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BOTH A GRAPH AND ITS COMPLEMENT ARE SELF-CENTERED WITH IDENTICAL RADIUS

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ABSTRACT. We show that a graph and its complement are self-centered with identical radius r only when $r = 2$. Further, we provide a construction of such a graph for any given order at least eight.

Keywords: Distance, Eccentricity, Self-centered graph, Complement (of a graph)
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1. INTRODUCTION

In this paper we restrict ourselves to simple and non-trivial graphs. We follow the notation of [11, 14], where definitions not included here may be found. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is denoted by p. The neighborhood of a vertex $v \in V(G)$, denoted $N_G(v)$, is the set of vertices adjacent to v in G. The degree $deg_G(v)$ of vertex v is the number of vertices adjacent to v in G. If $N_G(v) = V(G)$, then v is called a full degree vertex. If $N_G(v) = \emptyset$, then v is called an isolate. The distance $d_G(u, v)$ between two vertices u and v is the length of a shortest path between u and v if any exists; otherwise $d_G(u, v) = \infty$. The eccentricity of the vertex $v \in V(G)$, denoted $e_G(v)$, is the maximum distance from v to any vertex in G. The radius $rad(G)$ (diameter $diam(G)$) of G is the minimum (maximum) among the eccentricities. If $e_G(v) = rad(G)$ for all $v \in V(G)$, then G is called a self-centered graph with radius $rad(G)$. For detailed study of self-centered graphs readers are refer to [1, 9, 10, 15, 16]. The complement of a graph G is denoted by $\bar{G}$. A vertex v in a graph G is called an $a_1 - vertex$ if there is a vertex $u \in N_G(v)$ such that $N_G(v) \cup N_G(u) = V(G)$; otherwise called an $a_2 - vertex$. Note that any full degree vertex is an $a_1 - vertex$ in G; while any isolate is an $a_2 - vertex$ in G, see [12].

The study of common properties of both a graph and its complement attracts researchers, see [2–8, 13]. In [13], we proved that in case of identical radius the radius of the graph is 2, see Theorem 1.1; while the diameter is 2 or 3 in the case of identical diameter, see Theorem 1.2. And in case of identical center (resp. periphery) the center (resp. periphery) is whole vertex set. At this juncture, one can see that there are self-centered graphs for which their corresponding complements are self-centered with different

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radii. For example, the cycle graph C_p with $p \neq 5$ is self-centered with radius $\lfloor \frac{p}{2} \rfloor$ but its complement $\bar{C_p}$ is self-centered with radius ∞ if $p = 3, 4$; radius 2 otherwise. Here, we intend to show that both a graph and its complement are self-centered with identical radius if and only if both are self-centered with radius 2. Further, we provide a construction of such a graph for a given order $p \geq 8$ in the next section. To this end we need the following results.

Theorem 1.1. *[13] Let G be a graph. If the graph G and its complement $\bar{G}$ have same radius, then radius of the graph G is* 2.

Theorem 1.2. *[13] Let G be a graph. Then, $diam(G) = diam(\bar{G})$ implies $diam(G) \in \{2, 3\}$.*

The Theorems 1.1 and 1.2 ascertain that if both a graph and its complement are self-centered with identical radius then each of the graphs has radius 2. This leads us to the following theorem.

Theorem 1.3. *Let G and $\bar{G}$ be self-centered graphs. Then, $rad(G)$ equals $rad(\bar{G})$ if and only if $rad(G) = rad(\bar{G}) = 2$.*

Theorem 1.4. *[13] Let G be a graph. If the vertex u is an $a_2 - vertex$ in G with eccentricity* 2*, then u is an $a_2 - vertex$ in $\bar{G}$ with eccentricity* 2*.*

As a consequence, if all the vertices of a graph G are $a_2 - vertices$ of eccentricity 2, then both G and $\bar{G}$ are self-centered with radius 2. Akiyama [1] et al. obtained the following structural characterization of self-centered graph of radius 2 on order at least 5 with minimum number of edges.

Theorem 1.5. *[10] Let G be a self-centered graph with $p \geq 5$ vertices and diameter* 2*, having as few edges as possible among such graphs. Then, G is one of the following (refer Figure 1)*

a) *The Petersen graph.*
b) *The graph formed from $S_{a,b}$ by adding an additional vertex v and joining v to each vertex of degree* 1 *in $S_{a,b}$.*
c) *The graph formed from $K_3(a, b, c)$ by adding a new vertex w and joining w to each vertex of degree* 1 *in $K_3(a, b, c)$; $a + b + c = p - 4$ and $a, b, c \geq 1$.*

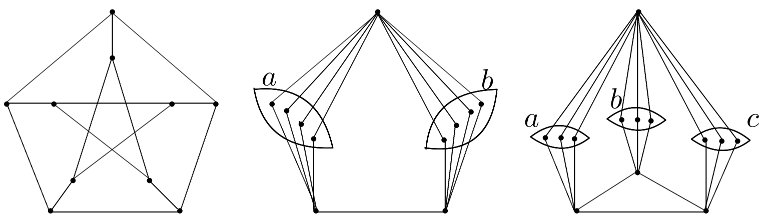

FIGURE 1. Graphs described in part a,b and c, respectively

One can easily verify that all the vertices of the graph described in part b of the theorem above are $a_2 - vertices$ of eccentricity 2. In view of Theorem 1.4, the corresponding complement is self-centered with radius 2 as well. Thus, we get a family of graphs for which graphs and its complements are self-centered with radius 2. Similarly, the family of graphs constructed in part c is such a family. In the next section we construct a new family of graphs for which each of its member and associated complement are self-centered with radius 2.

2. Construction

One can easily verify that there is no graph of fewer order than five such that the graph and its complement are self-centered with same radius. Now, for a given order p with $p \geq 8$ we construct a family of graphs called *Quotient Modulo-4 graph* (QM-4 graph) as follows:

Let $v_1, v_2, \ldots, v_p$ be the vertices and put

$$D_{i,j} = min\{|j-i|, p-|j-i|\} \quad \text{for } i,j \in \{1,2,\ldots,p\}.$$

Note that $D_{i,j} = D_{j,i}$ and $D_{i,j} \leq \lfloor \frac{p}{2} \rfloor$. Join the vertices v_i and v_j if

$$\begin{array}{rl} \lceil \frac{p}{4} \rceil + 1 \leq D_{i,j} \leq \lfloor \frac{p}{2} \rfloor \quad \text{or} \quad D_{i,j} = 1 & \text{for } p \equiv 1,2 \pmod 4; \\ \lceil \frac{p}{4} \rceil + 1 \leq D_{i,j} \leq \lfloor \frac{p}{2} \rfloor & \text{for } p \equiv 3,0 \pmod 4. \end{array} \tag{1}$$

We note that two vertices v_i and v_j are non-adjacent if

$$\begin{array}{ll} 2 \leq D_{i,j} \leq \lceil \frac{p}{4} \rceil & \text{for} \quad p \equiv 1,2 \pmod 4; \\ 1 \leq D_{i,j} \leq \lceil \frac{p}{4} \rceil & \text{for} \quad p \equiv 3,0 \pmod 4. \end{array} \tag{2}$$

Thus, for $v \in V(G)$

$$\begin{aligned} deg_G(v) &= \begin{cases} (p-1) - 2\lceil \frac{p}{4} \rceil + 2 & \text{if} \quad p \equiv 1,2 \pmod 4 \\ (p-1) - 2\lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 3,0 \pmod 4 \end{cases} \\ &= \begin{cases} p - 2\lceil \frac{p}{4} \rceil + 1 & \text{if} \quad p \equiv 1,2 \pmod 4 \\ p - 2\lceil \frac{p}{4} \rceil - 1 & \text{if} \quad p \equiv 3,0 \pmod 4, \end{cases} \end{aligned}$$

and so G is regular. We see that

$$\begin{array}{lll} p = 2\lfloor \frac{p}{2} \rfloor + 1 = 4\lceil \frac{p}{4} \rceil - 3 & \text{if} & p \equiv 1 \pmod 4; \\ p = 2\lfloor \frac{p}{2} \rfloor = 4\lceil \frac{p}{4} \rceil - 2 & \text{if} & p \equiv 2 \pmod 4; \\ p = 2\lfloor \frac{p}{2} \rfloor + 1 = 4\lceil \frac{p}{4} \rceil - 1 & \text{if} & p \equiv 3 \pmod 4; \\ p = 2\lfloor \frac{p}{2} \rfloor = 4\lceil \frac{p}{4} \rceil & \text{if} & p \equiv 0 \pmod 4. \end{array} \tag{3}$$

Based on this construction, graphs of order 9, 10, 11, and 8 are constructed in Figure 2.

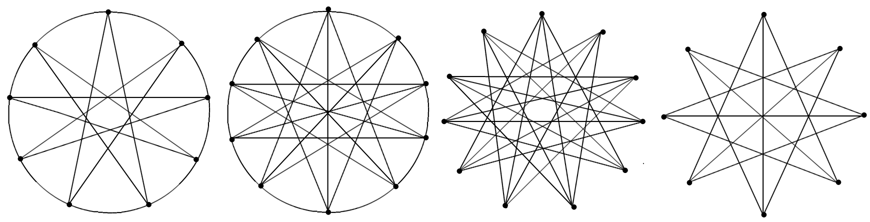

Figure 2. QM-4 graph of order 9,10,11 and 8

In the next section we prove that both *QM-4 graph* and its complement are self-centered with radius 2.

3. Proof

Let G be a *QM-4 graph.* In view of Theorem 1.4, it suffices to prove that each vertex is an $a_2 - vertex$ in G with eccentricity 2. Consider the following two cases:

Case i: $p \equiv 3,0 \pmod 4$

We first claim that each vertex is an $a_2 - vertex$. Let v_i be an arbitrary vertex and let $v_j \in N_G(v_i)$. Then, $D_{i,j} \geq \lceil \frac{p}{4} \rceil + 1$. Without loss of generality we may assume that $i < j$.

Consider the following two cases.
Case 1: $j - i \leq \lfloor \frac{p}{2} \rfloor$
Consider the vertex $v_{j-\lceil \frac{p}{4} \rceil}$. Obviously $D_{j,j-\lceil \frac{p}{4} \rceil} = \lceil \frac{p}{4} \rceil$. Further, $j - i \leq \lfloor \frac{p}{2} \rfloor$ implies that

$$\begin{aligned}(j - \lceil \tfrac{p}{4} \rceil) - i &\leq \lfloor \tfrac{p}{2} \rfloor - \lceil \tfrac{p}{4} \rceil \\ &= \begin{cases} (2\lceil \frac{p}{4} \rceil - 1) - \lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 3 \pmod 4 \\ 2\lceil \frac{p}{4} \rceil - \lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 3 \pmod 4 \end{cases} \\ &\leq \lceil \tfrac{p}{4} \rceil,\end{aligned}$$

and $D_{i,j-\lceil \frac{p}{4} \rceil} = (j - \lceil \frac{p}{4} \rceil) - i \leq \lceil \frac{p}{4} \rceil$. Therefore, (2) implies that $v_{j-\lceil \frac{p}{4} \rceil}$ is non-adjacent to both v_i and v_j, and so v_i is an $a_2 - vertex$.
Case 2: $j - i > \lfloor \frac{p}{2} \rfloor$
If $j + \lceil \frac{p}{4} \rceil \leq p$, it is readily seen that $D_{j,j+\lceil \frac{p}{4} \rceil} = \lceil \frac{p}{4} \rceil$, and

$$\begin{aligned}D_{i,j+\lceil \frac{p}{4} \rceil} &= \min\{j + \lceil \tfrac{p}{4} \rceil - i, p - (j + \lceil \tfrac{p}{4} \rceil - i)\} \\ &= \min\{(j - i) + \lceil \tfrac{p}{4} \rceil, p - \big((j - i) + \lceil \tfrac{p}{4} \rceil\big)\}.\end{aligned}$$

Since $(j - i) + \lceil \frac{p}{4} \rceil - \Big(p - \big((j - i) + \lceil \frac{p}{4} \rceil\big)\Big)$

$$\begin{aligned}&= 2(j - i) + 2\lceil \tfrac{p}{4} \rceil - p \\ &> 2\lfloor \tfrac{p}{2} \rfloor + 2\lceil \tfrac{p}{4} \rceil - p \\ &= \begin{cases} (4\lceil \frac{p}{4} \rceil - 2) + 2\lceil \frac{p}{4} \rceil - (4\lceil \frac{p}{4} \rceil - 1) & \text{if} \quad p \equiv 3 \pmod 4 \\ 4\lceil \frac{p}{4} \rceil + 2\lceil \frac{p}{4} \rceil - 4\lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 0 \pmod 4 \end{cases} \\ &= \begin{cases} 2\lceil \frac{p}{4} \rceil - 1 & \text{if} \quad p \equiv 3 \pmod 4 \\ 2\lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 0 \pmod 4 \end{cases} \\ &> 0,\end{aligned}$$

we have

$$\begin{aligned}D_{i,j+\lceil \frac{p}{4} \rceil} &= p - \big((j - i) + \lceil \tfrac{p}{4} \rceil\big) \\ &= p - (j - i) - \lceil \tfrac{p}{4} \rceil \\ &< p - \lfloor \tfrac{p}{2} \rfloor - \lceil \tfrac{p}{4} \rceil \\ &= \begin{cases} (4\lceil \frac{p}{4} \rceil - 1) - (2\lceil \frac{p}{4} \rceil - 1) - \lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 3 \pmod 4 \\ 4\lceil \frac{p}{4} \rceil - 2\lceil \frac{p}{4} \rceil - \lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 0 \pmod 4 \end{cases} \\ &= \lceil \tfrac{p}{4} \rceil.\end{aligned}$$

Therefore, (2) implies that $v_{j+\lceil \frac{p}{4} \rceil}$ is non-adjacent with v_i and v_j. On the other hand, if $j + \lceil \frac{p}{4} \rceil > p$, consider the vertex $v_{j+\lceil \frac{p}{4} \rceil - p}$. Now,

$$\begin{aligned}D_{j,j+\lceil \frac{p}{4} \rceil - p} &= \min\{j - (j + \lceil \tfrac{p}{4} \rceil - p), p - \big(j - (j + \lceil \tfrac{p}{4} \rceil - p)\big)\} \\ &= \min\{p - \lceil \tfrac{p}{4} \rceil, \lceil \tfrac{p}{4} \rceil\} \\ &= \lceil \tfrac{p}{4} \rceil\end{aligned}$$

and

$$\begin{aligned} D_{i,j+\lceil \frac{p}{4}\rceil -p} &= \min\{i-(j+\lceil \tfrac{p}{4}\rceil -p), p-\big(i-(j+\lceil \tfrac{p}{4}\rceil -p)\big)\} \\ &= \min\{p-\big((j-i)+\lceil \tfrac{p}{4}\rceil\big), (j-i)+\lceil \tfrac{p}{4}\rceil\} \\ &< \lceil \tfrac{p}{4}\rceil \quad \text{as in the calculation of } D_{i,j+\lceil \frac{p}{4}\rceil}. \end{aligned}$$

It follows that, $v_{j+\lceil \frac{p}{4}\rceil -p}$ is non-adjacent with v_i and v_j, and so v_i is an $a_2-vertex$. Thus, as we claimed each vertex of G is an $a_2-vertex$ for $p \equiv 0,3 \pmod 4$.

We now claim that each vertex of the graph G is of eccentricity 2. Let v_i be an arbitrary vertex of G and let v_j be any vertex non-adjacent to v_i. With no loss of generality assume that $i<j$. From inequalities (2), we have

$$1 \le D_{i,j} \le \lceil \tfrac{p}{4}\rceil.$$

That is,

$$1 \le \min\{j-i, p-(j-i)\} \le \lceil \tfrac{p}{4}\rceil.$$

This implies that

$$1 \le j-i \le \lceil \tfrac{p}{4}\rceil \quad \text{or} \quad 1 \le p-(j-i) \le \lceil \tfrac{p}{4}\rceil.$$

That is,

$$1 \le j-i \le \lceil \tfrac{p}{4}\rceil \quad \text{or} \quad p-\lceil \tfrac{p}{4}\rceil \le j-i \le p-1.$$

Case 1: $1 \le j-i \le \lceil \frac{p}{4}\rceil$
If $j+\lceil \frac{p}{4}\rceil +1 \le p$, then consider the vertex $v_{j+\lceil \frac{p}{4}\rceil+1}$. It is clear that v_j and $v_{j+\lceil \frac{p}{4}\rceil+1}$ are adjacent. Now,

$$\begin{aligned} D_{i,j+\lceil \frac{p}{4}\rceil+1} &= \min\{j+\lceil \tfrac{p}{4}\rceil+1-i, p-(j+\lceil \tfrac{p}{4}\rceil+1-i)\} \\ &= \min\{(j-i)+\lceil \tfrac{p}{4}\rceil+1, p-(j-i)-\lceil \tfrac{p}{4}\rceil-1\}. \end{aligned}$$

We see that $(j-i)+\lceil \frac{p}{4}\rceil+1 > \lceil \frac{p}{4}\rceil+1$ and

$$\begin{aligned} p-(j-i)-\lceil \tfrac{p}{4}\rceil-1 &\ge p-\lceil \tfrac{p}{4}\rceil-\lceil \tfrac{p}{4}\rceil-1 \\ &= p-2\lceil \tfrac{p}{4}\rceil-1 \\ &= \begin{cases} (4\lceil \frac{p}{4}\rceil-1)-2\lceil \frac{p}{4}\rceil-1 & \text{if} \quad p \equiv 3 \pmod 4 \\ 4\lceil \frac{p}{4}\rceil-2\lceil \frac{p}{4}\rceil-1 & \text{if} \quad p \equiv 0 \pmod 4 \end{cases} \\ &= \begin{cases} 2\lceil \frac{p}{4}\rceil-2 & \text{if} \quad p \equiv 3 \pmod 4 \\ 2\lceil \frac{p}{4}\rceil-1 & \text{if} \quad p \equiv 0 \pmod 4. \end{cases} \end{aligned}$$

Since $\lceil \frac{p}{4}\rceil \ge 3$ for $p \equiv 3 \pmod 4$, and since $\lceil \frac{p}{4}\rceil \ge 2$ for $p \equiv 0 \pmod 4$, we have

$$p-(j-i)-\lceil \tfrac{p}{4}\rceil-1 \ge \lceil \tfrac{p}{4}\rceil+1.$$

Thus,

$$D_{i,j+\lceil \frac{p}{4}\rceil+1} \ge \lceil \tfrac{p}{4}\rceil+1.$$

It follows that $v_{j+\lceil \frac{p}{4}\rceil+1}$ is adjacent to both v_i and v_j, so that $e_G(v_i)=2$. On the other hand, if $j+\lceil \frac{p}{4}\rceil+1>p$ then consider the vertex $v_{j+\lceil \frac{p}{4}\rceil+1-p}$. Clearly, $D_{j,j+\lceil \frac{p}{4}\rceil+1-p} =$

$\lceil \frac{p}{4} \rceil + 1$. Now

$$\begin{aligned} D_{i,j+\lceil \frac{p}{4} \rceil+1-p} &= \min\{i-(j+\lceil \tfrac{p}{4} \rceil+1-p), p-\big(i-(j+\lceil \tfrac{p}{4} \rceil+1-p)\big)\} \\ &= \min\{p-(j-i)-\lceil \tfrac{p}{4} \rceil-1, (j-i)+\lceil \tfrac{p}{4} \rceil+1\} \\ &\geq \lceil \tfrac{p}{4} \rceil+1 \quad \text{as in the calculation of } D_{i,j+\lceil \frac{p}{4} \rceil+1}. \end{aligned}$$

Thus $v_{j+\lceil \frac{p}{4} \rceil+1-p}$ is adjacent to both v_i and v_j, and so $e_G(v_i) = 2$.

Case 2: $p - \lceil \frac{p}{4} \rceil \leq j - i \leq p - 1$

We observe that $D_{j,j-\lceil \frac{p}{4} \rceil-1} = \lceil \frac{p}{4} \rceil + 1$, and

$$\begin{aligned} D_{i,j-\lceil \frac{p}{4} \rceil-1} &= \min\{(j-\lceil \tfrac{p}{4} \rceil-1)-i, p-\big((j-\lceil \tfrac{p}{4} \rceil-1)-i\big)\} \\ &= \min\{(j-i)-(\lceil \tfrac{p}{4} \rceil+1), p-(j-i)+\lceil \tfrac{p}{4} \rceil+1\}. \end{aligned}$$

But, $p - (j - i) + \lceil \frac{p}{4} \rceil + 1 > \lceil \frac{p}{4} \rceil + 1$ and

$$\begin{aligned} (j-i)-(\lceil \tfrac{p}{4} \rceil+1) &\geq (p-\lceil \tfrac{p}{4} \rceil)-(\lceil \tfrac{p}{4} \rceil+1) \\ &= p-2\lceil \tfrac{p}{4} \rceil-1 \\ &\geq \lceil \tfrac{p}{4} \rceil+1 \quad (\text{as in the calculation of } D_{i,j+\lceil \frac{p}{4} \rceil+1}) \end{aligned}$$

imply that $D_{i,j-\lceil \frac{p}{4} \rceil-1} \geq \lceil \frac{p}{4} \rceil + 1$, and so $v_{j-\lceil \frac{p}{4} \rceil-1}$ is adjacent to both v_i and v_j. That is, $e_G(v_i) = 2$. Thus, as we claimed each vertex of G is of eccentricity 2 for $p \equiv 0, 3 \pmod 4$.

Case ii: $p \equiv 1$ or $2 \pmod 4$

In this case too, we first claim that each vertex is an $a_2 - vertex$. Let v_i be any vertex of G and let v_j be any vertex adjacent to v_i. Then, it is clear from (1) that $\lceil \frac{p}{4} \rceil + 1 \leq D_{i,j} \leq \lfloor \frac{p}{2} \rfloor$ or $D_{i,j} = 1$. With no loss of generality we may assume that $i < j$ and consider the following three cases.

Case 1: $j - i \leq \lfloor \frac{p}{2} \rfloor$ and $D_{i,j} \neq 1$

We see that, $D_{j,j-\lceil \frac{p}{4} \rceil+1} = \lceil \frac{p}{4} \rceil - 1$. The fact $j - i \leq \lfloor \frac{p}{2} \rfloor$ implies that

$$\begin{aligned} (j-\lceil \tfrac{p}{4} \rceil+1)-i &\leq \lfloor \tfrac{p}{2} \rfloor-\lceil \tfrac{p}{4} \rceil+1 \\ &= \begin{cases} (2\lceil \frac{p}{4} \rceil-2)-\lceil \frac{p}{4} \rceil+1 & \text{if} \quad p \equiv 1 \pmod 4 \\ (2\lceil \frac{p}{4} \rceil-1)-\lceil \frac{p}{4} \rceil+1 & \text{if} \quad p \equiv 2 \pmod 4 \end{cases} \\ &= \begin{cases} \lceil \frac{p}{4} \rceil-1 & \text{if} \quad p \equiv 1 \pmod 4 \\ \lceil \frac{p}{4} \rceil & \text{if} \quad p \equiv 2 \pmod 4 \end{cases} \\ &\leq \lceil \tfrac{p}{4} \rceil \end{aligned}$$

consequently, $D_{i,j-\lceil \frac{p}{4} \rceil+1} \leq \lceil \frac{p}{4} \rceil$. It follows that, the vertex $v_{j-\lceil \frac{p}{4} \rceil+1}$ is non-adjacent to both v_i and v_j. In this case v_i is an $a_2 - vertex$.

Case 2: $j - i > \lfloor \frac{p}{2} \rfloor$ and $D_{i,j} \neq 1$.

If $j + \lceil \frac{p}{4} \rceil - 1 \leq p$, then $D_{j,j+\lceil \frac{p}{4} \rceil-1} = \lceil \frac{p}{4} \rceil - 1$ and we can show that $D_{i,j+\lceil \frac{p}{4} \rceil-1} < \lceil \frac{p}{4} \rceil$. Thus, $v_{j+\lceil \frac{p}{4} \rceil-1}$ is non-adjacent to both v_i and v_j; that is, v_i is an $a_2 - vertex$. Similarly, if $j + \lceil \frac{p}{4} \rceil - 1 > p$ then the vertex $v_{j+\lceil \frac{p}{4} \rceil-1-p}$ is non-adjacent to both v_i and v_j, and so v_i is an $a_2 - vertex$.

Case 3: $D_{i,j} = 1$

Case 3.1: $j \notin \{p - 1, p\}$

We see that $D_{j,j+2} = 2 < \lceil \frac{p}{4} \rceil + 1$ and $D_{i,j+2} = 3 < \lceil \frac{p}{4} \rceil + 1$. It follows that v_{j+2} is non-adjacent to both v_i and v_j.

Case 3.2: $j \in \{p-1, p\}$

If $j = p$ and $i = 1$, we see that v_{j-2} is non-adjacent to both v_i and v_j. Otherwise, v_{i-2} is non-adjacent to v_i and v_j.

In either case v_i is an $a_2 - vertex$.

Therefore, as we claimed each vertex of G is an $a_2 - vertex$.

We now claim that $e_G(v) = 2$ for all vertex of G. Let v_i be an arbitrary vertex and let $v_j \notin N_G(v_i)$. Then, $2 \leq D_{i,j} \leq \lceil \frac{p}{4} \rceil$; that is,

$$2 \leq \min\{j-i, p-(j-i)\} \leq \lceil \tfrac{p}{4} \rceil.$$

Thus,

$$2 \leq j-i \leq \lceil \tfrac{p}{4} \rceil \qquad \text{or} \qquad 2 \leq p-(j-i) \leq \lceil \tfrac{p}{4} \rceil;$$

that is,

$$2 \leq j-i \leq \lceil \tfrac{p}{4} \rceil \qquad \text{or} \qquad p - \lceil \tfrac{p}{4} \rceil \leq j-i \leq p-2.$$

Case 1: $2 \leq j-i \leq \lceil \frac{p}{4} \rceil$

If $j - i = \lceil \frac{p}{4} \rceil$, then the vertex v_{i-1} or v_{j+1} is adjacent to both v_i and v_j. If $j-i = 2$, then v_{i+1} is adjacent to both v_i and v_j. Hence we may now assume that

$$3 \leq j-i \leq \lceil \tfrac{p}{4} \rceil - 1.$$

Note that from this point onwards $p \notin \{9, 10\}$. If $j + \lceil \frac{p}{4} \rceil + 1 \leq p$, then it is readily seen that

$$D_{j,j+\lceil \frac{p}{4} \rceil+1} = \lceil \tfrac{p}{4} \rceil + 1.$$

Since $D_{i,j+\lceil \frac{p}{4} \rceil+1} = \min\{j+\lceil \frac{p}{4} \rceil+1-i, p-(j+\lceil \frac{p}{4} \rceil+1-i)\}$, and since $(j-i)+\lceil \frac{p}{4} \rceil+1 > \lceil \frac{p}{4} \rceil$, $p - \big((j-i)+\lceil \frac{p}{4} \rceil+1\big) > \lceil \frac{p}{4} \rceil$ it follows that

$$D_{i,j+\lceil \frac{p}{4} \rceil+1} > \lceil \tfrac{p}{4} \rceil.$$

Thus $v_{j+\lceil \frac{p}{4} \rceil+1}$ is adjacent to both the vertices v_i and v_j, and so $e_G(v_i) = 2$. On the other hand, if $j + \lceil \frac{p}{4} \rceil + 1 > p$, then the vertex $v_{j+\lceil \frac{p}{4} \rceil+1-p}$ is adjacent to both v_i and v_j. Consequently $e_G(v_i) = 2$.

Case 2: $p - \lceil \frac{p}{4} \rceil \leq j-i \leq p-2$

If $j-i = p-2$, then the vertex v_{j+1} or v_{i-1} is adjacent to both v_i and v_j. If $j-i = p - \lceil \frac{p}{4} \rceil$, then the vertex v_{i+1} is adjacent to both v_i and v_j. Hence we may now assume that

$$p - \lceil \tfrac{p}{4} \rceil + 1 \leq j-i \leq p-3.$$

Note that from this point onwards $p \notin \{9, 10\}$. Consider the vertex $v_{j-\lceil \frac{p}{4} \rceil-1}$. Then, it is clear that

$$D_{j,j-\lceil \frac{p}{4} \rceil-1} = \lceil \tfrac{p}{4} \rceil + 1.$$

And, we can show that

$$D_{i,j-\lceil \frac{p}{4} \rceil-1} > \lceil \tfrac{p}{4} \rceil.$$

Thus, $v_{j-\lceil \frac{p}{4} \rceil-1}$ is adjacent to both v_i and v_j so that $e_G(v_i) = 2$. Therefore, in the case $p \equiv 1, 2 \pmod 4$ each vertex of the graph G is an $a_2 - vertex$ of eccentricity 2 as well.

Thus *QM-4 graph* and its complement are self-centered with radius 2.

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