TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 89-100

IMPROVEMENTS ON SOME INEQUALITIES OF HERMITE HADAMARD INEQUALITIES FOR FUNCTIONS WHEN A POWER OF THE ABSOLUTE VALUE OF THE SECOND DERIVATIVE h AND P-CONVEX

ERDAL ÜNLÜYOL[1], SEREN SALAŞ[2], GÖRKEM DALKUN[2], §

Abstract. In this paper, firstly we obtain some improvements of Hermite-Hadamard integral inequalities via h and P-convex by using Hölder-İşcan inequality. Secondly new results are established. Thirdly, we determine some new inequalities for functions when a power of the absolute value of second derivatives are h and P-convex. Finally they are compared with the old ones.

Keywords:Hölder-İşcan integral inequality, improved power-mean integral inequality, Hermite Hadamard integral inequality, h and P-convexity.

AMS Subject Classification: 26A51, 26D10, 26D15

1. Introduction

The famous Young inequality for two scalars is the t-weighted arithmetic-geometric means inequality. This inequality says that if $x, y > 0$ and $t \in [0,1]$, then

$$x^t y^{1-t} \leq tx + (1-t)y \tag{1}$$

with equality if and only if $x = y$.

Definition 1.1. *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval I.

[1] Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey.
eunluyol@yahoo.com; ORCID: https://orcid.org/0000-0003-3465-6473.
[2] Department of Mathematics, Institute of Science, Ordu University, Ordu, Turkey.
serensalas@gmail.com; ORCID: https://orcid.org/0000-0003-4109-5635.
gdalkun52@hotmail.com; ORCID: https://orcid.org/0000-0002-7000-5443.
§ Manuscript received: April 10, 2019; accepted: July 16, 2019.

Definition 1.2. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. The following inequality*

$$f\Big(\frac{a+b}{2}\Big) \leq \int_0^1 f(ta+(1-t)b)dt \leq \frac{f(a)+f(b)}{2} \tag{2}$$

holds. This double inequality is known as Hermite-Hadamard integral inequality for convex functions in the literature. You can see ([3]-[6]), for the results of the generalization, improvement and extension of the famous integral inequality (2).

Definition 1.3. [7] *Let I, J be intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and $h : J \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. A non-negative function $f : I \longrightarrow \mathbb{R}$ is called h-convex (or that $f \in SX(h, I)$), if for all $x, y \in I$ and $t \in (0,1)$:*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

If the inequality is reversed then f is said to be h-concave and in this case f belongs to the class $SV(h, I)$.

Definition 1.4. [8] *Let $I \subseteq \mathbb{R}$ be an interval. The function $f : I \longrightarrow \mathbb{R}$ is said to be P-convex(or belong to the class $P(I)$) if it is non-negative function and, for all $x, y \in I$ and $t \in (0,1)$, satisfies the inequality*

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

Lemma 1.1. [10, Lemma 1] *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a twice differentiable mapping on $I^o, a, b \in I$ (I^o is interior of I) with $a < b$ and $f^{''} \in L^1([a,b])$, then the following equality holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \qquad (3)\\ = \;& \frac{(b-a)^2}{16}\int_a^b (1-t^2)\Big[f''\Big(\frac{1+t}{2}a + \frac{1-t}{2}b\Big) + f''\Big(\frac{1-t}{2}a + \frac{1+t}{2}b\Big)\Big]dt. \end{aligned}$$

Theorem 1.1. [11, Theorem 5] *Let $a, b \in I$ with $a < b$ and $f \in L^1([a,b])$. If $f \in SX(h, I)$ for $0 < t < 1$ then*

$$\frac{1}{2h\Big(\frac{1}{2}\Big)} f\Big(\frac{a+b}{2}\Big) \leq \int_0^1 f(ta+(1-t)b)dt \leq [f(a)+f(b)]\int_0^1 h(t)dt. \tag{4}$$

Theorem 1.2. [11, Theorem 7] *Let $f : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a twice differentiable function on I^o, such that $f^{''} \in L^1([a,b])$, where $a, b \in I^o$ with $a < b$. If $|f^{''}|^q$ for $q > 1$ with $p = \frac{q}{q-1}$ is h-convex on $[a,b]$, then for some fixed $t \in (0,1)$ the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16.2^{\frac{1}{p}}}\beta^{\frac{1}{p}}\Big(\frac{1}{2}, p+1\Big)\Big(\int_0^1 h(t)dt\Big)^{\frac{1}{q}} \times \qquad (5)\\ &\left[\left(|f''(a)|^q + \left|f''\Big(\frac{a+b}{2}\Big)\right|^q\right)^{\frac{1}{q}} + \left(|f''(b)|^q + \left|f''\Big(\frac{a+b}{2}\Big)\right|^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Theorem 1.3. [11, Theorem 8] *Let $f : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a twice differentiable function on I^o, such that $f^{''} \in L^1([a,b])$, where $a, b \in I^o$ with $a < b$. If $|f^{''}|^q$ for $q > 1$ be an*

h-convex on $[a,b]$ for some fixed $t \in (0,1)$ the following inequality holds:

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (6)\\ \leq\ & \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{\frac{1}{p}}\left[\left(\int_0^1 \left\{(1-t^2)|f''(a)|^q + t(2-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}h(t)dt\right)^{\frac{1}{q}}\right.\\ +\ & \left.\left(\int_0^1 \left\{(1-t^2)|f''(b)|^q + t(2-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}h(t)dt\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Theorem 1.4. [11, Theorem 9] *Let $f : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a twice differentiable function on I^o, such that $f'' \in L^1([a,b])$, where $a,b \in I^o$ with $a < b$: If $|f''|^q$ for $q > 1$ be an h-concave on $[a,b]$ for some fixed $t \in (0,1)$ the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} + \frac{a+b}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (7)\\ \leq\ & \frac{(b-a)^2}{32}\left(\beta\left(\frac{1}{2},p+1\right)\right)^{\frac{1}{p}}\left(\frac{1}{h(\frac{1}{2})}\right)^{\frac{1}{q}}\left[\left|f''\left(\frac{a+3b}{4}\right)\right|^q + \left|f''\left(\frac{3a+b}{4}\right)\right|^q\right]. \end{aligned}$$

Theorem 1.5. [9, Theorem 2.4] *Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function on I^o. Assume that $p \in \mathbb{R}$, $p > 1$ such that $|f''|^{p/(p-1)}$ is a P-convex on I. Suppose that $a,b \in I^o$ with $a < b$ and $f'' \in L^1([a,b])$. Then the following inequality holds for $\frac{1}{q} + \frac{1}{p} = 1$,*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (8)\\ \leq\ & \frac{(b-a)^2}{24}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}}\left[\left(|f''(a)|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(|f''(b)|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] \end{aligned}$$

Theorem 1.6. [9, Theorem 2.6] *Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function on I^o. Assume that $q > 1$ such that $|f''|^q$ is a P-convex on I. Suppose that $a,b \in I^o$ with $a < b$ and $f'' \in L^1([a,b])$. Then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (9)\\ \leq\ & \frac{(b-a)^2}{24}\left[\left(|f''(a)|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(|f''(b)|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Theorem 1.7. *(Hölder-İşcan Integral Inequality [1]). Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^q$, $|g|^q$ are integrable functions on $[a,b]$ then followings are held*

$$\begin{aligned} i.)\ \int_a^b |f(x)g(x)|dx \leq \frac{1}{b-a}\Bigg\{&\left(\int_a^b (b-x)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)|g(x)|^q dx\right)^{\frac{1}{q}} \qquad (10)\\ +&\left(\int_a^b (x-a)|f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)|g(x)|^q dx\right)^{\frac{1}{q}}\Bigg\} \end{aligned}$$

$$ii.)\frac{1}{b-a}\Big\{\Big(\int_a^b(b-x)|f(x)|^pdx\Big)^{\frac{1}{p}}\Big(\int_a^b(b-x)|g(x)|^qdx\Big)^{\frac{1}{q}} \quad (11)$$
$$+\Big(\int_a^b(x-a)|f(x)|^pdx\Big)^{\frac{1}{p}}\Big(\int_a^b(x-a)|g(x)|^qdx\Big)^{\frac{1}{q}}\Big\}$$
$$\leq\Big(\int_a^b|f(x)|^pdx\Big)^{\frac{1}{p}}\Big(\int_a^b(b-x)|g(x)|^qdx\Big).$$

Theorem 1.8. *(Improved power-mean integral inequality [2]). Let $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on $[a,b]$ and if $|f|^q$, $|g|^q$ are integrable functions on interval $[a,b]$ then followings are held*

$$i.)\int_a^b|f(x)g(x)|dx\leq\frac{1}{b-a}\Big\{\Big(\int_a^b(b-x)|f(x)|dx\Big)^{1-\frac{1}{q}}\Big(\int_a^b(b-x)|f(x)||g(x)|^qdx\Big)^{\frac{1}{q}} \quad (12)$$
$$+\Big(\int_a^b(x-a)|f(x)|dx\Big)^{1-\frac{1}{q}}\Big(\int_a^b(x-a)|f(x)||g(x)|^qdx\Big)^{\frac{1}{q}}\Big\}$$

$$ii.)\qquad\frac{1}{b-a}\Big\{\Big(\int_a^b(b-x)|f(x)|dx\Big)^{1-\frac{1}{q}}\Big(\int_a^b(b-x)|f(x)||g(x)|^qdx\Big)^{\frac{1}{q}} \quad (13)$$
$$+\quad\Big(\int_a^b(x-a)|f(x)|dx\Big)^{1-\frac{1}{q}}\Big(\int_a^b(x-a)|f(x)||g(x)|^qdx\Big)^{\frac{1}{q}}\Big\}$$
$$\leq\quad\Big(\int_a^b|f(x)|dx\Big)^{1-\frac{1}{q}}\Big(\int_a^b|f(x)||g(x)|^qdx\Big)^{\frac{1}{q}}\Big\}.$$

We note that, the Beta and Gamma function (see[12] pp. 908-910)) is defined as follow:

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1},\ x,y>0,\ \ \Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\ x>0.$$

And, $\beta(x,x)=2^{1-2x}\beta\Big(\frac{1}{2},x\Big)$, $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$; thus we can obtain below equality,

$$\beta(q+1,q+1)=2^{1-2(q+1)}\beta\Big(\frac{1}{2},q+1\Big)=2^{1-2(q+1)}\frac{\Gamma(\frac{1}{2})\Gamma(q+1)}{\Gamma(\frac{3}{2}+q)}$$

and $\Gamma(\frac{1}{2})=\sqrt{\pi},\ \Gamma(n+1)=n\Gamma(n)=n!$.

2. Main Result

Theorem 2.1. *Let $f:I\subset[0,\infty)\longrightarrow\mathbb{R}$ be a twice differentiable function on I^o, such that $f''\in L^1([a,b])$, where $a,b\in I^o$ with $a<b$. If $|f''|^q$ for $q>1$ with $p=\frac{q}{q-1}$ is a h-convex on $[a,b]$, then for some fixed $t\in(0,1)$ the following inequality holds:*

$$\Big|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\Big|\ \leq\ \frac{(b-a)^2}{16}\Big[\Big(\frac{1}{2}\beta\Big(\frac{1}{2},p+1\Big)-\frac{1}{2}\frac{1}{p+1}\Big)^{\frac{1}{p}}+\Big(\frac{1}{2}\Big(\frac{1}{p+1}\Big)\Big)^{\frac{1}{p}}\Big]$$
$$\times\ \Big[\Big(\int_0^1\Big\{t|f''(a)|^q+(1-t)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}h(t)dt\Big)^{\frac{1}{q}}$$
$$+\ \Big(\int_0^1\Big\{t|f''(b)|^q+(1-t)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}h(t)dt\Big)^{\frac{1}{q}}\Big].$$

Proof. By using Lemma 1.1 and Hölder-İşcan's inequality,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left\{\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|\right\}dt. \\
\leq\ & \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p(1-t)\right)^{\frac{1}{p}}\times\left[\left(\int_0^1 (1-t)\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
+\ & \left.\left(\int_0^1 (1-t)\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
+\ & \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p t\right)^{\frac{1}{p}}\times\left[\left(\int_0^1 t\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 t\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Since $|f''|^q \in SX(h,I)$, we obtained for $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p(1-t)\right)^{\frac{1}{p}}\times\left[\left(\int_0^1 (1-t)\left\{h(t)|f''(a)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 (1-t)\left\{h(t)|f''(b)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right] \\
+\ & \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p t\right)^{\frac{1}{p}}\times\left[\left(\int_0^1 t\left\{h(t)|f''(a)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 t\left\{h(t)|f''(b)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\left(\frac{1}{2}\beta\left(\frac{1}{2},p+1\right)-\frac{1}{2}\frac{1}{p+1}\right)^{\frac{1}{p}}\times\left[\left(\int_0^1 (1-t)h(t)|f''(a)|^q+th(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 (1-t)h(t)|f''(b)|^q+th(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
+\ & \frac{(b-a)^2}{16}\left(\frac{1}{2}\left(\frac{1}{p+1}\right)\right)^{\frac{1}{p}}\times\left[\left(\int_0^1 th(t)|f''(a)|^q+(1-t)h(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 th(t)|f''(b)|^q+(1-t)h(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Now we calculate $\int_0^1(1-t^2)^p(1-t)dt$ and $\int_0^1(1-t^2)^p t dt$. For $t^2=m$, we get

$$dt=\frac{1}{2}m^{-\frac{1}{2}}dm.$$

Namely,

$$\int_0^1(1-t^2)^p(1-t)dt=\frac{1}{2}\int_0^1(1-m)^p(m^{-\frac{1}{2}}-1)dm=\frac{1}{2}\int_0^1(1-m)^p m^{-\frac{1}{2}}-\frac{1}{2}\int_0^1(1-m)^p.$$

If we get the following fact that

$$\int_0^1(1-t^2)^p t dt=\frac{1}{2}\Big(\frac{1}{p+1}\Big),\quad \int_0^1 th(t)dt=\int_0^1(1-t)h(1-t)dt$$

and

$$\int_0^1 th(1-t)dt=\int_0^1(1-t)h(t)dt$$

then proof of the theorem is completed. □

Theorem 2.2. *Let $f:I\subset[0,\infty)\longrightarrow\mathbb{R}$ be a twice differentiable function on I^o, such that $f''\in L^1([a,b])$, where $a,b\in I^o$ with $a<b$. If $|f''|^q$ for $q>1$ be a h-convex on $[a,b]$ then for some fixed $t\in(0,1)$ the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (14)\\
\leq\ & \frac{(b-a)^2}{16}\Big(\frac{5}{12}\Big)^{\frac{1}{p}}\times\Bigg[\Bigg(\int_0^1\Big\{(1-t^2)(1-t)|f''(a)|^q+(1-(1-t)^2)t\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}h(t)dt\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1\Big\{(1-t^2)(1-t)|f''(b)|^q+(1-(1-t)^2)t\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}h(t)dt\Bigg)^{\frac{1}{q}}\Bigg]\\
+\ & \frac{(b-a)^2}{16}\Big(\frac{1}{4}\Big)^{\frac{1}{p}}\times\Bigg[\Bigg(\int_0^1\Big\{(1-t^2)t|f''(a)|^q+(1-(1-t)^2)(1-t)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}h(t)dt\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1\Big\{(1-t^2)t|f''(b)|^q+(1-(1-t)^2)(1-t)\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}h(t)dt\Bigg)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Proof. By using Lemma 1.1 and improved power-mean inequality,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
\leq\ & \frac{(b-a)^2}{16}\int_0^1(1-t^2)\Big\{\Big|f''\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)+f''\Big(\frac{1-t}{2}a+\frac{1+t}{2}b\Big)\Big|\Big\}dt.\\
\leq\ & \frac{(b-a)^2}{16}\Big(\int_0^1(1-t^2)(1-t)\Big)^{\frac{1}{p}}\times\Bigg[\Bigg(\int_0^1\Big|f''(1-t^2)(1-t)\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)\Big|^q dt\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1(1-t^2)(1-t)\Big|f''\Big(\frac{1-t}{2}a+\frac{1+t}{2}b\Big)\Big|^q dt\Bigg)^{\frac{1}{q}}\Bigg]\\
+\ & \frac{(b-a)^2}{16}\Big(\int_0^1(1-t^2)t\Big)^{\frac{1}{p}}\times\Bigg[\Bigg(\int_0^1\Big|f''(1-t^2)t\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)\Big|^q dt\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_0^1(1-t^2)t\Big|f''\Big(\frac{1-t}{2}a+\frac{1+t}{2}b\Big)\Big|^q dt\Bigg)^{\frac{1}{q}}\Bigg]
\end{aligned}$$

Since $|f''|^q \in SX(h,I)$, we obtained

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\; & \frac{(b-a)^2}{16}\left(\int_0^1(1-t^2)(1-t)\right)^{\frac{1}{p}}\times\left[\left(\int_0^1(1-t^2)(1-t)\left\{h(t)|f''(a)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t^2)(1-t)\left\{h(t)|f''(b)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right] \\
+\; & \frac{(b-a)^2}{16}\left(\int_0^1(1-t^2)t\right)^{\frac{1}{p}}\times\left[\left(\int_0^1(1-t^2)t\left\{h(t)|f''(a)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t^2)t\left\{h(t)|f''(b)|^q+h(1-t)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\; & \frac{(b-a)^2}{16}\left(\frac{5}{12}\right)^{\frac{1}{p}}\times\left[\left(\int_0^1(1-t^2)(1-t)h(t)|f''(a)|^q+(1-(1-t)^2)th(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t^2)(1-t)h(t)|f''(b)|^q+(1-(1-t)^2)th(t))\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
+\; & \frac{(b-a)^2}{16}\left(\frac{1}{4}\right)^{\frac{1}{p}}\times\left[\left(\int_0^1(1-t^2)th(t)|f''(a)|^q+(1-(1-t)^2)(1-t)h(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t^2)th(t)|f''(b)|^q+(1-(1-t)^2)(1-t)h(t)\left|f''\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$\int_0^1(1-t^2)(1-t)h(1-t)dt=\int_0^1(1-(1-t)^2)th(t)dt$$

and

$$\int_0^1(1-t^2)th(1-t)dt=\int_0^1(1-(1-t)^2)(1-t)h(t)dt$$

□

Theorem 2.3. *Let $f: I\subset[0,\infty)\longrightarrow\mathbb{R}$ be a twice differentiable function on I^o, such that $f''\in L^1([a,b])$, where $a,b\in I^o$ with $a<b$. If $|f''|^q$ for $q>1$ be a h-concave on $[a,b]$ then for some fixed $t\in(0,1)$ the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{32}\left(\frac{1}{2h(\frac{1}{2})}\right)^{\frac{1}{q}} \tag{15}$$
$$\left[\left(\frac{1}{2}\frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}-\frac{1}{2}\frac{1}{p+1}\right)^{\frac{1}{p}}+\left(\frac{1}{2}\left(\frac{1}{p+1}\right)\right)^{\frac{1}{p}}\right]\left[\left|f''\left(\frac{a+3b}{4}\right)\right|+\left|f''\left(\frac{3a+b}{4}\right)\right|\right].$$

Proof. By using Lemma 1.1 and Hölder-İşcan's inequality,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\int_0^1 (1-t^2)\left\{\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|\right\}dt. \\
\leq\ & \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p(1-t)\right)^{\frac{1}{p}} \times \left[\left(\int_0^1 (1-t)\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
+\ & \left.\left(\int_0^1 (1-t)\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
+\ & \frac{(b-a)^2}{16}\left(\int_0^1 (1-t^2)^p t\right)^{\frac{1}{p}} \times \left[\left(\int_0^1 t\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 t\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Since $|f''|^q \in SV(h,I)$; therefore by inequality Theorem 1.1

$$\int_0^1 t\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q = \frac{1}{4h(\frac{1}{2})}\left|f''\left(\frac{a+3b}{4}\right)\right|^q$$

$$\int_0^1 t\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q = \frac{1}{4h(\frac{1}{2})}\left|f''\left(\frac{3a+b}{4}\right)\right|^q$$

$$\int_0^1 (1-t)\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q = \frac{1}{4h(\frac{1}{2})}\left|f''\left(\frac{a+3b}{4}\right)\right|^q$$

$$\int_0^1 (1-t)\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q = \frac{1}{4h(\frac{1}{2})}\left|f''\left(\frac{3a+b}{4}\right)\right|^q$$

Thus, proof is completed. □

Theorem 2.4. *Let $f : I \longrightarrow \mathbb{R}$ be a twice differentiable function on I^o. Assume that $q > 1$ such that $|f''|^q$ is a P-convex function on I^o. Suppose that $a, b \in I^o$ with $a < b$ and $f'' \in L^1([a, b])$. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left[\left(\frac{1}{2}\frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}-\frac{1}{2}\frac{1}{p+1}\right)^{\frac{1}{p}}+\left(\frac{1}{2}\frac{1}{p+1}\right)^{\frac{1}{p}}\right] \\
\times\ & \left[\left(|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{16}$$

Proof. By assumption, Lemma 1.1 and Hölder-İşcan's inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\int_0^1(1-t^2)\Big\{\Big|f''\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)+f''\Big(\frac{1-t}{2}a+\frac{1+t}{2}b\Big)\Big|\Big\}dt. \\
\leq\ & \frac{(b-a)^2}{16}\Big(\int_0^1(1-t^2)^p(1-t)\Big)^{\frac{1}{p}}\times\Big[\Big(\int_0^1(1-t)\Big|f''\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)\Big|^q dt\Big)^{\frac{1}{q}} \\
&+\Big(\int_0^1(1-t)\Big|f''\Big(\frac{1-t}{2}a+\frac{1+t}{2}b\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Big] \\
+\ & \frac{(b-a)^2}{16}\Big(\int_0^1(1-t^2)^p t\Big)^{\frac{1}{p}}\times\Big[\Big(\int_0^1 t\Big|f''\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)\Big|^q dt\Big)^{\frac{1}{q}} \\
&+\Big(\int_0^1 t\Big|f''\Big(\frac{1-t}{2}a+\frac{1+t}{2}b\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

Since $|f''|^q$ is P-convex, we obtained

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\Big(\int_0^1(1-t^2)^p(1-t)\Big)^{\frac{1}{p}}\times\Big[\Big(\int_0^1(1-t)\{|f''(a)|^q+\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\}dt\Big)^{\frac{1}{q}} \\
&+\Big(\int_0^1(1-t)\Big\{|f''(b)|^q+\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}dt\Big)^{\frac{1}{q}}\Big] \\
+\ & \frac{(b-a)^2}{16}\Big(\int_0^1(1-t^2)^p t\Big)^{\frac{1}{p}}\times\Big[\Big(\int_0^1 t\Big\{|f''(a)|^q+\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}dt\Big)^{\frac{1}{q}} \\
&+\Big(\int_0^1 t\Big\{|f''(b)|^q+\Big|f''\Big(\frac{a+b}{2}\Big)\Big|^q\Big\}dt\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Now we calculate $\int_0^1(1-t^2)^p(1-t)dt$ and $\int_0^1(1-t^2)^p t dt$. For $t^2=m$, we get

$$dt=\frac{1}{2}m^{-\frac{1}{2}}dm.$$

Namely,

$$\int_0^1(1-t^2)^p(1-t)dt=\frac{1}{2}\int_0^1(1-m)^p(m^{-\frac{1}{2}}-1)dm=\frac{1}{2}\int_0^1(1-m)^p m^{-\frac{1}{2}}-\frac{1}{2}\int_0^1(1-m)^p$$

And we rewrite the above by using property of β as follow,

$$\frac{1}{2}\beta\Big(\frac{1}{2},p+1\Big)-\frac{1}{2}\frac{1}{p+1}.$$

If we get the following fact that

$$\int_0^1(1-t^2)^p t dt=\frac{1}{2}\Big(\frac{1}{p+1}\Big).$$

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\left(\frac{1}{2}\left[\beta\left(\frac{1}{2},p+1\right)-\frac{1}{2}\frac{1}{p+1}\right]\right)^{\frac{1}{p}}\times\left[\left(\frac{1}{2}\left(|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left(\frac{1}{2}|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right] \\
+\ & \frac{(b-a)^2}{16}\left(\frac{1}{2}\left(\frac{1}{p+1}\right)\right)^{\frac{1}{p}}\times\left[\left(\frac{1}{2}\left(|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2}\left(|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.5. *Let $f : I \longrightarrow \mathbb{R}$ be a twice differentiable function on I^o. Assume that $q > 1$ such that $|f''|^q$ is a P-convex function on I^o. Suppose that $a, b \in I^o$ with $a < b$ and $f'' \in L^1([a,b])$. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (17)\\
\leq\ & \frac{(b-a)^2}{24}\left[\left(|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. By using Lemma 1.1 and improved power-mean inequality,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16}\int_0^1(1-t^2)\left\{\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|\right\}dt. \\
\leq\ & \frac{(b-a)^2}{16}\left(\int_a^b(1-t^2)(1-t)\right)^{\frac{1}{p}}\times\left[\left(\int_0^1(1-t^2)(1-t)\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t^2)(1-t)\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
+\ & \frac{(b-a)^2}{16}\left(\int_a^b(1-t^2)t\right)^{\frac{1}{p}}\times\left[\left(\int_0^1(1-t^2)t\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t^2)t\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

Since $|f''|^q \in P(I)$, we reach

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{(b-a)^2}{16}\left(\frac{5}{12}\right)^{\frac{1}{p}}\times\left[\left(\int_0^1(1-t)(1-t^2)\{|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\}dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1(1-t)(1-t^2)\left\{|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right]\\ +\ & \frac{(b-a)^2}{16}\left(\frac{1}{4}\right)^{\frac{1}{p}}\times\left[\left(\int_0^1 t(1-t^2)\left\{|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1 t(1-t^2)\left\{|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right\}dt\right)^{\frac{1}{q}}\right]\end{aligned}$$

If we do necessary operation, then the proof is completed. □

Corollary 2.1. *If we take* $q=2$ *in the inequality* (17)*, then The inequality* (18) *coincides with the inequality* (9) *in Theorem* 8*, namely*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (18)\\ \leq\ & \frac{(b-a)^2}{24}\left[\left(|f''(a)|^2+\left|f''\left(\frac{a+b}{2}\right)\right|^2\right)^{\frac{1}{q}}+\left(|f''(b)|^2+\left|f''\left(\frac{a+b}{2}\right)\right|^2\right)^{\frac{1}{2}}\right].\end{aligned}$$

References

[1] İşcan İ, (2019), New refinements for integral and sum forms of Hölder inequality, Researchgate, DOI: 10.13140/RG.2.2.19356.54409, Preprint January.

[2] Kadakal M., İşcan İ, Kadakal H. and Bekar K. (2019), On Improvement Of Some Integral Inequalities, Researchgate, DOI: 10.13140/RG.2.2.15052.46724, Preprint January.

[3] Alomari M., Darus M. and Kirmaci U. S., (2011), Some inequalities of Hermite-Hadamard type for s-convex functions, Acta Math. Scientia, vol. 31B, no 4, pp. 1643-1652.

[4] Dragomir S. S.,and Pearce C. E. M., (2000), Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University.

[5] İşcan İ., (2013), On generalization of some integral inequalities for quasi-convex and their applications, Inter- national Journal of Engineering and Applied Sciences, 3 (1) pp. 37-42.

[6] Kirmaci U. S., (2004), Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput., 147, pp. 137-146.

[7] S. Varošanec, (2007), On h-convexity, J. Math. Appl., 326, 303-311.

[8] Godunova E. K. and Levin V. L, (1985), Neravenstva dlja funkcii širokogo klassa, soder*žaščego* vypuklye, monotonnye i nekotorye drugie vidy funkii, Vycislitel. Mat. i. Fiz. mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 138-142.

[9] Barani A., Barani S., and Dragomir S. S., (2012), Refinements of Hermite-Hadamard inequalities for functions when a power of the absolute value of the second derivative is P-Convex, Journal of Applied Mathematics, Volume 2012, Article ID 615737, 10 pages doi:10.1155615737.

[10] Barani A., Barani S., and Dragomir S. S., (2011), Refinements of Hermite-Hadamard inequality for functions whose second derivatives absolute value are quasicvonvex, RGMIA Research Report Collection, vol. 14, article 69.

[11] Iqbal M., Muddass M., Bhatti M. I., On Hermite-Hadamard Type Inequalities Via h-convexity, arXiv:1511.05281.

[12] Gradshteyn I. S., Ryzhik I. M., (2007), Table of Integrals, Series, and Products, 7th ed., Academic Press, Elsevier Inc.

Erdal Ünlüyol received his B.Sc. degree from 19 May University, Ordu Faculty of Arts and Sciences, Department of Mathematics, Ordu, Turkey in 2003 and M.Sc. degree from Karadeniz Technical University, Science Institute, Trabzon, Turkey in 2006 and then his Ph.D. degree from Karadeniz Technical University, Science Institute, Trabzon, Turkey in 2011. Currently he works as an asst. prof. in the Department of Mathematics, Art and Science Faculty, Ordu University. He has many research papers about the Theory of inequalities, Multipoint Normal Differential Operators.

Seren Salaş is a Ph.D. student in the Department of Mathematics, Institute of Science, Ordu University. She received her B.Sc. degree from Department of Mathematics, Faculty of Arts and Sciences, Ordu University in 2014 and M.Sc. degree in the Department of Mathematics, Institute of Science, Ordu University (2016). She has many research papers about the Operator Theory and Inequalities. Moreover, she is interested in Non-Newtonian Calculus.

Görkem Dalkun is a master's student in the Department of Mathematics, Institute of Science, Ordu University. She received her B. Sc. degree from the Department of Mathematics, Faculty of Arts and Sciences, Ordu University in 2018. She is interested in inequalities of real valued functions.