldots,n-1\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2\}\ j \in \{4,7,10,\ldots,n-2\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$.
Case 4. $n \equiv 1 \mod 3$, $n \neq 4$.
Let us define, $w(e) = 2$ if $e \in \{v_{1(n-1)}v_{1n}\} \bigcup v_{ij}v_{i(j+1)} : i \in \{2,3\}, j \in \{3,6,9,\ldots,n-4\}\} \bigcup \{v_{2j}v_{3j} : j \in \{1,2\}\} \bigcup \{v_{2j}v_{3j} : j \in \{3,6,9\ldots,n-4\}\} \bigcup \{v_{2j}v_{3j} : j \in \{4,7,10\ldots,n-3\}\} \bigcup \{v_{3j}v_{4j} : j \in \{1,2,n-1,n\}\} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2\}\ j \in \{3,6,9\ldots,n-1\}\}$; Next, we assign $w(e) = 3$ if $e \in \{v_{1j}v_{2j} : j \in \{1,2,3,\ldots,n-1\}\} \bigcup \{v_{2(n-1)}v_{3(n-1)} \bigcup \{v_{ij}v_{(i+2)j} : i \in \{1,2\}\ j \in \{4,7,10,\ldots,n-3\}\}$; $w(e) = 1$ otherwise. Consequently, for any edge $e = uv$, $c(u) \neq c(v)$. Thus $sd((P_4 \Box P_n)^2) = 3$.

3. Conclusion

In this paper, we have determined the vertex coloring edge weighings of square of Cartesian product of paths, except $(P_3 \Box P_n)^2$.

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