TWMS J. App. and Eng. Math. V.11, N.1, 2021, pp. 122-133

A NEW CLASS OF MIXED MONOTONE OPERATORS WITH CONCAVITY AND APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

H. SHOJAAT[1], H. AFSHARI[2], M. S. ASGARI[1], §

Abstract. In this article, we investigate a class of mixed monotone operators with concavity on ordered Banach spaces. As applications, we utilize the main results obtained in this paper to study for solutions of fractional differential equations. An example is also considered to illustrate the main result.

Keywords: Fractional differential equation; normal cone; positive solution.

AMS Subject Classification: 83-02, 99A00

1. Introduction

As an important branch of nonlinear functional analysis, the nonlinear operators and their application in nonlinear differential equations are taken into consideration (see [7, 8, 9]). In 2015 the sum operator

$$T_1u + T_2u + T_3(u, u) = u, \tag{1}$$

has been considered by Wang and Zhang, where T_1 is a decreasing operator, T_2 is an increasing sub-homogeneous operator and T_3 is mixed monotone operator. In this paper we study (1) with different conditions. As an application, we apply our main fixed point theorem to solution of the boundary value problems via nonlinear fractional differential equations.

Suppose $(E, \| \,.\, \|)$ be a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $u \leq v$ if and only if $v - u \in P$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $u \in P,\ \lambda \geq 0 \Longrightarrow \lambda u \in P$; (ii) $u \in P,\ -u \in P \Longrightarrow u = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq u \leq v$ implies $\| u \| \leq N \| v \|$, also we define the order interval

[1] Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
e-mail: hadishojaat@yahoo.com; ORCID: https://orcid.org/ 0000-0002-6838-072x.
e-mail: msasgari@yahoo.com; ORCID: https://orcid.org/ 0000-0002-0675-0262.
[2] Faculty of Basic Sciences, University of Bonab, Bonab - 5551761167, Iran.
e-mail: hojat.afshari@yahoo.com; ORCID: https://orcid.org/0000-0003-1149-4336.
§ Manuscript received: April 20, 2019; accepted: June 26, 2019.

$[u_1, u_2] = \{u \in E | u_1 \leq u \leq u_2\}$ for all $u_1, u_2 \in E$ and
$P_h = \{u \in E | \exists \lambda, \mu > 0 \text{ such that } \lambda h \leq u \leq \mu h\}$ for $h > \theta$.

2. Preliminaries

Definition 2.1. [2, 3] *$T_1 : P \times P \to P$ is said to be a mixed monotone operator if T_1 is increasing in u and decreasing in v, i.e., u_i, v_i $(i = 1, 2) \in P$, $u_1 \leq u_2, v_1 \geq v_2$ imply $T_1(u_1, v_1) \leq T_1(u_2, v_2)$. The element $u \in P$ is called a fixed point of T_1 if $T_1(u, u) = u$.*

Theorem 2.1. [9] *Let P be a normal cone in a real Banach space E.
Assume that $T_1 : P \times P \to P$ is a mixed monotone operator and that satisfy the following conditions:*

(i) $\exists\, h \in P$ with $h \neq \theta$ such that $T_1(h, h) \in P_h$;

(ii) for $u, v \in P$ and $t \in (0, 1)$ there exists $\phi(t) \in (t, 1]$ such that

$$T_1(tu, \frac{1}{t}v) \geq \phi(t) T_1(u, v).$$

Then

(1) *$T_1 : P_h \times P_h \to P_h$;*

(2) *$\exists\, x_0, y_0 \in P_h$ and $r \in (0, 1)$ such that*

$$ry_0 \leq x_0 < y_0, x_0 \leq T_1(x_0, y_0) \leq T_1(y_0, x_0) \leq y_0;$$

(3) *the operator equation $T_1(u, u) = u$ has a unique solution u^* in P_h;*

(4) *for initial values $u_0, v_0 \in P_h$, construct*

$$u_n = T_1(u_{n-1}, v_{n-1})$$
$$v_n = T_1(v_{n-1}, u_{n-1}), n = 1, 2, \ldots,$$

then $u_n \to u^*$ and $v_n \to u^*$.

In paper [6], Sun and Zhao studied the equation

$$D^{\nu}_{0+} x(t) + g(t) f(t, x(t)) = 0, \quad 0 < t < 1,$$
$$x(0) = x'(0) = 0, \quad x(1) = \int_0^1 q(\zeta) x(\zeta) d\zeta,$$

where $2 < \nu \leq 3$, D^{ν}_{0+} is the Riemann-Liouville fractional derivative.
Motivated by [6], in paper [1], Feng and Zhai considered the following form:

$$D^{\nu}_{0+} x(t) + f(t, x(t)) + g(t, x(t)) = 0, \quad 0 < t < 1, \tag{2}$$
$$x(0) = x'(0) = 0, \quad x(1) = \int_0^1 q(\zeta) x(\zeta) d\zeta,$$

where $2 < \nu \leq 3$, D^{ν}_{0+} is the Riemann-Liouville fractional derivative. The function $q(t)$ satisfies the following conditions:

$$q : [0, 1] \to [0, \infty) \quad \text{with} \quad q \in L^1[0, 1] \quad \text{and}$$
$$\omega_1 = \int_0^1 \zeta^{\nu-1}(1 - \zeta) q(\zeta) d\zeta > 0, \quad \omega_2 = \int_0^1 \zeta^{\nu-1} q(\zeta) d\zeta < 1.$$

In that paper the authors obtained some alternative answers to the them main results by using a sum operator.
In this paper we study the equation

$$D_{0^+}^{\nu}x(s,t)+f(t,\frac{\partial}{\partial s}x(s,t))+g(t,x(s,t))+e(t,x(s,t),\frac{\partial}{\partial s}x(s,t))=0, \quad (3)$$
$$0<s,t<1, \quad x(s,0)=\frac{\partial}{\partial t}x(s,0)=0, \quad x(s,1)=\int_0^1 q_1(s,\zeta)x(s,\zeta)d\zeta,$$

where q_1 satisfies the following:

$$(Q) \quad q_1:[0,1]\times[0,1]\to[0,\infty) \quad \text{with} \quad q_1\in L^1([0,1]\times[0,1]) \quad \text{and}$$
$$\omega_1=\int_0^1 \zeta^{\nu-1}(1-\zeta)q_1(s,\zeta)d\zeta>0, \quad \omega_2=\int_0^1 \zeta^{\nu-1}q_1(s,\zeta)d\zeta<1.$$

Definition 2.2. [4, 5] *The Riemann-Liouville fractional derivative for a continuous function f is defined by*

$$D^{\nu}f(t)=\frac{1}{\Gamma(n-\nu)}(\frac{d}{dt})^n\int_0^t \frac{f(\zeta)}{(t-\zeta)^{\nu-n+1}}d\zeta, \ (n=[\nu]+1),$$

where the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. [4, 5] *Let $[a,b]$ be an interval in $\mathbb{R}$ and $\nu>0$. The Riemann-Liouville fractional order integral of a function $f\in L^1([a,b],\mathbb{R})$ is defined by*

$$I_a^{\nu}f(t)=\frac{1}{\Gamma(\nu)}\int_a^t \frac{f(\zeta)}{(t-\zeta)^{1-\nu}}d\zeta,$$

whenever the integral exists.

Suppose;

$$G(t,\zeta)=G_1(t,\zeta)+G_2(t,\zeta), \quad (t,\zeta)\in[0,1]\times[0,1], \quad (4)$$

where

$$G_1(t,\zeta)=\frac{1}{\Gamma(\nu)}\begin{cases} t^{\nu-1}(1-\zeta)^{\nu-1}-(t-\zeta)^{\nu-1}, & 0\le\zeta\le t\le 1, \\ t^{\nu-1}(1-\zeta)^{\nu-1}, & 0\le t\le\zeta\le 1 \end{cases} \quad (5)$$

and

$$G_2(t,\zeta)=\frac{t^{\nu-1}}{1-\omega_2}\int_0^1 G_1(\tau,\zeta)q_1(\zeta,\tau)d\tau. \quad (6)$$

Lemma 2.1. [8] *The function $G_1(t,\zeta)$ defined by (5) has the following properties:*

$$\frac{t^{\nu-1}(1-t)\zeta(1-\zeta)^{\nu-1}}{\Gamma(\nu)}\le G_1(t,\zeta)\le\frac{\zeta(1-\zeta)^{\nu-1}}{\Gamma(\nu-1)}, \quad t,\zeta\in[0,1].$$

From [6] and Lemma 2.1, we have

$$\frac{\omega_1\zeta(1-\zeta)^{\nu-1}t^{\nu-1}}{(1-\omega_2)\Gamma(\nu)}\le G(t,\zeta)\le\frac{t^{\nu-1}(1-\zeta)^{\nu-1}}{(1-\omega_2)\Gamma(\nu)}, \quad t,\zeta\in[0,1].$$

Theorem 2.2. [1] *Assume (Q) and*

(H_1) $f:[0,1]\times[0,\infty)\to[0,\infty)$ *is continuous and increasing with respect to the second argument, $f(t,0)\not\equiv 0$;*

(H_2) *$g : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and decreasing with respect to the second argument, $g(t,1) \not\equiv 0$;*

(H_3) *for $\lambda \in (0,1)$, $\exists\ \phi_i(\lambda) \in (\lambda,1), i = 1,2$ with*

$$f(t,\lambda u) \geq \phi_1(\lambda) f(t,u), \quad g(t,\lambda u) \leq \frac{1}{\phi_2(\lambda)} g(t,u)$$

for $t \in (0,1), u \in [0,\infty)$.

Then the problem (2) has a unique positive solution x^* in P_h, where $h(t) = t^{\nu-1}, t \in [0,1]$ and for $u_0, v_0 \in P_h$,

$$u_{n+1}(t) = \int_0^1 G(t,\zeta)[f(\zeta,u_n(\zeta)) + g(\zeta,v_n(\zeta))]d\zeta,$$

$$v_{n+1}(t) = \int_0^1 G(t,\zeta)[f(\zeta,v_n(\zeta)) + g(\zeta,u_n(\zeta))]d\zeta,$$

$n = 0,1,2,...,$ we have $u_n(t) \to u^*(t)$, $v_n(t) \to u^*(t)$, where $G(t,\zeta)$ is given as (4).

Recall that $T_1 : P \to P$ is said to be homogeneous if $T_1(tu) = tT_1(tu)$ for $t > 0, u \in E$. $T_1 : P \to P$ is said to be sub-homogeneous if $T_1(tu) \geq tT_1(u)$ for all $t > 0, u \in E$.

Theorem 2.3. [7] *Let P be a normal cone in a real Banach space E, $T_1 : P \to P$ is an increasing sub-homogeneous operator, $T_2 : P \to P$ is a decreasing operator and $T_3 : P \times P \to P$ is a mixed monotone operator that satisfy the following conditions:*

$$T_2(\frac{1}{t}v) \geq tT_2 v, \quad T_3(tu, \frac{1}{t}v) \geq t^\gamma T_3(u,v), \quad t \in (0,1), \gamma \in (0,1), u,v \in P.$$

Assume that

(i) *$\exists\ h_0 \in P_h$ such that $T_1 h_0 \in P_h, T_2 h_0 \in P_h, T_3(h_0,h_0) \in P_h$;*

(ii) *$\exists\ \delta_0 > 0$ with $T_3(u,v) \geq \delta_0(T_1 u + T_2 u)$ for $u,v \in P$.*

Then

(1) *$T_1 : P_h \to P_h$, $T_2 : P_h \to P_h$ and $T_3 : P_h \times P_h \to P_h$;*

(2) *$\exists\ x_0, y_0 \in P_h$ and $r \in (0,1)$ with*

$$ry_0 \leq x_0 < y_0, x_0 \leq T_1 x_0 + T_2 y_0 + T_3(x_0,y_0) \leq T_1 y_0 + T_2 x_0 + T_3(y_0,x_0) \leq y_0;$$

(3) *the operator equation $T_1 u + T_2 u + T_3(u,u) = u$ has a unique solution u^* in P_h;*

(4) *for $u_0, v_0 \in P_h$, construct*

$$u_n = T_1 u_{n-1} + T_2 v_{n-1} + T_3(u_{n-1}, v_{n-1})$$
$$v_n = T_1 v_{n-1} + T_2 u_{n-1} + T_3(v_{n-1}, u_{n-1}), n = 1,2,\ldots,$$

then $u_n \to u^*$, $v_n \to u^*$.

3. MAIN RESULTS

In this section we consider the generalization of Theorem 2.3.

Theorem 3.1. *Let P be a normal cone, in a real Banach space E, $T_1 : P \to P$ be a decreasing, $T_2 : P \to P$ be a increasing, $T_3 : P \times P \to P$ be a mixed monotone operators and*

(H_1) *For* $u, v \in P$ *and* $t \in (0,1)$, $\exists\ \phi_1(t), \phi_2(t), \phi_3(t) \in (t,1)$ *with*

$$T_1(tv) \leq \frac{1}{\phi_1(t)} T_1 v, \quad T_2(tu) \geq \phi_2(t) T_2 u \tag{7}$$

and

$$T_3(tu, \frac{1}{t}v) \geq \phi_3(t) T_3(u,v); \tag{8}$$

(H_2) $\exists\ h_0 \in P_h$ *such that* $T_1 h_0 + T_2 h_0 + T_3(h_0, h_0) \in P_h$.

Then

(i) $\exists\ x_0, y_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$ry_0 \leq x_0 < y_0, x_0 \leq T_1 y_0 + T_2 x_0 + T_3(x_0, y_0) \leq T_1 x_0 + T_2 y_0 + T_3(y_0, x_0) \leq y_0;$$

(ii) *the equation* $T_1 u + T_2 u + T_3(u,u) = u$ *has a unique solution* u^* *in* P_h*;*

(iii) *for* $u_0, v_0 \in P_h$*, construct*

$$u_n = T_1 v_{n-1} + T_2 u_{n-1} + T_3(u_{n-1}, v_{n-1})$$
$$v_n = T_1 u_{n-1} + T_2 v_{n-1} + T_3(v_{n-1}, u_{n-1}), n = 1, 2, \dots,$$

then $u_n \to u^*$ *and* $v_n \to u^*$.

Proof. From (7) we obtain

$$T_1(\frac{1}{t}v) \geq \phi_1(t) T_1 v, \quad T_2(tu) \geq \phi_2(t) T_2 u, \tag{9}$$
$$T_3(tu, \frac{1}{t}v) \geq \phi_3(t) T_3(u,v), t \in (0,1), u, v \in P.$$

Since $T_1 h_0 + T_2 h_0 + T_3(h_0, h_0) \in P_h$, $\exists\ \lambda_1, \lambda_2 > 0$ with

$$\lambda_1 h \leq T_1 h_0 + T_2 h_0 + T_3(h_0, h_0) \leq \lambda_2 h.$$

From $h_0 \in P_h$, $\exists\ t_0 \in (0,1)$ such that

$$t_0 h \leq h_0 \leq \frac{1}{t_0} h.$$

Let $\phi(t) = \min\{\phi_1(t), \phi_2(t), \phi_3(t)\}$. Then $\phi(t) \in (t,1)$ for $t \in (0,1)$. From (H_1) and (9),

$$\begin{aligned} T_1 h + T_2 h + T_3(h,h) &\geq T_1(\frac{1}{t_0} h_0) + T_2(t_0 h_0) + T_3(t_0 h_0, t_0^{-1} h_0) \\ &\geq \phi_1(t_0) T_1 h_0 + \phi_2(t_0) T_2 h_0 + \phi_3(t_0) T_3(h_0, h_0) \\ &\geq \phi(t_0)[T_1 h_0 + T_2 h_0 + T_3(h_0, h_0)] \\ &\geq \lambda_1 \phi(t_0) h, \end{aligned}$$

$$\begin{aligned} T_1 h + T_2 h + T_3(h,h) &\leq T_1(t_0 h_0) + T_2(\frac{1}{t_0} h_0) + T_3(t_0^{-1} h_0, t_0 h_0) \\ &\leq \frac{1}{\phi_1(t_0)} T_1 h_0 + \frac{1}{\phi_2(t_0)} T_2 h_0 + \frac{1}{\phi_3(t_0)} T_3(h_0, h_0) \\ &\leq \frac{1}{\phi(t_0)}[T_1 h_0 + T_2 h_0 + T_3(h_0, h_0)] \\ &\leq \frac{\lambda_2}{\phi(t_0)} h. \end{aligned}$$

Note that $\lambda_1\phi(t_0), \frac{\lambda_2}{\phi(t_0)} > 0$, we get $T_1h + T_2h + T_3(h,h) \in P_h$.
We define $T = T_1 + T_2 + T_3$ by $T(u,v) = T_1v + T_2u + T_3(u,v)$, then $T : P \times P \to P$ is a mixed monotone and $T(h,h) = T_1h + T_2h + T_3(h,h) \in P_h$.
Moreover, for $u, v \in P$ and $t \in (0,1)$, we have

$$\begin{aligned} T(tu, t^{-1}v) &= T_1(t^{-1}v) + T_2(tu) + T_3(tu, t^{-1}v) \\ &\geq \phi_1(t)T_1v + \phi_2(t)T_2u + \phi_3(t)T_3(u,v) \\ &\geq \phi(t)[T_1v + T_2u + T_3(u,v)] \\ &= \phi(t)T(u,v). \end{aligned}$$

Hence, all the conditions of Theorem 2.1 are satisfied. Application of Theorem 2.1 implies that:
$\exists\ x_0, y_0 \in P_h$ and $r \in (0,1)$ with

$$ry_0 \leq x_0 < y_0, x_0 \leq T(x_0, y_0) \leq T(y_0, x_0) \leq y_0$$

and $T(u,u) = u$ has a unique solution u^* in P_h; for $u_0, v_0 \in P_h$, construct

$$\begin{aligned} u_n &= T(u_{n-1}, v_{n-1}) \\ v_n &= T(v_{n-1}, u_{n-1}), n = 1, 2, \ldots, \end{aligned}$$

then $u_n \to u^*$ and $v_n \to u^*$. That is,

(i) $\exists\ x_0, y_0 \in P_h$ and $r \in (0,1)$ with

$$ry_0 \leq x_0 < y_0, x_0 \leq T_1y_0 + T_2x_0 + T_3(x_0, y_0) \leq T_1x_0 + T_2y_0 + T_3(y_0, x_0) \leq y_0;$$

(ii) equation $T_3(u,u) + T_1u + T_2u = u$ has a unique solution u^* in P_h;

(iii) for $u_0, v_0 \in P_h$, construct

$$\begin{aligned} u_n &= T_1v_{n-1} + T_2u_{n-1} + T_3(u_{n-1}, v_{n-1}) \\ v_n &= T_1u_{n-1} + T_2v_{n-1} + T_3(v_{n-1}, u_{n-1}), n = 1, 2, \ldots, \end{aligned}$$

we have $u_n \to u^*$ and $v_n \to u^*$.

□

If in Theorem 3.1, we put $\phi_1(t) = \phi_2(t) = t$ and $\phi_3(t) = t^\gamma$, then we can obtain the following result.

Corollary 3.1. *Let P be a normal cone in a real Banach space E, $T_1 : P \to P$ be a decreasing operator, $T_2 : P \to P$ be a sub-homogeneous operator, $T_3 : P \times P \to P$ be a mixed monotone operators and $\gamma \in (0,1)$, that satisfies the following conditions:*

(H_1) *For $u, v \in P$ and $t \in (0,1)$*

$$T_1(\frac{1}{t}v) \geq tT_1v, \quad T_3(tu, \frac{1}{t}v) \geq t^\gamma T_3(u,v); \tag{10}$$

(H_2) $\exists\ h_0 \in P_h$ *such that* $T_1h_0 + T_2h_0 + T_3(h_0, h_0) \in P_h$.

Then

(i) $\exists\ x_0, y_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$ry_0 \leq x_0 < y_0, x_0 \leq T_1y_0 + T_2x_0 + T_3(x_0, y_0) \leq T_1x_0 + T_2y_0 + T_3(y_0, x_0) \leq y_0;$$

(ii) *equation $T_1u + T_2u + T_3(u,u) = u$ has a unique solution u^* in P_h;*

(iii) for $u_0, v_0 \in P_h$, construct

$$u_n = T_1 v_{n-1} + T_2 u_{n-1} + T_3(u_{n-1}, v_{n-1})$$
$$v_n = T_1 u_{n-1} + T_2 v_{n-1} + T_3(v_{n-1}, u_{n-1}), n = 1, 2, \dots,$$

then $u_n \to u^*$ and $v_n \to u^*$.

Lemma 3.1. *Assume (Q) holds. Let $v \in C([0,1] \times [0,1]), 2 < \nu \leq 3$, then the problem*

$$D^{\nu}_{0+} x(s,t) + v(s,t) = 0, \tag{11}$$
$$0 < s, t < 1, x(s,0) = \frac{\partial}{\partial t} x(s,0) = 0, \quad x(s,1) = \int_0^1 q_1(s,\zeta) x(s,\zeta) d\zeta,$$

has the solution

$$x(s,t) = \int_0^1 G(t,\zeta) v(s,\zeta) d\zeta,$$

where $G(t,\zeta)$ is given as (4).

Proof. We reduce problem (11) to an equivalent integral equation

$$x(s,t) = -I^{\nu}_{0+} v(s,t) + c_1 t^{\nu-1} + c_2 t^{\nu-2} + c_3 t^{\nu-3},$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Consequently the general solution of the problem (11) is

$$x(s,t) = -\int_0^t \frac{(t-\zeta)^{\nu-1}}{\Gamma(\nu)} v(s,\zeta) d\zeta + c_1 t^{\nu-1} + c_2 t^{\nu-2} + c_3 t^{\nu-3}.$$

By $x(s,0) = \frac{\partial}{\partial t} x(s,0) = 0, \quad x(s,1) = \int_0^1 q_1(s,\zeta) x(s,\zeta) d\zeta$, we have

$$c_2 = c_3 = 0, \quad c_1 = \int_0^1 \frac{(1-\zeta)^{\nu-1}}{\Gamma(\nu)} v(s,\zeta) d\zeta + \int_0^1 q_1(s,\zeta) u(s,\zeta) d\zeta.$$

Hence the unique solution of (11) is

$$\begin{aligned} x(s,t) = &-\int_0^t \frac{(t-\zeta)^{\nu-1}}{\Gamma(\nu)} v(s,\zeta) d\zeta + \frac{t^{\nu-1}}{\Gamma(\nu)} \int_0^1 (1-\zeta)^{\nu-1} v(s,\zeta) d\zeta \\ &+ t^{\nu-1} \int_0^1 q_1(s,\zeta) u(s,\zeta) d\zeta \\ &= \int_0^1 G_1(t,\zeta) v(s,\zeta) d\zeta + t^{\nu-1} \int_0^1 q_1(s,\zeta) u(s,\zeta) d\zeta. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 q_1(s,t) x(s,t) dt &= \int_0^1 q_1(s,t) (\int_0^1 G_1(t,\zeta) v(s,\zeta) d\zeta) dt \\ &+ \int_0^1 (q_1(s,t) t^{\nu-1} \int_0^1 q_1(s,\zeta) x(s,\zeta) d\zeta) dt \\ &= \int_0^1 (\int_0^1 q_1(s,t) G_1(t,\zeta) dt) v(s,\zeta) d\zeta \\ &+ (\int_0^1 t^{\nu-1} q_1(s,t) dt)(\int_0^1 q_1(s,\zeta) x(s,\zeta) d\zeta), \end{aligned}$$

$$\begin{aligned}\int_0^1 q_1(s,\zeta)x(s,\zeta)d\zeta =&\frac{1}{1-\omega_1}\int_0^1(\int_0^1 G_1(t,\zeta)q_1(s,t)dt)v(s,\zeta)d\zeta\\ =&\frac{1}{1-\omega_1}\int_0^1(\int_0^1 G_1(\tau,\zeta)q_1(s,\tau)d\tau)v(s,\zeta)d\zeta.\end{aligned}$$

We have

$$\begin{aligned}x(s,t) =&\int_0^1 G_1(t,\zeta)v(s,\zeta)d\zeta + \frac{t^{\nu-1}}{1-\omega_2}\int_0^1(\int_0^1 G_1(\tau,\zeta)q_1(s,\tau)d\tau)v(s,\zeta)d\zeta\\ =&\int_0^1 G_1(t,\zeta)v(s,\zeta)d\zeta + \int_0^1 G_2(t,\zeta)v(s,\zeta)d\zeta\\ =&\int_0^1 G(t,\zeta)v(s,\zeta)d\zeta.\end{aligned}$$

This completes the proof. □

In this section we consider the Banach space E as the follows,

$$E = \{y(s,t) \in C([0,1]\times[0,1]) | \frac{\partial}{\partial s}y(s,t) \in C([0,1]\times[0,1])\},$$

with the norm

$$\|y\| = \max\{\max_{s,t\in[0,1]}\{|y(s,t)|, \max_{s,t\in[0,1]}|\frac{\partial}{\partial s}y(s,t)|\}\},$$

also let E be endowed with an order relation $\frac{\partial}{\partial s}y(s,t)) \leq \frac{\partial}{\partial s}y'(s,t))$ if $y(s,t) \leq y'(s,t)$. let

$$P = \{y \in E : y(s,t), \frac{\partial}{\partial s}y(s,t)) \geq 0,\ s,t \in [0,1]\}. \tag{12}$$

It's easy to see that, P is a normal cone and $P_h \subseteq E$.
We can obtain the following consequences.

Theorem 3.2. *Assume (Q) and*

- (H_1) $f : [0,1]\times[0,\infty) \to [0,\infty)$ *is continuous and decreasing with respect to the second argument,* $f(t,1) \not\equiv 0$*;*
- (H_2) $g : [0,1]\times[0,\infty) \to [0,\infty)$ *is continuous and increasing with respect to the second argument,* $g(t,0) \not\equiv 0$*;*
- (H_3) $e : [0,1]\times[0,\infty)\times[0,\infty) \to [0,\infty)$ *is continuous and increasing with respect to the second argument, also decreasing with respect to the third argument,* $e(t,0,1) \not\equiv 0$*;*
- (H_4) *for* $\lambda \in (0,1)$, $\exists\ \phi_i(\lambda) \in (\lambda,1), i = 1,2,3$ *such that*

$$f(t,\frac{1}{\lambda}v) \geq \phi_1(\lambda)f(t,v),\quad g(t,\lambda u) \leq \frac{1}{\phi_2(\lambda)}g(t,u),\qquad e(t,\lambda u,\frac{1}{\lambda}v) \geq \phi_3(\lambda)e(t,u,v)$$

for $t \in (0,1), u,v \in [0,\infty)$.

Then (2) has a solution x^* *in* P_h*, where* $h(t) = t^{\alpha-1}, t \in [0,1]$ *and for* $u_0 \in P_h$*, construct*

$$u_{n+1}(s,t) = \int_0^1 G(t,\xi)[f(s,\frac{\partial}{\partial s}v_n(s,\xi)) + g(s,u_n(s,\xi)) + e(s,u_n(s,\xi),\frac{\partial}{\partial s}v_n(s,\xi))]d\xi,$$

$$v_{n+1}(s,t) = \int_0^1 G(t,\xi)[f(s,u_n(s,\xi)) + g(s,\frac{\partial}{\partial s}v_n(s,\xi)) + e(s,\frac{\partial}{\partial s}v_n(s,\xi),u_n(s,\xi))]d\xi,$$

$n = 0,1,2,...,$ *and* $u_n(s,t) \to u^*(s,t)$*,* $v_n(s,t) \to u^*(s,t)$ *where* $G(t,s)$ *is given as (4).*

Proof. From Lemma (3.1) we know that problem (3) has an integral formulation given by

$$x(s,t) = \int_0^1 G(t,\zeta)[f(\zeta, \frac{\partial}{\partial s} y(s,\zeta)) + g(\zeta, x(s,\zeta)) + e(s, x(s,\zeta), \frac{\partial}{\partial s} y(s,\zeta))]d\zeta,$$

Define, $T_1 : P \to P$, $T_2 : P \to P$ and $T_3 : P \times P \to P$ by

$$T_1 y(s,t) = \int_0^1 G(t,\zeta) f(\zeta, \frac{\partial}{\partial s} y(s,\zeta))d\zeta, \quad T_2 x(s,t) = \int_0^1 G(t,\zeta) g(\zeta, x(s,\zeta))d\zeta$$

$$T_3(x(s,t), y(s,t)) = \int_0^1 G(t,\zeta) e(s, x(s,\zeta), \frac{\partial}{\partial s} y(s,\zeta))d\zeta.$$

Then u is the solution of problem (3) if and only if

$$u = T_1 u + T_2 u + T_3(u,u).$$

T_1 is decreasing, T_2 is increasing. We show that T_3 is increasing operator respect to the second argument, also decreasing respect to third argument. For $(x,y), (x',y') \in P \times P$ with $x \geq x'$ and $y \leq y'$, we have

$$\begin{aligned} T_3(x(s,t), y(s,t)) &= \int_0^1 G(t,\zeta) f(\zeta, x(s,\zeta), \frac{\partial}{\partial s} y(s,\zeta))d\zeta \\ &\geq_1 \int_0^1 G(t,\zeta) f(\zeta, x'(s,\zeta), \frac{\partial}{\partial s} y'(s,\zeta))d\zeta \\ &= T_3(x'(s,t), y'(s,t)). \end{aligned}$$

We can prove that T_1, T_2 and T_3 are satisfies (7) and (13). So we only need to prove that $T_1 h + T_2 h + T_3(h,h) \in P_h$. From H_1, H_2, H_3 and 2.1,

$$T_1 h(t) + T_2 h(t) + T_3(h(t), h(t)) = \int_0^1 G(t,\zeta)[f(\zeta,0) + g(\zeta, \zeta^{\nu-1}) + e(\zeta, \zeta^{\nu-1}, 0)]d\zeta$$

$$\leq \frac{t^{\nu-1}}{(1-\omega_2)\Gamma(\nu)} \int_0^1 (1-\zeta)^{\nu-1}[f(\zeta,0) + g(\zeta,1) + e(\zeta,1,0)]d\zeta,$$

$$T_1 h(t) + T_2 h(t) + T_3(h(t), h(t)) = \int_0^1 G(t,\zeta)[f(\zeta,0) + g(\zeta, \zeta^{\nu-1}) + e(\zeta, \zeta^{\nu-1}, 0)]d\zeta$$

$$\geq \frac{\omega_1 t^{\nu-1}}{(1-\omega_2)\Gamma(\nu)} \int_0^1 \zeta(1-\zeta)^{\nu-1}[f(\zeta,1) + g(\zeta,0) + e(\zeta,0,1)]d\zeta.$$

From (H_3) and (H_1) we have

$$f(\zeta,0) + g(\zeta,1) + e(\zeta,1,0) \geq f(\zeta,1) + g(\zeta,0) + e(\zeta,0,1) > 0.$$

Note that $\nu - 1 > 0$ and $f(\zeta,1) + g(\zeta,0) + e(\zeta,0,1) \not\equiv 0$, we get

$$\begin{aligned} &\int_0^1 (1-\zeta)^{\nu-1}[f(\zeta,0) + g(\zeta,1) + e(\zeta,1,0)]d\zeta \\ &\geq_1 \int_0^1 \zeta(1-\zeta)^{\nu-1}[f(\zeta,1) + g(\zeta,0) + e(\zeta,0,1)]d\zeta > 0. \end{aligned}$$

Let

$$l_1 := \frac{\omega_1}{(1-\omega_2)\Gamma(\nu)} \int_0^1 \zeta(1-\zeta)^{\nu-1}[f(\zeta,1) + g(\zeta,0) + e(\zeta,0,1)]d\zeta > 0,$$

$$l_2 := \frac{1}{(1-\omega_2)\Gamma(\nu)} \int_0^1 (1-\zeta)^{\nu-1}[f(\zeta,0) + g(\zeta,1) + e(\zeta,1,0)]d\zeta > 0.$$

Then $l_2 \geq l_1 > 0$ and thus $l_1 h(t) \leq T_1 h(t) + T_2 h(t) + T_3(h(t), h(t)) \leq l_2 h(t), t \in [0,1]$, hence $T_1 h(t) + T_2 h(t) + T_3(h(t), h(t)) \in P_h$.
Finally, by Theorem 3.1, $T_1 u + T_2 u + T_3(u,u) = u$ has a unique solution $x^* \in p$; for $u_0, v_0 \in P_h$, construct

$$u_n = T_1 v_{n-1} + T_2 u_{n-1} + T_3(u_{n-1}, v_{n-1})$$
$$v_n = T_1 u_{n-1} + T_2 v_{n-1} + T_3(v_{n-1}, u_{n-1}), n = 1, 2, \dots,$$

then $u_n \to x^*$ and $v_n \to x^*$. That is, problem (3) has a unique positive solution $x^* \in P_h$, where $h(t) = t^{\nu-1}, t \in [0,1]$ and for $u_0, v_0 \in P_h$, construct

$$u_{n+1}(s,t) = \int_0^1 G(t,\zeta)[f(s, \frac{\partial}{\partial s} v_n(s,\zeta)) + g(s, u_n(s,\zeta)) + e(s, u_n(s,\zeta), \frac{\partial}{\partial s} v_n(s,\zeta))]d\zeta,$$

$$v_{n+1}(s,t) = \int_0^1 G(t,\zeta)[f(s, u_n(s,\zeta)) + g(s, \frac{\partial}{\partial s} v_n(s,\zeta)) + e(s, \frac{\partial}{\partial s} v_n(s,\zeta), u_n(s,\zeta))]d\zeta,$$

$n = 0, 1, 2, ...$, then $u_n(s,t) \to x^*(s,t)$, $v_n(s,t) \to x^*(s,t)$. □

From the previous theorem and Corollary 3.1, we obtain the following result.

Corollary 3.2. *Assume (Q) and*

(H_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous and decreasing with respect to the second argument,* $f(t,1) \not\equiv 0$*;*
(H_2) $g : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous and increasing with respect to the second argument,* $g(t,0) \not\equiv 0$*;*
(H_3) $e : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *is continuous and increasing with respect to the second argument, also decreasing with respect to the third argument,* $e(t,0,1) \not\equiv 0$*;*
(H_4) *there exists* $\lambda \in (0,1)$ *such that*

$$f(t, \frac{1}{\lambda} v) \geq \lambda f(t,v), \quad g(t, \lambda u) \leq \frac{1}{\lambda} g(t,u), \qquad e(t, \lambda u, \frac{1}{\lambda} v) \geq \lambda^\gamma e(t,u,v)$$

for $t, \gamma \in (0,1), u, v \in [0,\infty)$.

Then the problem (2) has a unique positive solution x^* *in* P_h*, where* $h(t) = t^{\nu-1}, t \in [0,1]$ *and for* $x_0 \in P_h$*, construct*

$$u_{n+1}(s,t) = \int_0^1 G(t,\zeta)[f(s, \frac{\partial}{\partial s} v_n(s,\zeta)) + g(s, u_n(s,\zeta)) + e(s, u_n(s,\zeta), \frac{\partial}{\partial s} v_n(s,\zeta))]d\zeta,$$

$$v_{n+1}(s,t) = \int_0^1 G(t,\zeta)[f(s, u_n(s,\zeta)) + g(s, \frac{\partial}{\partial s} v_n(s,\zeta)) + e(s, \frac{\partial}{\partial s} v_n(s,\zeta), u_n(s,\zeta))]d\zeta,$$

$n = 0, 1, 2, ...$, *then* $u_n(s,t) \to x^*(s,t)$, $v_n(s,t) \to x^*(s,t)$ *where* $G(t,s)$ *is given as (4).*

Example 3.1. *Consider*

$$D_{0^+}^{2.6} x(s,t) + \frac{1}{\frac{\partial}{\partial s} x(s,t)} + x(s,t) + \Big(\frac{x(s,t)}{\frac{\partial}{\partial s} x(s,t)}\Big)^{2.6} e^t + a = 0, \tag{13}$$

$$0 < s < \frac{1}{2}, \quad 0 < t < 1$$

$$x(s,0) = \frac{\partial}{\partial t} x(s,0) = 0, \quad x(s,1) = \int_0^1 q_1(s,\zeta) x(s,\zeta) d\zeta,$$

where $a > 0$. *In this example,* $q_1(s,t) = (s+t)^2$. *Then* $q_1 : [0,1] \times [0,1] \to [0,\infty)$ *with* $q_1 \in L^1([0,1] \times [0,1])$, $\omega_1 = \int_0^1 \zeta^{1.6}(1-\zeta)(\zeta+s)^2 d\zeta > 0$ *and* $\omega_2 = \int_0^1 \zeta^{1.6}(\zeta+s)^2 d\zeta < 1$. *Take* $0 < b < a$ *and* $f, g : [0,1] \times (0,\infty) \times (0,\infty) \to [0,\infty)$ *defined by:*

$$f(t,v) = \frac{1}{v}, \quad g(t,u) = u + b, \quad e(t,u,v) = (\frac{u}{v})^{2.6} e^t + a - b.$$

f *is decreasing respect to the second argument,* g *is increasing respect to the second argument and* e *is increasing with respect to the second argument, also decreasing respect to third argument,* $f(t,1) > 0$, $g(t,0) = b > 0$ *and* $e(t,0,1) = a - b > 0$ *for* $\lambda \in (0,1)$, $t \in (0,1)$, $u, v \in (0,\infty)$, *also*

$$f(t, \frac{1}{\lambda} v) \geq \lambda f(t,v) \quad , g(t, \lambda u) \geq \lambda g(t,u)$$
$$e(t, \lambda u, \frac{1}{\lambda} v) \geq \lambda^{2.6} e(t,u,v).$$

So the conditions of corollary 3.2 are satisfied. Hence problem (13) has a solution in P_h, *where* $h(t,s) = (t+s)^{1.6}$, $0 < s < \frac{1}{2}$ *and* $0 < t < 1$.

References

[1] Feng, H. and Zhai, Ch., (2017), Existence and uniqueness of positive solutions for a class of fractional differential equation with integral boundary conditions, Nonlinear Analysis: modelling and control, 22(2), pp. 160-172.

[2] Guo, D. and Lakskmikantham, V., (1987), Coupled fixed points of nonlinear operators with applications. Nonlinear Anal., 11(5), pp. 623-632.

[3] Guo, D., (1988), Fixed points of mixed monotone operators with application, Appl. Anal., 34(34), pp. 215-224.

[4] Kilbas, A.A., Srivastava H.M. and Trujillo, j.j., (2006), Theory and applications of fractiona differential equations, North-Holland Mathematics Studies. 204(204), pp. 7-10.

[5] Podlubny, I., (1999), Fractional Differential Equations, Academic Press, New york.

[6] Sun, Y. and Zhao, M., (2014), Positive solutions for a class of fractional differential equations with integral boundary conditions, Appl. Math. Lett., 34(34), pp. 17-21.

[7] Wang, H. and Zhang, L., (2015), The solution for a class of sum operator equation and its application to fractional differential equation boundary value problems, Boundary Value Problem, 2015(2015).

[8] Yuan, C., (2013), Two positive solutions for (n-1,1)-type semigroup integral boundary value problems for coupled systems of nonlinear fractional differential equations, Commun. Nonlinear Sci. Numer. Simul., 18(18), pp. 858-866.

[9] Zhai, C.B. and Zhang, L.L., (2011), New fixed point theorems for a mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems, J. Math. Anal. Appl., 382(382), pp. 594-614.

Hadi Shojaat received his B.Sc. degree from Payam Noor University, M.Sc. degree from Urmia University. He is currently a Ph.D. candidate in Islamic Azad University, Tehran Branch, all in mathematics. His research interests include fixed point theory, functional analysis and differential equations.

Hojjat Afshari for the photograph and short biography, see TWMS J. Appl. and Eng. Math., V.6, No.2, 2016.

M. Sadegh Asgari received his B.S. degree in Arak University, M.S. in Shahid Beheshti University, and Ph.D. in Science and Research Branch, Islamic Azad University, all in mathematics. He published more than 60 ISI-indexed papers in well-known journals.