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RECIPROCAL VERSION OF PRODUCT DEGREE DISTANCE OF CACTUS GRAPHS

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ABSTRACT. The reciprocal version of product degree distance is a product degree weighted version of Harary index defined for a connected graph G as $RDD_*(G) = \sum_{\{x,y\}\subseteq V(G)} \frac{(d_G(x).d_G(y))}{d_G(x,y)}$, where $d_G(x)$ is the degree of the vertex x and $d_G(x,y)$ is the distance from x to y in G. This article is attain the value of RDD_* of different types of cactus such as triangular, square and hexagonal chain cactus graphs.

Keywords: Topological index, Degree, distance, cactus graph.
AMS Subject Classification: 05C12

1. Introduction

All graphs considered in this paper are simple and connected. One of the oldest and well-studied distance based graph invariants associated with a connected graph G is the *Wiener number* $W(G)$, also termed as Wiener index in chemical or mathematical chemistry literature, which is defined [13] as the sum of distance over all unordered vertex pairs in G. The motivation for studying the quantity that we intend to call reciprocal product degree distance of a graph, comes from the following observation. The sum of distances between all pairs of vertices in a graph, namely, $W(G) = \sum_{u,v\in V(G)} d_G(u,v)$. For more results on Wiener index one may be referred to those in Dobrynin and Kochetova [2] and its references.

Another distance-based graph invariant defined [15, 16] in a fully analogous manner to Wiener index is the Harary index which is equal to the sum of reciprocal distances over all unordered vertex pairs in G, that is, $H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)}$.

Dobrynin and Kochetova [2] and Gutman [4] independently proposed a vertex-degree-weighted version of Wiener index called degree distance, which is defined for a connected graph G as $DD(G) = \frac{1}{2}\sum_{u,v\in V(G)} (d_G(u) + d_G(v))d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Tomescu [9] showed that the star is the unique graph with minimum degree distance within the class on n-vertex connected graphs. Tomescu [9] deduced properties

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of graphs with minimum degree distance in the class of n-vertex connected graphs with $m \geq n-1$ edges. Similarly, the *product degree distance* or *Gutman index* of a connected graph G is defined as $DD_*(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d_G(u) d_G(v) d_G(u,v)$.

The reciprocal degree distance is defined in [1] as $RDD(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. Similarly, Su et al. [17] introduce the reciprocal product degree distance of graphs, which is defined as $RDD_*(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} \frac{d_G(u)d_G(v)}{d_G(u,v)}$. In [18], Hamzeh et al. recently introduced generalized degree distance of graphs. Hua and Zhang [5] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. The chemical applications and mathematical properties of the reciprocal degree distance are well studied in [1,6,10].

Cactus chain is a class of linear polymers it is known initially as Husimi tree [5,6,10]. Some mathematical aspects of this chain studied by various authors in [3,7]. A connected graph with no edges lies in more than one cycle is called cactus graph, that is, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus are triangle, then it is called triangular cactus. If each C_3 of a triangular cactus has at most two cut vertices and each cut vertex is shared by exactly two triangles. If we replacing triangles in $\mathbb{G}$ by C_4 we obtain cacti whose block is C_4, and it is called square cacti. If the cut vertices of square Cacti are adjacent, we call such a cacti an Ortho-square, if the cut vertices are not adjacent, then we say it is a para-square [1]. In this paper, we obtain the upper bounds for reciprocal product degree distance of some classes of cactus graph such as triangular, square and hexagonal chain cactus graphs.

2. Different cactus graphs

In this section, initially we take a chain triangular cactus. The length of the chain is the number of triangles in the cactus. One can easily observe that all chain triangular cacti of the same length are isomorphic. The length z of the chain triangular cactus graph is denoted by T_k. First we find the exact value of reciprocal product degree distance of T_k.

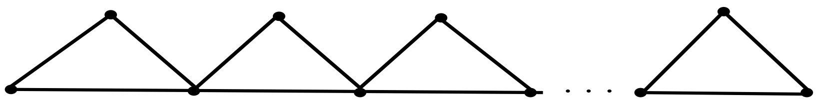

Figure 1. Chain triangular graph

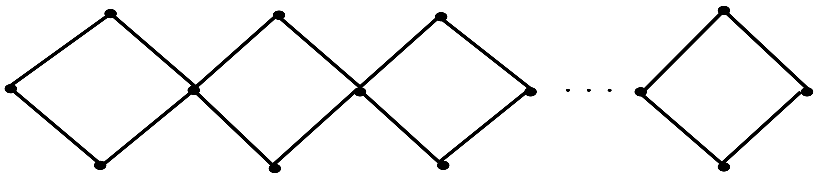

Figure 2. Para-chain square cactus graph

Theorem 2.1. *The reciprocal product degree distance of the chain triangular cactus T_k, $k \geq 2$ is $RDD_*(T_k) = \frac{4(4+9k-10k^2+3k(1+3k))\alpha_1}{k}$, where $\alpha_1 = \sum\limits_{i}^{k-1} \frac{1}{i}$.*

Proof: Suppose u and v be any two vertices of T_k. Let $d_G(u,v) = r$. We have the following three cases:

(i) If $d_G(u,v)=1$, then there exists two pairs of vertices with $d_G(u)=d_G(v)=2, 2k$ pairs of vertices with $d_G(u)=2$ and $d_G(v)=4$, and $k-2$ pairs of vertices with $d_G(u)=d_G(v)=4$, such as $u,v\in V(G)$.
(ii) If $d_G(u,v)=r$, such that $2\le r\le k-1$, then there exists $k-r+3$ pairs of vertices with $d_G(u)=d_G(v)=2$, $2(k-r+1)$ pair of vertices with $d_G(u)=2$, and $d_G(v)=4$, and $k-r-1$ pairs of vertices with $d_G(u)=d_G(v)=4$,like as $u,v\in V(G)$.
(iii) If $d_G(u,v)=k$, then there are four pairs of vertices with $d_G(u)=d_G(v)=2$, like as $u,v\in V(G)$.

By the definition of reciprocal product degree distance of a graph, we obtain

$$RDD_*(T_k)=\sum_{\{u,v\}\subseteq V(G)}\frac{(d_G(u).d_G(v))}{d_G(u,v)}$$

$$=\sum_{\{u,v\}\subseteq V(G),\ d(u,v)=r,\ r\in\{2,\ldots,k-1\}}\frac{(d_G(u).d_G(v))}{d_G(u,v)}+\sum_{\{u,v\}\subseteq V(G),\ d(u,v)=1}\frac{(d_G(u).d_G(v))}{d_G(u,v)}$$

$$+\sum_{\{u,v\}\subseteq V(G),\ d(u,v)=k}\frac{(d_G(u).d_G(v))}{d_G(u,v)}$$

$$=\sum_{r=2}^{k-1}\frac{4(k-r+3)}{r}+\sum_{r=2}^{k-1}\frac{16(k-r+1)}{r}+\sum_{r=2}^{k-1}\frac{16(k-r-1)}{r}$$

$$+2(4)+2k(8)+16(k-2)+\frac{4(2)(2)}{k}$$

$$=4(-1-2k+(3+k)\sum_{i}^{k-1}\frac{1}{i})+16(1-2k+(1+k)\sum_{i}^{k-1}\frac{1}{i})$$

$$+16(3-2k+(-1+k)\sum_{i}^{k-1}\frac{1}{i})+8+16k+16(k-2)+\frac{16}{k}.$$

$$=\frac{4(4+9k-10k^2+3k(1+3k)\sum_{i}^{k-1}\frac{1}{i})}{k}.$$

Polygamma functions gives the n^{th} derivative of the digamma function $\psi^n(z)$.The Polygamma functions PolyGamma$[n,z]$ are given by $\psi^n(z)=\frac{d^n\psi(z)}{dz^n}$. Notice that the Di-gamma function corresponds to $\psi^0(z)$. The general form $\psi^n(z)$ is the $(n+1)^{th}$, not the n^{th}, logarithmic derivative of the gamma function. The polygamma functions satisfy the relation $\psi^n(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{(n+1)}}$. For negative polygamma functions satisfy the relation $\psi^n(z)=(\psi^r(1)+\sum_{k=1}^{n}\frac{r!}{(k)^{(r+1)}})$.

Theorem 2.2. *The Reciprocal product degree distance of the Para-chain square cactus graph* $P_k,\ k\ge 2$ *is* $RDD*(P_k)\le\frac{(2-4k-46k^2+28k^3)}{(k-2k^2)}+16k\alpha_1-8\Big(-4-\psi^0\big(\frac{1}{2}\big)+2k\big(1+\psi^0\big(\frac{1}{2}\big)\big)+\psi^0\big(\frac{3}{2}\big)-2k\big(\psi^0(1)+\beta_1\big)\Big)$, *where* $\alpha_1=\sum_{i}^{k-1}\frac{1}{i}$ *and* $\beta_1=\sum_{k=1}^{\frac{1}{2}+k}\frac{0!}{k^{(0+1)}}$.

Proof: Suppose u and v be two arbitrary vertices of P_k, and $d_G(u,v)=r$. We have the following cases:

(i) If $d_G(u,v) = r$ and $r = 2s+1 (0 \le s \le k-2)$ then we have the following pairs of vertices:
(a) There are four pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $4(k-s-1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.

(ii) If $d_G(u,v) = 2$ then we have the following pairs of vertices:
(a) There are $5(k-1)+1$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are two pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(c) There are $k-2$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 4$.

(iii) If $d_G(u,v) = r$ and $r = 2s (2 \le s \le k-1)$ then we have the following pairs of vertices:
(a) There are $4(k-s)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are two pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(c) There are $k-s-1$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 4$.
(iv) If $d_G(u, v = 2k-1$, then there are four pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.

(v) If $d_G(u,v) = 2k$, then there is one pair of vertices such as $u, v \in V(G)$ *with* $d_G(u) = d_G(v) = 2$. Hence

$$
\begin{aligned}
RDD_*(P_k) &= \sum_{\{u,v\}\subseteq V(G)} \frac{(d_G(u).d_G(v))}{d_G(u,v)} \\
&= \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2s+1,\ s\in\{0,\ldots,k-2\}} \frac{(d_G(u).d_G(v))}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2} \frac{(d_G(u).d_G(v))}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2s,\ s\in\{2,\ldots,k-1\}} \frac{(d_G(u).d_G(v))}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2k-1} \frac{(d_G(u).d_G(v))}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2k} \frac{(d_G(u).d_G(v))}{d_G(u,v)} \\
&= \sum_{s=0}^{k-2} \frac{4(4)}{2s+1} + \sum_{s=0}^{k-2} \frac{4(k-s-1)(8)}{2s+1} + \frac{(5(k-1)+1)(4)}{2} \\
&+ \frac{2(8)}{2} + \frac{(k-2)(16)}{2} + \sum_{s=2}^{k-1} \frac{4(k-s)(4)}{2s} + \sum_{s=2}^{k-1} \frac{2(8)}{2s} \\
&+ \sum_{s=2}^{k-1} \frac{(k-s-1)(16)}{2s} + \frac{4(4)}{2s-1} + \frac{(4)}{2s} \\
&\le -8\Big(-4-\psi^0\Big(\frac{1}{2}\Big) + 2k\Big(1+\psi^0\Big(\frac{1}{2}\Big)\Big) + \psi^0\Big(\frac{3}{2}\Big) - 2k\Big(\psi^0(1) + \sum_{k=1}^{\frac{1}{2}+k} \frac{0!}{k^{(0+1)}}\Big)\Big) \\
&+ 8(2-2k+k\sum_i^{k-1} \frac{1}{i}) + 8(-1+\sum_i^{k-1} \frac{1}{i}) + 8\Big(3-2k+(-1+k)\sum_i^{k-1} \frac{1}{i}\Big) \\
&+ \frac{(5(k-1)+1)((2)(2))}{2} + \frac{2((2)(4))}{2} + \frac{(k-2)((4)(4))}{2} + \frac{4((2)(2))}{2k-1} + \frac{((2)(2))}{2k} \\
&\le -8\Big(-4-\psi^0\Big(\frac{1}{2}\Big) + 2k\Big(1+\psi^0\Big(\frac{1}{2}\Big)\Big) + \psi^0\Big(\frac{3}{2}\Big) - 2k\Big(\psi^0(1) + \sum_{k=1}^{\frac{1}{2}+k} \frac{0!}{k^{(0+1)}}\Big)\Big) \\
&+ 16(2-2k+k\sum_i^{k-1} \frac{1}{i}) + \frac{2-36k+50k^2-36k^3}{k-2k^2}.
\end{aligned}
$$

$$\leq \frac{(2-4k-46k^2+28k^3)}{(k-2k^2)} + 16k\sum_{i}^{k-1}\frac{1}{i} - 8\Big(-4-\psi^0\Big(\frac{1}{2}\Big)+2k\Big(1+\psi^0\Big(\frac{1}{2}\Big)\Big)$$

$$+\psi^0\Big(\frac{3}{2}\Big) - 2k\Big(\psi^0(1)+\sum_{k=1}^{\frac{1}{2}+k}\frac{0!}{k^{(0+1)}}\Big)\Big).$$

Next we find the reciprocal product degree distance of another kind of ortho-chain square cactus graph $O_k,\ k \geq 5$, shown in Figure 3.

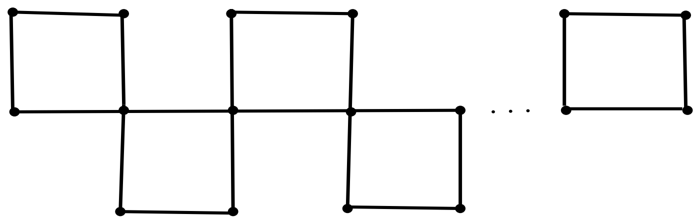

Figure 3. Ortho-chain square cactus graph

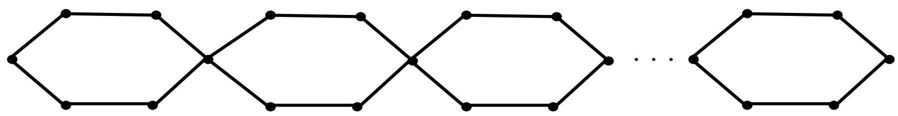

Figure 4. Para-chain hexagonal cactus graph

Theorem 2.3. *The reciprocal product degree distance of the ortho-chain square cactus graph* $Q_k,\ k \geq 5$ *is* $RDD_*(Q_k) = \frac{(408+1140k+250k^2-708k^3-298k^4)}{(6k+9k^2+3k^3)} + 16(3+4k)\alpha_1$, *where* $\alpha_1 = \sum_{i}^{k-1}\frac{1}{i}$.

Proof: Consider two arbitrary u and v vertices of Q_k, let $d_G(u,v) = r$. We have the following cases:
(i) If $d_G(u,v) = 1$ then we have the following pairs of vertices:
(a) There are $k+2$ of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $2k$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(c) There are $k-2$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 4$.
(ii) If $d_G(u,v) = 2$ then we have the following pairs of vertices:
(a) There are $k+3$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $4(k-1)$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(c) There are $k-3$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 4$.
(iii) If $d_G(u,v) = 3$ then we have the following pairs of vertices:
(a) There are $3k$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $4(k-3)+2$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(b) There are $k-4$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 4$.
(iv) If $d_G(u,v) = r(4 \leq r \leq k-1)$ then we have the following pairs of vertices:
(a) There are $4(k-r+3)$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $4(k-r)+2$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(b) There are $k-r-1$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = 4$.
(v) If $d_G(u,v) = k$ then we have the following pairs of vertices:

(*a*) There are thirteen pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(*b*) There are two pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(*vi*) If $d_G(u, v) = k + 1$ then there are six pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(*vii*) If $d_G(u, v) = k + 2$ then there is one pair of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.

$$
\begin{aligned}
RDD_*(Q_k) &= \sum_{\{u,v\}\subseteq V(G)} \frac{(d_G(u).d_G(v))}{d_G(u,v)} \\
&= \sum_{\{u,v\}\subseteq V(G), d(u,v)=1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G), d(u,v)=2} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G), d(u,v)=3} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=r,\ r\in\{4,\ldots,k-1\}} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=k} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=k+1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=k+2} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&= (k+2)((2)(2)) + 2k((2)(4)) + (k-2)((4)(4)) + \frac{(k+3)((2)(2))}{2} + \frac{4(k-1)((2)(4))}{2} \\
&+ \frac{(k-3)((4)(4))}{2} + \frac{3k((2)(2))}{3} + \frac{(4(k-3)+2)((2)(4))}{3} + \frac{(k-4)((4)(4))}{3} \\
&+ \sum_{r=4}^{k-1} \frac{4(k-r+3)((2)(2))}{r} + \sum_{r=4}^{k-1} \frac{(4(k-r)+2)((2)(4))}{r} + \sum_{r=4}^{k-1} \frac{(k-r-1)((4)(4))}{r} \\
&+ \frac{13((2)(2))}{k} + \frac{2((2)(4))}{k} + \frac{6((2)(2))}{k+1} + \frac{((2)(2))}{k+2} \\
&= (k+2)((2)(2)) + 2k((2)(4)) + (k-2)((4)(4)) + \frac{(k+3)((2)(2))}{2} + \frac{4(k-1)((2)(4))}{2} \\
&+ \frac{(k-3)((4)(4))}{2} + \frac{3k((2)(2))}{3} + \frac{(4(k-3)+2)((2)(4))}{3} + \frac{(k-4)((4)(4))}{3} \\
&+ \frac{13((2)(2))}{k} + \frac{2((2)(4))}{k} + \frac{6((2)(2))}{k+1} + \frac{((2)(2))}{k+2} + \frac{8}{3}(-9 - 17k + 6(3+k)) \sum_{i}^{k-1} \frac{1}{i}) \\
&+ \frac{8}{3}(37 - 34k + 6(1+2k)) \sum_{i}^{k-1} \frac{1}{i} + \frac{8}{3}(35 - 17k + 6(-1+k)) \sum_{i}^{k-1} \frac{1}{i}) \\
&= \frac{(2(68 + 22k - 29k^2 + 70k^3 + 41k^4))}{(k(2+3k+k^2))} + \frac{8}{3}(63 - 68k + 6(3+4k)) \sum_{i}^{k-1} \frac{1}{i} \\
&= \frac{(408 + 1140k + 250k^2 - 708k^3 - 298k^4)}{(6k + 9k^2 + 3k^3)} + 16(3+4k) \sum_{i}^{k-1} \frac{1}{i}.
\end{aligned}
$$

3. RDD_* of chain hexagonal cactus

Replacing triangles in the definitions of triangular cactus, by cycles of length 6 we have cacti whose every block is C_6, and it is called hexagonal cacti, see Figure 4. One can see that the internal hexagonal may differ in the way they connect to their neighbors, if their cut-vertices are adjacent, we say that such a square is an Ortho-hexagonal and if the cut-vertices are not adjacent, we call the square a para-hexagonal. We consider a para-chain of length k, which is denoted by L_k as shown in Figure 4. Now we obtain the value of reciprocal product degree distance of para-chain hexagonal cactus graph L_k.

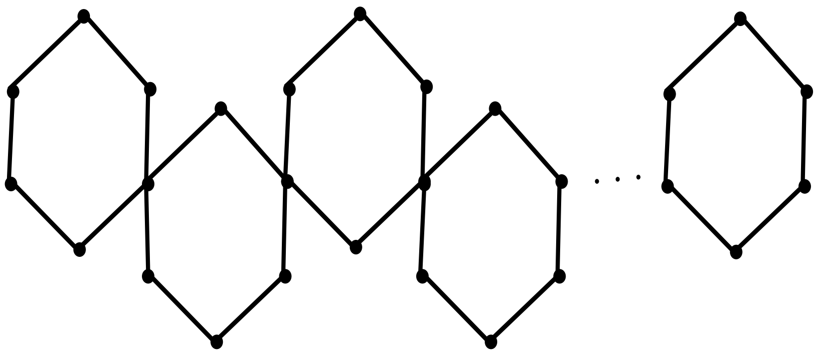

Figure 5. Meta-chain graph

Theorem 3.1. *The reciprocal degree distance of the para-chain hexagonal cactus graph* L_k $k \geq 3$ *is* $RDD_*(L_k) \leq \frac{4(2-129k+571k^2-909k^3+585k^4)}{3k(2-9k+9k^2)} + 16\left(7 - 4k + k\alpha_2 - k\psi^0\left(\frac{4}{3}\right) - k\psi^0\left(\frac{5}{3}\right) + k(\psi^0(1)+\beta_2) + k(\psi^0(1)+\beta_3)\right)$, *where* $\alpha_2 = \sum_{i}^{k-2} \frac{1}{i}$, $\beta_2 = \sum_{k=1}^{\frac{2}{3}+k} \frac{0!}{k^{(0+1)}}$, $\beta_3 = \sum_{k=1}^{\frac{1}{3}+k} \frac{0!}{k^{(0+1)}}$.

Proof: Suppose u and v be two arbitrary vertices of L_k, and $d_G(u,v) = r$. We have the following cases:
(i) If $d_G(u,v) = 1$ then we have the following pairs of vertices:
(a) There are $2k+4$ of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $4(k-1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(ii) If $d_G(u,v) = 2$ then we have the following pairs of vertices:
(a) There are $6k$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
$(ii)_(b)$ There are $4(k-1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(iii) If $d_G(u,v) = 3$ then we have the following pairs of vertices:
(a) There are $10(k-1)+2$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are 2 pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(c) There are $k-2$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 4$.
(iv) If $d_G(u,v) = r$ and $r = 3s+1 (1 \leq s \leq k-2)$ then we have the following pairs of vertices:
(a) There are $4(k-s+1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $4(k-s-1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(v) If $d_G(u,v) = r$ and $r = 3s+2 (1 \leq s \leq k-2)$ then we have the following pairs of vertices:
(a) There are $4(k-s)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are $4(k-s-1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(vi) If $d_G(u,v) = r$ and $r = 3s (2 \leq s \leq k-1)$ then there are following pairs of vertices
(a) There are $8(k-s)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
(b) There are 2 pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
(c) There are $k-s-1$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 4$.
(vii) If $d_G(u,v) = 3k-2$ then there are eight pair of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.

$(viii)$ If $d_G(u,v)=3k-1$ then there are four pair of vertices such as $u,v\in V(G)$ with $d_G(u)=d_G(v)=2$.
(ix) If $d_G(u,v)=3k$ then there is one pair of vertices such as $u,v\in V(G)$ with $d_G(u)=d_G(v)=2$.

$$
\begin{aligned}
RDD_*(L_k) &= \sum_{\{u,v\}\subseteq V(G)} \frac{[d_G(u).d_G(v)]}{d_G(u,v)}\\
&= \sum_{\{u,v\}\subseteq V(G),d(u,v)=1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),d(u,v)=2} \frac{[d_G(u).d_G(v)]}{d_G(u,v)}\\
&+ \sum_{\{u,v\}\subseteq V(G),d(u,v)=3} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=3s+1,\ s\in\{1,\ldots,k-2\}} \frac{[d_G(u).d_G(v)]}{d_G(u,v)}\\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=3s+2,\ s\in\{1,\ldots,k-2\}} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{d(u,v)=3s,\ s\in\{2,\ldots,k-1\}} \frac{[d_G(u).d_G(v)]}{d_G(u,v)}\\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=3k-2} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=3k-1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)}\\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=3k} \frac{[d_G(u).d_G(v)]}{d_G(u,v)}\\
= \ & (2k+4)((2)(2)) + 4(k-1)((2)(4)) + \frac{6k((2)(2))}{2} + \frac{4(k-1)((2)(4))}{2} + \frac{(10(k-1)+2)((2)(2))}{3}\\
&+\frac{2((2)(4))}{3} + \frac{(k-2)((4)(4))}{3} + + \sum_{s=1}^{k-2} \frac{4(k-s+1)((2)(2))}{(3s+1)} + \sum_{s=1}^{k-2} \frac{4(k-s-1)((2)(4))}{(3s+1)}\\
&+\sum_{s=1}^{k-2} \frac{4(k-s)((2)(2))}{(3s+2)} + \sum_{s=1}^{k-2} \frac{4(k-s-1)((2)(4))}{(3s+2)} + \sum_{s=2}^{k-2} \frac{8(k-s)((2)(2))}{3s} + \sum_{s=2}^{k-2} \frac{2((2)(4))}{3s}\\
&+\sum_{s=2}^{k-2} \frac{(k-s-1)((4)(4))}{3s} + \frac{8((2)(2))}{3n-2} + \frac{4((2)(2))}{3n-1} + \frac{((2)(2))}{3n}\\
&\leq (2k+4)((2)(2)) + 4(k-1)((2)(4)) + \frac{6k((2)(2))}{2} + \frac{4(k-1)((2)(4))}{2} + \frac{(10(k-1)+2)((2)(2))}{3}\\
&+\frac{2((2)(4))}{3} + \frac{(k-2)((4)(4))}{3} - \frac{16}{9}\Big(-6+4\psi^0\Big(\frac{4}{3}\Big) + 3k(1+\psi^0\Big(\frac{4}{3}\Big) - (4+3k)(\psi^0(1) + \sum_{k=1}^{\frac{2}{3}+k} \frac{0!}{k^{(0+1)}}\Big)\\
&-\frac{32}{9}\Big(3k(1+\psi^0\Big(\frac{4}{3}\Big)) - 2(3+\psi^0\Big(\frac{4}{3}\Big)) + (2-3k)(\psi^0(1) + \sum_{k=1}^{\frac{2}{3}+k} \frac{0!}{k^{(0+1)}}\Big)) - \frac{16}{9}\Big(2\Big(-3+\psi^0\Big(\frac{5}{3}\Big)\Big)\\
&+3k(1+\psi^0\Big(\frac{5}{3}\Big)) - (2+3k)(\psi^0(1) + \sum_{k=1}^{\frac{1}{3}+k} \frac{0!}{k^{(0+1)}}\Big)) - \frac{32}{9}\Big(-6-\psi^0\Big(\frac{5}{3}\Big) + 3k(1+\psi^0\Big(\frac{5}{3}\Big))\\
&+(1-3k)(\psi^0(1) + \sum_{k=1}^{\frac{1}{3}+k} \frac{0!}{k^{(0+1)}}\Big)) + \frac{32}{3}(3-2k+k\sum_{i}^{k-2}\frac{1}{i}) + \frac{16}{3}(-1+\sum_{i}^{k-2}\frac{1}{i})\\
&+\frac{16}{3}(4-2k+(-1+k)\sum_{i}^{k-2}\frac{1}{i})) + \frac{8((2)(2))}{3n-2} + \frac{4((2)(2))}{3n-1} + \frac{((2)(2))}{3n}\\
&= \frac{4(2-129k+571k^2-909k^3+585k^4)}{3k(2-9k+9k^2)}\\
&+16\Big(7-4k+k\sum_{i}^{k-2}\frac{1}{i} - k\psi^0\Big(\frac{4}{3}\Big) - k\psi^0\Big(\frac{5}{3}\Big) + k(\psi^0(1) + \sum_{k=1}^{\frac{2}{3}+k}\frac{0!}{k^{(0+1)}} + k(\psi^0(1) + \sum_{k=1}^{\frac{1}{3}+k}\frac{0!}{k^{(0+1)}}\Big)\Big).
\end{aligned}
$$

Theorem 3.2. *The reciprocal product degree distance of the para-chain hexagonal cactus graph M_k, $k \geq 4$ is* $RDD_*(M_k) \leq \frac{-60+351k+868k^2-521k^3+272k^4+1268k^5}{3k(-1-k+4k^2+4k^3)} - 18(-6+\psi^0(\frac{5}{2}) + 2k(1+\psi^0(\frac{5}{2})\Big) - (1+2k)(\psi^0(1)+\beta_1)\Big) + 9\Big(9-10k+(2+4k)(\alpha_1)\Big)$, *where* $\alpha_1 = \sum_{i}^{k-1} \frac{1}{i}$, *and* $\beta_1 = \sum_{k=1}^{\frac{1}{2}+k} \frac{0!}{k^{(0+1)}}$.

Proof: Suppose u and v be two arbitrary vertices of M_k, and $d_G(u,v) = r$. We have the following cases:

(i) If $d(u,v) = 1$ then we have the following pairs of vertices:
 (a) There are $2k+4$ of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
 (b) There are $4(k-1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.

(ii) If $d_G(u,v) = 2$ then we have the following pairs of vertices:
 (a) There are $5k$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
 (b) There are $2k$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
 (c) There are $(k-2)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 4$.

(iii) If $d_G(u,v) = 3$ then we have the following pairs of vertices:
 (a) There are $5k+2$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
 (b) There are $6k-10$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.

(iv) If $d_G(u,v) = 4$ then we have the following pairs of vertices:
 (a) There are $9(k-1)-2$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
 (b) There are $2(k-1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
 (c) There are $(k-3)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 4$.

(v) If $d_G(u,v) = r$ and $r = 2s+1 (2 \leq s \leq k-2)$ then we have the following pairs of vertices:
 (a) There are $6(k-s+1)+2$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
 (b) There are $6(k-s-1)+2$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.

(vi) If $d_G(u,v) = r$ and $r = 2s (3 \leq s \leq k-1)$ then there are following pairs of vertices
 (a) There are $10(k-s+1)-1$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
 (b) There are $2(k-s+1)$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.
 (c) There are $k-s-1$ pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 4$.

(vii) If $d_G(u,v) = 2k-1$ then there are following pair of vertices
 (a) There are fourteen pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.
 (b) There are 2 pairs of vertices such as $u, v \in V(G)$ with $d_G(u) = 2$ and $d_G(v) = 4$.

$(viii)$ If $d_G(u,v) = 2k$ then there are ten pair of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.

(ix) If $d_G(u,v) = 2k+1$ then there are four pair of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.

(x) If $d_G(u,v) = 2k+2$ then there is one pair of vertices such as $u, v \in V(G)$ with $d_G(u) = d_G(v) = 2$.

By the definition of reciprocal product degree distance of graph G;

$$
\begin{aligned}
RDD_*(M_k) &= \sum_{\{u,v\}\subseteq V(G)} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&= \sum_{\{u,v\}\subseteq V(G), d(u,v)=1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G), d(u,v)=2} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G), d(u,v)=3} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G), d(u,v)=4} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2s,\ s\in\{2,\dots,k-2\}} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{d(u,v)=2s+1,\ s\in\{3,\dots,k-1\}} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2k-1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2k} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G),\ d(u,v)=2k+1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G)\ ,d(u,v)=2k+2} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&= (2k+4)((2)(2)) + 4(k-1)((2)(4)) + \frac{5k((2)(2))}{2} + \frac{2k((2)(4))}{2} + \frac{(k-2)((4)(4))}{2} \\
&+ \frac{(5k+2)((2)(2))}{3} + \frac{(6k-10)((2)(4))}{3} + \frac{(9(k-1)-2)((2)(2))}{4} + \frac{2(k-1)((2)(4))}{4} \\
&+ \frac{(k-3)((4)(4))}{4} + \sum_{s=2}^{k-2} \frac{(6(k-s+1)+2)((2)(2))}{(2s+1)} + \sum_{s=2}^{k-2} \frac{(6(k-s-1)+2)((2)(4))}{(2s+1)} \\
&+ \sum_{s=3}^{k-1} \frac{(10(k-s+1)-1)((2)(2))}{(2s)} + \sum_{s=3}^{k-2} \frac{2(k-s+1)((2)(4))}{(2s)} + \sum_{s=3}^{k-1} \frac{(k-s-1)((4)(4))}{(2s)} \\
&+ \frac{14((2)(2))}{(2k-1)} + \frac{2((2)(4))}{(2k-1)} + \frac{10((2)(2))}{(2k)} + \frac{4((2)(2))}{(2k+1)} + \frac{((2)(2))}{(2k+2)} \\
&\le \frac{-60+351k+868k^2-521k^3+272k^4+1268k^5}{3k(-1-k+4k^2+4k^3)} - 2\Big(-18 + 11\psi^0\Big(\frac{5}{2}\Big) + 6k(1+\psi^0\Big(\frac{5}{2}\Big)) \\
&-(11+6k)(\psi^0(1) + \sum_{k=1}^{\frac{1}{2}+k} \frac{0!}{k^{(0+1)}})\Big) - 4\Big(-18 - \psi^0\Big(\frac{5}{2}\Big) + 6k(1+\psi^0\Big(\frac{5}{2}\Big)) + (1-6k)(\psi^0(1) + \sum_{k=1}^{\frac{1}{2}+k} \frac{0!}{k^{(0+1)}})\Big) \\
&+\Big(33 - 50k + 2(9+10k)\sum_{i}^{k-1} \frac{1}{i}\Big) + 4(3-5k+2(1+k)\sum_{i}^{k-1} \frac{1}{i}) + 4(9-5k+2(-1+k)\sum_{i}^{k-1} \frac{1}{i}) \\
&\le \frac{-60+351k+868k^2-521k^3+272k^4+1268k^5}{3k(-1-k+4k^2+4k^3)} - 18\Big(-6 + \psi^0\Big(\frac{5}{2}\Big) + 2k(1+\psi^0\Big(\frac{5}{2}\Big)) \\
&-(1+2k)(\psi^0(1) + \sum_{k=1}^{\frac{1}{2}+k} \frac{0!}{k^{(0+1)}})\Big) + 9\Big(9 - 10k + (2+4k)\sum_{i}^{k-1} \frac{1}{i}\Big).
\end{aligned}
$$

Let G be a connected graph constructed from pairwise disjoint connected graphs $G_1, \dots, G_r$ as follows;

Select a vertex of G_1, a vertex of G_2, and identify these two vertices. Then continue in this manner inductively. Note that the graph G constructed in this way has a tree-like structure, the G_is being its building stones. Usually say that G is obtained by point-attaching from $G_1, \dots, G_r$ and that G_is are the primary subgraphs of G. A particular case of this construction is the decomposition of a connected graph into blocks (see [3]). Cactus chains which we considered in this paper are particular cases of point attaching of cycles of length three, four and six. As another example consider the graph $Q(m,k)$ constructed in the following manner: consider the graph K_m and m copies of K_k. The graph $Q(m,k)$ is obtained by identifying each vertex of K_m with a vertex of a unique K_k . Finally, we compute the value of reciprocal product degree distance of the graph $Q(m,k)$, $m,k \geq 2$.

Theorem 3.3. *The reciprocal product degree distance of the graph* $Q(m,k)(m,k \geq 2)$ *is* $RDD_*(Q(m,k)) = \frac{1}{6}m\Big(-13 + k(10-13m) + 22m - 12m^2 + 3m^3 + k^4(2+m) - k^3(8+m) + 3k^2(2+m^2)\Big)$.

Proof: Suppose u and v be two arbitrary vertices of $Q(m,k)$, and $d_G(u,v) = r$. We have the following cases:
(i) If $d_G(u,v) = 1$ then there are $\frac{m(k-1)(k-2)}{2}$ of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = k-1$, $m(k-1)$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = k-1$ and $d_G(v) = m+k-2$.
And there are $\frac{m(m-1)}{2}$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = m+k-2$.
(ii) If $d_G(u,v) = 2$ then there are $m(m-1)(k-1)$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = k-1$ and $d_G(v) = m+k-2$.
(iii) If $d_G(u,v) = 3$ then there are $\frac{m(k-1)^2(m-1)}{2}$ pairs of vertices such as $u,v \in V(G)$ with $d_G(u) = d_G(v) = k-1$.
We know reciprocal degree distance of a connected graph is,

$$\begin{aligned}
RDD_*(Q(m,k)) &= \sum_{\{u,v\}\subseteq V(G)} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} = \sum_{\{u,v\}\subseteq V(G), d(u,v)=1} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&+ \sum_{\{u,v\}\subseteq V(G), d(u,v)=2} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} + \sum_{\{u,v\}\subseteq V(G), d(u,v)=3} \frac{[d_G(u).d_G(v)]}{d_G(u,v)} \\
&= \frac{m(k-1)(k-2)}{2}(k-1)^2 + m(k-1)^2(m+k-2) + \frac{m(m-1)}{2}(m+k-2)^2 \\
&+ \frac{m(m-1)(k-1)^2(m+k-2)}{2} + \frac{\frac{m(m-1)(k-1)^2}{2}(k-1)^2}{3} \\
&= \frac{1}{6}m\Big(-13 + k(10-13m) + 22m - 12m^2 + 3m^3 + k^4(2+m) - k^3(8+m) + 3k^2(2+m^2)\Big).
\end{aligned}$$

References

[1] Alikhani, S., Jahari, S., Mehryar M., and Hasni, M. (2014), Counting the number of dominating sets of cactus chains, Opt. Adv. Mat. Rapid Comm., 8,(9-10), 955-960.
[2] Dobrynin, A.A., Kochetova, A.A. (1994), degree distance of a graph: a degree analogue of the Wiener index, J. Chem. Inf. Comput. Fci., 34, 1082-1086.
[3] Deutsch, E., Klavẑar, S. (2013), Computing the Hosoya polynomial of graphs from primary subgraphs, MATCH Commun. Math. Comput. Chem., 70, 627-644.
[4] Gutman, I. (1994), Felected properties of the Fchultz molecular topological index, J. Chem. Inf. Comput. Fci., 34, 1087-1089.
[5] Harary, F., Uhlenbeck, B. (1953), On the number of Husimi trees I, Proc. Nat. Acad. Sci. 39 , 315-322.
[6] Husimi, K. (1950), Note on Mayer's theory of cluster integrals, J. Chem. Phys., 18, 682-684.
[7] Majstorović, S., Doŝlić, T., and Klobuĉar, A. (2012), k-domination on hexagonal cactus chains, Kragujevac J. Math. 36,(2), 335-347.
[8] Mukwembi, S. (2012), On the upper bound of Gutman index of graphs, MATCH Commun. Math. Comput. Chem., 68, 343-348.

[9] Tomescu, I. (2008), Properties of connected graphs having minimum degree distance, Discrete Math., 309, 2745-2748.
[10] Riddell, R.J. (1951), Contributions to the theory of condensation, Ph.D. Thesis, Univ. of Michigan, Ann Arbor.
[11] Sheeba Agnes, V. (2015), Degree distance and Gutman index of corona product of graphs, Trans. Combin., 4, (3), 11-23.
[12] Yan, W., Yang, B.y., and Yeh, Y.N. (2007), The behavior of wiener indices and polynomials of graphs under five graph decorations, Appl.Math.Lett., 20, (3), 290-295.
[13] Gutman, I. (1997), A property of the Wiener number and its modification, Indian, J.chem., 36, 128-132.
[14] Dobrynin, A.A., Kochetova, A.A. (1994), degree distance of a graph: a degree analogue of the Wiener index, J. Chem. Inf. Comput. Fci., 34, 1082-1086.
[15] Ivanciuc, O., Balaban, T.S., and Balaban, A.T. (1993), Reciprocal distance matrix, related local vertex invariants and topological indices, J. Math. Chem., 12, 309-318.
[16] Plavic, D., Nikolic, S., Trinajstic, N., and Mihalic, Z. (1993), On the Harary index for the characterization of chemical graphs, J. Math. Chem., 12, 235-250.
[17] Su, G., Gutman, I., Xiong, L., and Xu, L., Reciprocal product degree distance of graphs, Manuscript.
[18] Hamzeh, A., Iranmanesh, A., Hossein-Zadeh, S., and Diudea, M.V. (2012), Generalized degree distance of trees, unicyclic and bicyclic graphs, Studia Ubb Chemia, LVII, 4, 73-85.

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