TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 34-43

CRITICAL POINT APPROACHES TO SECOND-ORDER DIFFERENTIAL SYSTEMS GENERATED BY IMPULSES

M. OMIDI[1], A. SALARI[1], §

ABSTRACT. Using variational methods and critical point theory, we establish multiplicity results of solutions for second-order differential systems generated by impulses. Indeed, employing two sorts of three critical points theorems, we establish the multiplicity results for weak solutions of the problem and verify that these solutions are generated by impulses.

Keywords: Multiple solutions; second-order impulsive differential equation; Critical point theory; Variational methods

AMS Subject Classification: 235J40; 35J35.

The aim of this paper is to investigate the existence of two weak solutions for the following second-order impulsive differential equqtions;

$$\begin{cases} u''(t) + V_u(t,u(t)) = 0, & t \in (s_{k-1}, s_k), \\ \Delta u'(s_k) = \lambda f_k(u(s_k)), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1}$$

where s_k , $k = 1, 2, \ldots$, instants in which the impulses occur and $0 = s_0 < s_1 < s_2 ... < s_m < s_{m+1} = T$, and $\Delta u'(s_k) = u'({s_k}^+) - u'({s_k}^-)$ with $u'({s_k}^+) = \lim_{t \to {s_k}^+} u'(t)$, $f_k(u) = grad_u F_k(u)$, $F_k \in C^1(\mathbb{R}^N, \mathbb{R})$ such that $F_k(0, \ldots, 0) = 0$ $V \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$, $V_u(t, u(t)) = grad_u V(t,u)$, $\lambda > 0$ is a constant.

Impulsive differential equations emerge from the real world problems and are acclimated to be employed as handy means for the description of the processes which are endowed with abrupt discontinuous jumps. As for this, these processes are used in such a vast array of fields as control theory, biology, impact mechanics, physics, chemistry, chemical engineering, population dynamics, biotechnology, economics, optimization theory and inspection process in operations research. That's why, the theory of impulsive differential equations is now highly appreciated as a natural theoretical basis for the mathematical modeling of the natural phenomena of various kinds. For a comprehensive background in the theory and the applications of the impulsive differential equations, we hereby refer the interested readers to [2, 3, 13, 17, 21, 25].

[1] Department of Sciences, Kermanshah University of Technology, Kermanshah, Iran.
e-mail: m.omidi@kut.ac.ir, ORCID: https://orcid.org/0000-0002-5001-9610.
e-mail: amjads45@yahoo.com; ORCID: https://orcid.org/0000-0001-9170-9856.

§ Manuscript received: April 4, 2019; accepted: August 7, 2019.
TWMS Journal of Applied and Engineering Mathematics, Vol.11, No.1

There is already a large body of research on the notion of impulsive differential equations in the literature. The findings of most of these studies are mainly achieved through some such theories as fixed point theory, topological degree theory (including continuation method and coincidence degree theory) and comparison method (including upper and lower solutions method and monotone iterative method)(see, for example, [12, 14] and references therein). Recently, the existence and multiplicity of solutions for impulsive problems have been thoroughly investigated by [1, 16, 22, 23, 26, 27] using variational methods and the critical point theory, the whole findings of which can be considered as but generalizations of the corresponding ones for the second order ordinary differential equations. Put differently, the aforementioned achievements can be applied to impulsive systems in the absence of the impulses and still give the existence of solutions in this situation. This is, somehow, to say that the nonlinear term V_u functions more significantly as compared to the role played by the impulsive terms f_k in guaranteeing the existence of solutions in these results. Based on the variational methods and the critical point theory, [24] has examined problem (1), by means of which the authors have proved that such a problem admits at least one non-zero, two non-zeros, or an infinite number of periodic solutions as yielded by the impulses under different assumptions, respectively. Most particularly, in [11] using a smooth version of [8, Theorem 2.1] which is a more precise version of Ricceri's Variational Principle [20, Theorem 2.5] under some hypotheses on the behavior of the nonlinear terms at infinity, under conditions on the potentials of f_k and g_k, proved that the existence of definite intervals about λ and μ, in which problem (1) yields an unbounded sequence of solutions generated by impulses. Moreover, it has been proved that replacing the conditions at infinity of the nonlinear terms with a similar one at zero admits the same results.

In the present paper, employing two sorts of three critical points theorems obtained in [5, 7], which we will recall in the next section (Theorems 1.1 and 1.2), establish the multiplicity results for weak solutions of problem (1). We also verify that these solutions are generated by impulses; see Theorems 2.1 and 2.2. Along the same lines of reasoning, these theorems (Theorems 1.1 and 1.2) have been successfully employed by [6] to ensure the presence of at least three solutions for the perturbed boundary value problems.

The organization of the present paper is as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple solutions for the impulsive differential problem (1).

1. Preliminaries

Our main tools are Theorems 1.1 and 1.2, consequences of a local minimum theorem [4, Theorem 3.1] which is inspired by Ricceri's variational principle (see [20]).

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\vartheta(r_1, r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho_1(r_1, r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, and

$$\rho_2(r) = \sup_{v\in\Phi^{-1}(]r,\infty[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r \in \mathbb{R}$.

Theorem 1.1. [4, Theorem 5.1] *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that $\vartheta(r_1, r_2) < \rho_1(r_1, r_2)$. Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in]\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\vartheta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.*

Theorem 1.2. [4, Theorem 5.3] *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that $\rho_2(r) > 0$, and for each $\lambda > \frac{1}{\rho_2(r)}$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive. Then for each $\lambda \in]\frac{1}{\rho_2(r)}, +\infty[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r, +\infty[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ $\forall u \in \Phi^{-1}(]r, +\infty[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.*

For a thorough study on the subject, we also refer the reader to [9, 19].

In order to study the problem (1), the variational setting is the space

$$X := H^1_T = \{u : [0,T] \to \mathbb{R}^N | u \text{ is absolutly continuous,} \\ u(0) = u(T),\ \ u' \in L^2([0,T], \mathbb{R}^n)\}.$$

Then H^1_T is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_0^T \left[(u(t), v(t)) + (u'(t), v'(t))\right] \mathrm{d}t, \quad \forall u, v \in H^1_T,$$

where $(.,.)$ denotes the inner product in $\mathbb{R}^N$. The corresponding norm is

$$\|u\| = \left(\int_0^T [|u'(t)|^2 + |u(t)|^2]\mathrm{d}t\right)^{\frac{1}{2}}, \quad \forall u \in H^1_T. \tag{2}$$

Since $(H^1_T, \|.\|)$ is compactly embedded in $C([0,T], \mathbb{R}^N)$ (see [15]), there exists a positive constant C such that

$$\|u\|_\infty \le C\|u\|, \tag{3}$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$. we say that a functional $u \in X$ is a weak solution of the problem (1) if

$$\int_0^T \left[u'(t)v'(t) - V_u(t, u(t))v(t)\right] \mathrm{d}t + \lambda \sum_{k=1}^m f_k u(s_k)v(s_k) = 0$$

for every $v \in X$.

To study problem (1), we consider the functional I define by

$$I(u) = \int_0^T \left[\frac{1}{2}|u'|^2 - V(t, u)v\right] \mathrm{d}t + \lambda \sum_{k=1}^m F_k(u(s_k))v(s_k). \tag{4}$$

Lemma 1.1. *Suppose $V \in C^1([0,T]) \times \mathbb{R}^N \longrightarrow \mathbb{R}$ and $F_k \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $k = 1, 2, \dots m$ Then, I is Frechet differentiable with*

$$I'(u)v = \int_0^T \left[u'v' - V_u(t, u)v\right] \mathrm{d}t + \lambda \sum_{k=1}^m f_k(u(s_k))v(s_k) \tag{5}$$

for any u and v in H^1_T. Furthermore, u is a solution of (1) if and only if u is a critical point of I in H^1_T.

Now assume that

(A_1) V is a continuous differentiable and there exist positive constants $b_1, b_2 > 0$ such that $b_1|u|^2 < -V(t,u) \leq b_2|u|^2$ for all $(t,u) \in [0,T] \times \mathbb{R}^N$;

(A_1') V is continuous differentiable and there exist positive constant $b > 1$ and $\gamma \in (1,2]$ such that $-V(t,u) \geq b|u|^\gamma$ for all $(t,u) \in [0,T] \times \mathbb{R}^N$;

(A_2) $-V(t,u) \leq -V_u(t,u)u \leq -2V(t,u)$ for all $(t,u) \in [0.T] \times \mathbb{R}^N$;

And we consider the functionals Φ, $\Psi : X \to \mathbb{R}$ for each $u \in X$, as follows

$$\Phi(u) = \int_0^T \left[\frac{1}{2}|u'(t)|^2 - V(t,u(t))\right] \mathrm{d}t, \tag{6}$$

$$\Psi(u) = -\sum_{k=1}^{m} F_k\left(u(s_k)\right). \tag{7}$$

Obviously, $X = H_T^1$ is a separable and uniformly convex Banach space. By (A_1) and (2), we have

$$b_3\|u\|^2 \leq \Phi(u) \leq b_4\|u\|^2 \tag{8}$$

where $b_3 = \min\{\frac{1}{2}, b_1\}$ and $b_4 = \min\{\frac{1}{2}, b_2\}$. Hence Φ is coercive, and bounded on each bounded sub set of X. Moreover, since $V(t,u)$ is continuous, we see that Φ is continuous. Furthermore, Φ is sequentially weakly lower semicontinuous. Indeed, let u_n be a weakly convergent sequence to u in X. Then, $\liminf_{n\to+\infty} \|u_n\| \geq \|u\|$ and $u_n \to u$ uniformly on $[0,T]$. Hence

$$\lim_{n\to+\infty} \frac{1}{2}\int_0^T [|u_n'(t)|^2 - V(t,u_n(t))]\mathrm{d}t \geq \frac{1}{2}\int_0^T [|u'(t)|^2 - V(t,u(t))]\mathrm{d}t,$$

namely $\liminf_{n\to+\infty} \Phi(u_n) \geq \Phi(u)$ which means Φ is sequentially weakly lower semicontinuous.

It is well Known that Ψ is a Gâteaux differentiable functional and sequentially weakly upper semicontinuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\langle \Psi'(u), v\rangle = -\sum_{k=1}^{m} f_k(u(s_k))v(s_k). \tag{9}$$

Let $u_n \longrightarrow u \in X$, then $u_n \longrightarrow u \in C[0,T]$. Hence $\Psi'(u_n) \longrightarrow \Psi'(u)$ as $n \longrightarrow \infty$ therefore, we have that Ψ' is a compact operator. Moreover, Φ is a Gateaux differentiable functional whose Gateaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\langle \Phi'(u), v\rangle = \int_0^T |\left(u'(t), v'(t)\right) - \left(V_u(t,u(t)), v(t)\right)|\mathrm{d}t. \tag{10}$$

for every $v \in X$.

We need the following proposition in the proof of Theorem 2.1.

Proposition 1.1. *The operator $T : X \to X^*$ defined by*

$$\langle T(u), v\rangle = \int_0^T |\left(u'(t), v'(t)\right) - \left(V_u(t,u(t)), v(t)\right)|\mathrm{d}t.$$

for every $v \in X$ admits a continuous inverse on X^*.

Proof. We have

$$\begin{aligned}\langle \Phi'(u), u\rangle &= \int_0^T |\big(u'(t), u'(t)\big) - (V_u(t, u(t)), u(t))\,|\mathrm{d}t \\ &\geq \int_0^T \left[(u'(t), u'(t)) + b_1|u(t)|^2\right]\mathrm{d}t \geq \min\{1, b_1\}\|u\|^2.\end{aligned}$$

So $\lim_{u\longrightarrow\infty} \frac{\langle\Phi'(u),u\rangle}{\|u\|} = +\infty$, that is, Φ' is coercive. For any $u, v \in X$ one has

$$\begin{aligned}&\langle \Phi'(u) - \Phi'(v), u - v\rangle = \int_0^T (u'(t) - v'(t), u'(t) - v'(t))dt \\ &- \int_0^T (V_u(t, u(t)) - V_v(t, v(t)), u(t) - v(t))\mathrm{d}t \\ &\geq \int_0^T (u'(t) - v'(t)), u'(t) - v'(t))dt + \int_0^T b_1|u(t) - v(t)|^2\mathrm{d}t \\ &\geq \min\{1, b_1\}\|u - v\|^2.\end{aligned}$$

So Φ' is uniformly monotone. Hence, $(\Phi')^{-1}$ there exists and is continuous on X^*. □

Corresponding to f_k we introduce the function $F_k \in C^1(\mathbb{R}^N, \mathbb{R})$, where

$$f_k = grad_u F_k, \quad \forall\ k = 1, 2, ...m.$$

Remark 1.1. *We say that a solution of the problem* (1) *is called a solution generated by impulses if this solution is nontrivial when impulsive terms $f_k \neq 0$ for some $1 \leq k \leq m$, but it is trivial when impulsive term is zero. For example, if the problem* (1) *does not possess non-zero weak solution when $f_k \equiv 0$ for all $1 \leq k \leq m$, then a non-zero weak solution for problem* (1) *with $f_k \neq 0$ for some $1 \leq k \leq m$ is called a weak solution generated by impulses.*

2. Main results

In order to introduce our result, for a given non-negative constant θ and a given positive constant η with $\frac{\theta^2 b_3}{C^2} \neq \min\{T\eta^2, bC^{\gamma-2}(\sqrt{T}\eta)^\gamma\}$, put

$$a_\eta(\theta) := \frac{\max_{|t|\leq\theta}\left[-\sum_{k=1}^m F_k(t) - F_k(\eta)\right]}{\min\{T\eta^2, bC^{\gamma-2}(\sqrt{T}\eta)^\gamma\} - \frac{\theta^2 b_3}{C^2}},$$

and

$$b_\eta(\theta) := \frac{\max_{|t|\leq\theta}\left[-\sum_{k=1}^m F_k(t) - F_k(\eta)\right]}{\frac{\theta^2 b_3}{C^2} - b_4 T\eta^2}$$

where C is given in (3). Now, we formulate our main result as follows.

Theorem 2.1. *Assume that $f_k \neq 0$ for some $1 \leq k \leq m$ and there exist a non-negative constant θ_1 and two positive constants θ_2 and η with $\frac{\theta_1}{C}\sqrt{\frac{1}{T}} < \eta < \frac{\theta_2}{C}\sqrt{\frac{b_3}{Tb_4}}$ such that*

(B_1) *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu F_k(t) \leq t f_k(t)), \quad \textit{for all } |t| \geq R \textit{ and for all } k = 1, \ldots, m;$$

(B_2) $b_\eta(\theta_2) < a_\eta(\theta_1)$.

Then, for each $\lambda \in \left]\frac{1}{a_\eta(\theta_1)}, \frac{1}{b_\eta(\theta_2)}\right[$ the problem (1) *admits at least two non-trivial weak solutions $u_1, u_2 \in X$ generated by impulses such that $\frac{\theta_1^2 b_3}{b_4 C^2} < \|u_1\| < \frac{\theta_2^2}{C^2}$.*

Proof. In order to apply Theorem 1.1 to our problem, we introduce the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ for each $u \in X$, as (6) and (7). Let us prove that the functionals Φ and Ψ satisfy the required conditions. It is clear that, Ψ is a sequentially weakly upper semicontinuous and differentiable functional, whose differential at the point $u \in X$ obtain in (9). we proved that Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ obtain in (10), while Proposition 1.1 gives that Φ' admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Clearly, the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$. Put $r_1 := \frac{\theta_1^2 b_3}{C^2}$ and $r_2 := \frac{\theta_2^2 b_3}{C^2}$, and

$$w := (\eta, 0, \dots, 0). \tag{11}$$

It is easy to see that $w \in X$ and, in particular, one has $\|w\|^2 = T\eta^2$. Taking into account $\frac{\theta_1}{C}\sqrt{\frac{1}{T}} < \eta < \frac{\theta_2}{C}\sqrt{\frac{b_3}{Tb_4}}$, using (8), we observe that $r_1 < \Phi(w) < r_2$. Now taking A_2' into account, it follows that, for each $u \in X$,

$$\begin{aligned}
\Phi(u) &= \int_0^T \left[|u'|^2 - V_u(t,u)u\right] \mathrm{d}t \\
&\geq \int_0^T \left[|u'|^2 + b_1|u|^\gamma\right] \mathrm{d}t \\
&\geq \int_0^T |u'|^2 \mathrm{d}t + b_1\|u\|_\infty^{\gamma-2} \int_0^T |u|^2 \mathrm{d}t \\
&\geq \int_0^T |u'|^2 \mathrm{d}t + b_1 C^{\gamma-2}\|u\|^{\gamma-2} \int_0^T |u|^2 \mathrm{d}t \\
&\geq \min\{1, b_1 C^{\gamma-2}\|u\|^{\gamma-2}\}\|u\|^2 = \min\{\|u\|^2, b_1 C^{\gamma-2}\|u\|^\gamma\}
\end{aligned}$$

and we have

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r])} \left[-\sum_{k=1}^m F_k\left(u(s_k)\right)\right] \leq \max_{|t|\leq\theta} \left[-\sum_{k=1}^m F_k(t)\right].$$

On the other hand since $w(x) = (\eta, 0, \dots, 0)$, we infer $\Psi(w) = -\sum_{k=1}^m F_k(\eta)$. Therefore, one has

$$\begin{aligned}
\vartheta(r_1, r_2) \leq& \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\
\leq& \frac{\max_{|t|\leq\theta_2}\left[-\sum_{k=1}^m F_k(t)\right] - \Psi(w)}{r_2 - \Phi(w)} \\
\leq& \frac{\max_{|t|\leq\theta_2}\left[-\sum_{k=1}^m F_k(t) - F_k(\eta)\right]}{\frac{\theta_2^2 b_3}{C^2} - b^4 T\eta^2} = b_\eta(\theta_2).
\end{aligned}$$

On the other hand, arguing as before, one has

$$\begin{aligned}
\rho_1(r_1, r_2) \geq& \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{\Phi(w) - r_1} \\
\geq& \frac{\Psi(w) - \max_{|t|\leq\theta_1}\left[-\sum_{k=1}^m F_k(t)\right]}{\Phi(w) - r_1} \\
\geq& \frac{\max_{|t|\leq\theta_1}\left[-\sum_{k=1}^m F_k(t) - F_k(\eta)\right]}{\min\{T\eta^2, bC^{\gamma-2}(\sqrt{T}\eta)^\gamma\} - \frac{\theta_1^2 b_3}{C^2}} = a_\eta(\theta_1).
\end{aligned}$$

Hence, from Assumption (B_2), one has $\vartheta(r_1, r_2) < \rho_1(r_1, r_2)$. Therefore, applying Theorem 1.1, for each $\lambda \in \left]\frac{1}{a_\eta(\theta_1)}, \frac{1}{b_\eta(\theta_2)}\right[$, the functional $\Phi - \lambda\Psi$ admits at least one critical point $u_1 \in X$ such that $r_1 < \Phi(u_1) < r_2$, that is $\frac{\theta_1^2 b_3}{b_4 C^2} < \|u_1\| < \frac{\theta_2^2}{C^2}$.
Now we prove the existence of the second local minimum distinct from the first one. To this purpose, we verify the hypotheses of the mountain pass theorem for the functional $\Phi - \lambda\Psi$.
Clearly the functional $\Phi - \lambda\Psi$ is of class C^1 and $(\Phi - \lambda\Psi)(0, \dots, 0) = 0$. The first part of proof guarantees that $u_1 \in X$ is a local nontrivial local minimum for $\Phi - \lambda\Psi$ in X. Therefore there is $s > 0$ such that

$$\inf_{||u-u_1||=s} (\Phi - \lambda\Psi)(u) > (\Phi - \lambda\Psi)(u_1).$$

So the condition [18, (I_1), Theorem 2.2] is verified. Now choosing $u \neq 0$, from (B_1) condition, there exist constants a_1 and a_2 such that $F_k(t) \geq a_1|t|^\nu + a_2$ for all $t \in \mathbb{R}^n$. Now, choosing any $u \in X\backslash\{0, \dots, 0\}$, one has

$$\begin{aligned}(\Phi - \lambda\Psi)(tu) &= \int_0^T \left[|tu'|^2 - V_u(t, tu)tu\right] \mathrm{d}t + \lambda \sum_{k=1}^m F_k\left(tu(s_k)\right) \\ &\leq b_4 t^2\|u\|^2 - \lambda a_1 t^\nu \int_0^T |u|^\nu \mathrm{d}x - Ta_2 \longrightarrow -\infty\end{aligned}$$

as $t \to \infty$, since $\nu > 2$. So the condition [18, (I_2), Theorem 2.2] is verified. Moreover, by standard computations $\Phi - \lambda\Psi$ satisfies (PS) condition. Hence the classical theorem of Ambrosetti and Rabinowitz ensures a critical point u_2 of $\Phi - \lambda\Psi$ such that $(\Phi - \lambda\Psi)(u_2) > (\Phi - \lambda\Psi)(u_1)$. So by Remark 1.1 u_1 and u_2 are distinct weak solutions generated by impulses of the problem (1). □

An immediate consequence of Theorem 2.1 is the following result.

Corollary 2.1. *Assume that $f_k \neq 0$ for some $1 \leq k \leq m$ and there exist two positive constants η and θ with $0 < \eta < \theta$ and*

$$\frac{\theta^2 b_3}{TC^2(b^4 - 1)} < \eta^2 \leq \frac{bC^{\gamma-2}\eta^\gamma}{T^{1-\frac{\gamma}{2}}}$$

such that Assumption (B1) in Theorem 2.1 hold. Then, for every

$$\lambda \in \left]\frac{\frac{\theta^2 b_3}{C^2} - b^4 T\eta^2}{\max_{|t|\leq\theta}\left[-\sum_{k=1}^m F_k(t) - F_k(\eta)\right]}, \frac{T\eta^2}{-\sum_{k=1}^m F_k(\eta)}\right[$$

the problem (1) *admits at least two non-trivial weak solutions u_1, $u_2 \in X$ generated by impulses.*

Proof. The conclusion follows from Theorem 2.1 by taking $\theta_1 = 0$ and $\theta_2 = \theta$. Indeed, owing to our assumptions, one has

$$\begin{aligned} b_\eta(\theta) =& \frac{\max_{|t|\le\theta}\left[-\sum_{k=1}^m (F_k(t) - F_k(\eta))\right]}{\frac{\theta^2 b_3}{C^2} - b^4 T\eta^2} \\ \le& \frac{-\sum_{k=1}^m (F_k(0,\dots,0) - F_k(\eta))}{\frac{\theta^2 b_3}{C^2} - b^4 T\eta^2} \\ =& \frac{-\sum_{k=1}^m F_k(\eta)}{b^4 T\eta^2 - \frac{\theta^2 b_3}{C^2}} \\ \le& \frac{-\sum_{k=1}^m F_k(\eta)}{T\eta^2} \\ \le& \frac{-\sum_{k=1}^m F_k(\eta)}{\min\{T\eta^2, bC^{\gamma-2}(\sqrt{T}\eta)^\gamma\}} = a_\eta(0). \end{aligned}$$

Hence, Theorem 2.1 ensures the desired conclusion. □

Now, we present an application of Theorem 1.2 which we will use to obtain multiple solutions.

Theorem 2.2. *Assume that $f_k \neq 0$ for some $1 \le k \le m$ and there exist two positive constants η and θ with $0 < \eta < \theta$ and $\eta^2 \le \frac{bC^{\gamma-2}\eta^\gamma}{T^{1-\frac{\gamma}{2}}}$ such that Assumption (B_1) in Theorem 2.1 holds. Furthermore, suppose that*

$(B_1)'$ $\max_{|t|\le\theta}\left[-\sum_{k=1}^m F_k(t)\right] \le \sum_{k=1}^m \left[-F_k(\eta)\right]$;

$(B_2)'$ $\limsup_{|t|\to+\infty} \dfrac{\sup_{k=1}^m[-F_k(t)]}{|t|^2} \le 0.$

Then, for each

$$\lambda \in \left]\frac{\min\{T\eta^2, bC^{\gamma-2}(\sqrt{T}\eta)^\gamma\} - \frac{\theta^2 b_3}{C^2}}{\max_{|t|\le\theta}\left[-\sum_{k=1}^m (F_k(t) - F_k(\eta))\right]}, +\infty\right[,$$

the problem (1) *admits at least one non-trivial weak solution $\bar{u} \in X$ generated by impulses such that $\|\bar{u}\| > \frac{\theta}{C}$.*

Proof. The functionals Φ and Ψ satisfy all regularity assumptions requested in Theorem 1.2. Fix $0 < \epsilon < \dfrac{b_3}{C^2\lambda}$. From $(B2)'$ there are constants h_k such that

$$-F_k(t) \le \epsilon|t|^2 + h_k, \tag{12}$$

for every $k = 1, \dots, m$. Now we have, $\Phi(u) \ge \min\{\|u\|^2, bC^{\gamma-2}\|u\|^\gamma\}$ then,

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &= \int_0^T \left[\frac{1}{2}|u'|^2 - V(t,u)\right] dt - \lambda\left[-\sum_{k=1}^m [F_k(u(s_k))]\right] \\ &\ge \min\{\|u\|^2, bC^{\gamma-2}\|u\|^\gamma\} - \lambda\epsilon\sum_{k=1}^m |u|^2 - \sum_{k=1}^m h_k \\ &\ge \min\{\|u\|^2, bC^{\gamma-2}\|u\|^\gamma\} - \frac{\lambda C^2\epsilon}{b_3}\|u\|^2 - \lambda\sum_{k=1}^m h_k, \end{aligned}$$

and thus $\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$, which means the functional $\Phi - \lambda\Psi$ is coercive. Put $r := \frac{b_3\theta^2}{C^2}$ and choose w as given in (11). Using the same arguments as in the proof of

Theorem 2.1, one has

$$\rho_2(r) \geq \frac{\max_{|t|\leq\theta}\left[-\sum_{k=1}^{m}(F_k(t) - F_k(t))\right]}{\min\{T\eta^2, bC^{\gamma-2}(\sqrt{T}\eta)^\gamma\} - \frac{\theta^2 b_3}{C^2}}.$$

So, from our assumptions, it follows that $\rho_2(r) > 0$. Hence from the Theorem 1.2 and Remark 1.1 the functional $\Phi - \lambda\Psi$ admits at least one local minimum $\bar{u} \in X$ generated by impulses such that $\|\bar{u}\| \geq \frac{\theta}{C}$ and our conclusion is achieved. □

Example 2.1. *we consider the problem:*

$$\begin{cases} u''(t) + V_u(t, u(t)) = 0, \qquad t \in (0,1), \\ \Delta u'(s_k) = \lambda f_1(u), \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{13}$$

that $V(t,u) = -|u|^2$ *and* $V_u(t,u) = grad_u V(t,u)$. *It is easy to see that conditions* (A_1) *,*$(A_1)'$ *and* (A_2) *hold. now let* $T = 1$, $m = 1$ *and*

$$F_1(u) = \begin{cases} 2|u|^3, & |u| < 1, \\ 6|u| - 4, & |u| \geq 1. \end{cases} \tag{14}$$

then F *is* C^1 *function with* $f_1(u) = grad_u F_1(u)$. *In this example one has* $b = b_1 = b_3 = b_4 = \frac{1}{2}$ *and* $b_2 = 1$ *and we can consider* $C = \sqrt{2}$. *we let* $\eta = 0$, $\theta = 1$ *then by theorem (3.1) for each* $\lambda \in (\frac{1}{8}, +\infty)$ *the problem (13) admits at least two nan-trivial weak solutions* $u_1, u_2 \in X$ *generated by impulses such that* $\|u_1\| = \frac{1}{\sqrt{2}}$.

References

[1] L. Bai, B. Dai, Application of variational method to a class of Dirichlet boundary value problems with impulsive effects, J. Franklin Inst. 348 (2011) 2607-2624.

[2] D. Bainov, P. Simeonov, Systems with Impulse Effect, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester (1989).

[3] M. Benchohra, J. Henderson, S. Ntouyas, Theory of Impulsive Differential Equations, Contemporary Mathematics and Its Applications, 2. Hindawi Publishing Corporation, New York (2006).

[4] G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal. TMA 75 (2012) 2992-3007.

[5] G. Bonanno, P. Candito, Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, J. Differ. Eqs. 244 (2008) 3031-3059.

[6] G. Bonanno, A. Chinnì, Existence of three solutions for a perturbed two-point boundary value problem, Appl. Math. Lett. 23 (2010) 807-811.

[7] G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Appl. Anal. 89 (2010) 1-10.

[8] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, Bound. Value Probl. 2009 (2009) 1-20.

[9] M. Ferrara, G. Molica Bisci, D. Repovš, Existence results for nonlinear elliptic problems on fractal domains, Adv. Nonlinear Anal. 5 (2016) 75-84.

[10] P. K. George, A. K. Nandakumaran, A. Arapostathis, A note on controllability of impulsive systems, J. Math. Anal. Appl. 241 (2000) 276-283.

[11] S. Heidarkhani, M. Ferrara, A. Salari, Infinitely many periodic solutions for a class of perturbed second-order differential equations with impulses, Acta Appl. Math. 139 (2015) 81-94.

[12] E. Hernandez, H. R. Henriquez, M. A. Mckibben, Existence results for abstract impulsive second-order neutral functional differential equations, Nonlinear Anal. TMA 70 (2009) 2736-2751.

[13] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, Impulsive differential equations and inclusions, World Scientific, Singapore (1989).

[14] X. N. Lin, D. Q. Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, J. Math. Anal. Appl. 321 (2006)501-514.

[15] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, Berlin (1989).

[16] J. J. Nieto, D. O'Regan, Variational approach to impulsive differential equations, Nonlinear Anal. RWA 10 (2009) 680-690.
[17] J. J. Nieto, R. Rodríguez-López, Boundary value problems for a class of impulsive functional equations, Comput. Math. Appl. 55 (2008) 2715-2731.
[18] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65. Amer. Math. Soc., Providence (1986)
[19] V. Rădulescu, D. Repovš, Combined effects in nonlinear problems arising in the study of anisotropic continuous media, Nonlinear Anal. TMA 75 (2012) 1524-1530.
[20] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401-410.
[21] A. M. Samoilenko, N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore (1995).
[22] J. Sun, H. Chen, J. J. Nieto, M. Otero-Novoa, The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects, Nonlinear Anal. TMA 72 (2010) 4575-4586.
[23] Y. Tian, W. Ge, Applications of variational methods to boundary value problem for impulsive differential equations, Proc. Edinb. Math. Soc. 51 (2008) 509-527.
[24] L. Yang, H. Chen, Existence and multiplicity of periodic solutions generated by impulses, Abstr. Appl. Anal. 2011 (2011) Article ID 310957, 15 pages.
[25] S. T. Zavalishchin, A. N. Sesekin, Dynamic Impulse System, Theory and Applications, Kluwer Academic, Dordrecht (1997).
[26] D. Zhang, B. Dai, Existence of solutions for nonlinear impulsive differential equations with Dirichlet boundary conditions, Math. Comput. Modelling 53 (2011) 1154-1161.
[27] J. Zhou, Y. Li, Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, Nonlinear Anal. TMA 71 (2009) 2856-2865.

Mohammad Reza Omidi received his B.S. degree from Razi University of Kermanshah, M.S. degree from Kharazmi University of Tehran and Ph.D. degree from Kharazmi University of Tehran. He is an assistant professor in the Department of Basic Sciences, Kermanshah University of Technology, Iran. His area of interest includes Functional Analysis, Automatic continuity and Nonlinear Analysis.

Amjad Salari is a post-doctorate researcher of mathematics at the University of Mazandaran, Babolsar/Iran. He is a lecturer of mathematics in the Department of Mathematics at Razi University and Kermanshah University of Technology, Kermanshah/Iran. He received his PhD degree in mathematical analysis from Razi University, Kermanshah/Iran (2017). His research interests include Nonlinear Analysis, Differential Equations, Applied Mathematics, Fractional Differential Equations, Impulsive Systems and Bio Math.