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EXISTENCE OF FIXED POINT AND BEST POINT OF PROXIMITY FOR MULTIFUNCTIONAL NON SELF MAPPINGS IN A PARTIAL METRIC SPACE

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Abstract. In this paper we give some theorems of existence of best points of proximity for a multifunctional non-self-mapping in a partial metric space and some approximations on the sets of the best points of proximity. Other results are also given.

Keywords: Partial metric space; Fixed point; best point; proximity.

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1. Introduction

Let A, B be two nonempty subsets of a metric space (X, d). The aim of this paper is to establish the existence theorems of a best proximity point $\overline{x} \in A$, which satisfies $inf\{p(\overline{x}, y) : y \in F(\overline{x})\} = dist(A, B)$ for a non-self-mapping multifunction $F : A \to 2^B$. In this article, we prove that the results obtained in [3] can be enhanced and managed on partial metric spaces. We define

$$d(x, B) = \inf_{y \in B} d(x, y),\ e(A, B) = \sup_{x \in A} d(x, B)$$

and

$$D(A, B) = \max(e(A, B), e(B, A)).$$

Here $e(A, B)$ is the excess of A over B and $D(A, B)$ is the Pompeiu-Hausdorff distance between A and B. And let

$$A_0 = \{x \in A : d(x, y) = d(A, B), for\, some\, y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B), for\, some\, x \in A\}.$$

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Definition 1.1. [3] *Let (X,d) be a partial metric space and (A,B) a pair of nonempty subsets of X, and $A \neq \emptyset$.*
We say that the pair (A,B) satisfies the P-property if and only if

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,y_1) = d(x_2,y_2).$$

Where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Theorem 1.1. [3] *Let (X,d) be a complete metric space and A, B two closed and nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A,B) satisfies the P-property. We assume that $F : A \to 2^B$ a multifunction with bounded and closed value. If there exists a constant $\theta \in (0,1)$ such that $\Big[D(F(x),F(y)) \leq \theta d(x,y), \qquad \forall x,y \in A\Big]$ and $F(x) \subseteq B_0$ for all $x \in A_0$. Then F has a best proximity point $\overline{x}$ in A.*

Definition 1.2. *We say that the function $p : X \times X \to [0,+\infty[$ is a partial metric on X if the following conditions are satisfied:*

(a) $p(x,x) = p(y,y) = p(x,y)$ *if and only if* $x = y$ *for all* $x,y \in X$,
(b) $p(x,x) \leq p(x,y)$ *for all* $x,y \in X$,
(c) $p(x,y) = p(y,x)$ *for all* $x,y \in X$,
(d) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ *for all* $x,y,z \in X$.

Then (X,p) is called a partial metric space.

For a partial metric p on X, the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1}$$

and

$$p^w(x,y) = p(x,y) - min\{p(x,x), p(y,y)\} \tag{2}$$

are metrics on X. Each partial metric p on X Generates a T_0-topology τ_p with a basis the families of open p-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.3. [17]

(i) *A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.*
(ii) *A sequence $\{x_n\}$ in a partial metric space (X,p) is called Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and finite.*
(iii) *A partial metric space (X,p) est is complete if any Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.*

Lemma 1.1. [17]

($a1$) *A sequence $\{x_n\}$ is Cauchy in a partial metric space (X,p) if and only if $\{x_n\}$ is Cauchy in the space (X,p^s).*
($a2$) *A partial metric space (X,p) is complete if and only if the space (X,p^s) Is complete. Morewer*

$$\lim_{n\to\infty} p^s(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{3}$$

Let A, B be two nonempty subsets of a partial metric space (X, p), the partial surplus $e_p(A, B)$ from A to B and $D_p(A, B)$ are the partial distance of Pompeiu-Hausdorff between A and B defined as follows:

$$p(x, B) = \inf_{y \in B} p(x, y),$$

$$p(A, B) = \sup_{x \in A} p(x, B)$$

and

$$p(A, B) = \max(e_p(A, B), e_p(B, A)).$$

Remark 1.1. [6] *Let (X, p) be a partial metric space and A a nonempty subset of X, then $a \in \overline{A} \Leftrightarrow p(a, A) = p(a, a)$.*

Lemma 1.2. [7] *Let (X, p) be a partial metric space and A, B two closed, nonempty and bounded subsets of X, and $h > 1$. Then for each $a \in A$, there exists $b \in B$, such that $p(a, b) \leq hD_p(A, B)$.*

We next apply the notations

$$A_0 = \{x \in A : p(x, y) = p(A, B), for\, some\, y \in B\}$$

and

$$B_0 = \{y \in B : p(x, y) = p(A, B), for\, some\, x \in A\}.$$

Definition 1.4. *Let (X, p) be a partial metric space and (A, B) a pair of nonempty subsets of X, and $A \neq \emptyset$.*
We say that the pair (A, B) satisfies the P-property if and only if

$$\begin{cases} p(x_1, y_1) = p(A, B) \\ p(x_2, y_2) = p(A, B) \end{cases} \Rightarrow \quad p(x_1, y_1) = p(x_2, y_2).$$

Where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

2. Main results

2.1. Existence of best proximity point for non-self mapping multifunction.

Theorem 2.1. *Let (X, p) be a complete partial metric space and A, B two closed and nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A, B) satisfies the P-property . We assume that $F : A \to 2^B$ a multifunction with bounded and closed values. If there exists a constant $\theta \in (0, 1)$ such that*

$$D_p(F(x), F(y)) \leq \theta p(x, y), \quad \forall x, y \in A.$$

And $F(x) \subseteq B_0$ for each $x \in A_0$. Then F has a best proximity point $\overline{x}$ in A.

Proof. Let $x_0 \in A_0$, and $y_0 \in F(x_0) \subseteq B_0$. Then there exists $x_1 \in A_0$, such that $p(x_1, y_0) = p(A, B)$. On the other hand, by lemma 1.2 we obtain $y_1 \in F(x_1) \subseteq B_0$ such that

$$p(y_0, y_1) \leq D_p(F(x_0), F(x_1)) + \theta.$$

Futher more, there exists $x_2 \in A_0$, such that $p(x_1, y_1) = p(A, B)$, and $y_2 \in F(x_2) \subseteq B_0$ such that

$$p(y_1, y_2) \leq D_p(F(x_1), F(x_2)) + \theta^2.$$

One continues for getting $x_n \in A_0$, such that $p(x_n, y_{n-1}) = p(A, B)$, and $y_n \in F(x_n) \subseteq B_0$ such that

$$p(y_{n-1}, y_n) \leq D_p(F(x_{n-1}), F(x_n)) + \theta^n.$$

Since $p(x_{n+1}, y_n) = p(A, B)$ and $p(x_n, y_{n-1}) = p(A, B)$, by the P-property, we obtain $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$. Which gives

$$\begin{aligned} p(x_n, x_{n+1}) = p(y_{n-1}, y_n) &\leq D_p(F(x_{n-1}), F(x_n)) + \theta^n \\ &\leq \theta p(x_{n-1}, x_n) + \theta^n \\ &\leq \theta p(y_{n-2}, y_{n-1}) + \theta^n) \\ &\leq \theta(D_p(F(x_{n-2}), F(x_{n-1})) + \theta^{n-1}) + \theta^n \\ &\leq \theta^2 p(x_{n-2}, x_{n-1}) + 2\theta^n \\ &\vdots \\ &\leq \theta^n p(x_0, x_1) + n\theta^n \to 0, \ when\, n \to +\infty. \end{aligned}$$

Next by definition of p^s, we get $p^s(x_n, x_{n+1}) \leq 2p(x_n, x_{n+p}) \to 0$, when $n \to +\infty$. We now prove that (x_n) is a Cauchy sequence in (X, p^s). Suppose that there exist an $\epsilon > 0$ and $k \in \mathbb{N}$, there exist $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and $p(x_{m_k}, x_{n_k}) > \epsilon$. We have

$$\begin{aligned} p(A, B) &\leq p(y_{n_k-1}, x_{m_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) \\ &\leq p(y_{n_k-1}, x_{n_k}) + p(x_{n_k}, x_{n_k+1}) + ... + p(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Taking the limits on k we get

$$\lim_{k\to\infty} p(y_{n_k-1}, x_{n_k}) + \lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = p(A, B).$$

Which gives $\lim_{k\to\infty} p(x_{n_k}, x_{m_k}) = 0$. Since $p^s(x_{n_k}, x_{m_k}) < 2p(x_{n_k}, x_{m_k}) \to 0$, then (x_n) is a Cauchy sequence in (X, p^s), and since (X, p) is complete then by lemma 1.1, (X, p^s) is a complete metric space and so the sequence (x_n) converges in X. Let $\overline{x} = \lim_n x_n$. Since A is closed, we have $\overline{x} \in A$.

Now, since $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$, the sequence (y_n) is convergent in X. Let $\overline{y} = \lim_n y_n$.

Sine B is closed, we have $\overline{y} \in B$. On the other hand, by lemma 1.1, we have

$$p(\overline{x}, \overline{x}) = \lim_{n \to +\infty} p(x_n, \overline{x}) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0$$

and

$$p(\overline{y}, \overline{y}) = \lim_{n \to +\infty} p(y_n, \overline{y}) = \lim_{n,m \to +\infty} p(y_n, y_m) = 0$$

and $p(\overline{x}, \overline{y}) = p(A, B)$. Hence,

$$\begin{array}{rcl} 0 & \leq & p(y_n, F(\overline{x})) \\ & \leq & D_p(F(x_n), F(\overline{x})) \\ & \leq & \theta p(x_n, \overline{x}). \end{array}$$

We take the limit $n \to +\infty$, we obtain $p(\overline{y}, F(\overline{x})) = 0$ which gives $p(\overline{y}, F(\overline{x})) = p(\overline{y}, \overline{y})$, using remark 1.1, to get $\overline{y} \in \overline{F(\overline{x})} = F(\overline{x})$. One obtains that $\overline{x}$ is a best proximity point in A which satisfies $p(\overline{x}, F(\overline{x})) = p(A, B)$. □

As a direct result of Theorem 2.1, we have the following result.

Corollary 2.1. [7] *Let (X, p) be a complete partial metric space and $F : X \to 2^X$ be a multifunction with bounded and closed values. If there exists a constant $\theta \in (0, 1)$ such that*

$$D_p(F(x), F(y)) \leq \theta p(x, y), \ \ \forall x \in X \text{ and } \forall y \in X.$$

Then F has a fixed point $\overline{x}$ in X.

In the following, let PF_i be the set of best proximity points for a multifunction F_i.

Theorem 2.2. *Let (X, d) be a complete metric space and A, B two closed nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A, B) satisfies the P-property. Let $F_i : A \to 2^B, i = 1, 2$ be two multifunctions with compact non empty values. If there exist two constants $\theta_1, \theta_2 \in (0, 1)$ such that*

$$D(F_1(x), F_1(y)) \leq \theta_1 d(x, y) \ \forall x \in A \text{ and } \forall y \in A,$$

$$D(F_2(x), F_2(y)) \leq \theta_2 d(x, y) \ \forall x \in A \text{ and } \forall y \in A$$

and $F_i(x) \subseteq B_0, i = 1, 2,$ for each $x \in A_0$. Then

$$D(PF_1, PF_2) \leq \frac{1}{1 - max\{\theta_1, \theta_2\}} [\sup_{x \in A} D(F_1(x), F_2(x))].$$

Proof. Let $\varepsilon > 0$, we choose $\beta > 0$ such that $\beta \sum n\theta_2^n < 1$ and $\alpha = \dfrac{\beta\varepsilon}{(1 - \theta_2)}$. Let $x_{0,1} = x_0 \in PF_1$, there exists $y_{0,1} \in F_1(x_0) \subseteq B_0$ such that $d(x_0, y_{0,1}) = d(A, B)$. On the other hand, there exist $y_0 \in F_2(x_0) \subseteq B_0$ and $x_1 \in A_0$ such that $d(x_1, y_0) = d(A, B)$. By the P-property, we get $d(x_0, x_1) = d(y_{0,1}, y_0)$. Which gives

$$d(x_0, x_1) \leq D(F_1(x_0), F_2(x_0)) + \varepsilon.$$

We take $y_1 \in F(x_1) \subseteq B_0$ such that

$$d(y_0, y_1) \leq D(F_2(x_0), F_2(x_1)) + \theta_2 \alpha.$$

Furthermore, there exists $x_1 \in A_0$, such that $d(x_1, y_1) = d(A, B)$, and there exists $y_2 \in F_2(x_2) \subseteq B_0$ such that

$$d(y_1, y_2) \leq D(F_2(x_1), F_2(x_2)) + \theta_2^2\alpha.$$

We continue to find $x_n \in A_0$, such that $d(x_n, y_{n-1}) = d(A, B)$, and there exists $y_n \in F_2(x_n) \subseteq B_0$ such that

$$d(y_{n-1}, y_n) \leq D(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n\alpha.$$

Since $d(x_{n+1}, y_n) = d(A, B)$ and $d(x_n, y_{n-1}) = d(A, B)$, by the P-property, we get $d(x_n, x_{n+1}) = d(y_{n-1}, y_n)$. Which gives

$$\begin{aligned} d(x_n, x_{n+1}) = d(y_{n-1}, y_n) &\leq D(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n\alpha \\ &\leq \theta_2 d(x_{n-1}, x_n) + \theta_2^n\alpha \\ &\leq \theta_2 d(y_{n-2}, y_{n-1}) + \theta_2^n\alpha \\ &\leq \theta_2(D(F_2(x_{n-2}), F_2(x_{n-1})) + \theta_2^{n-1}) + \theta_2^n\alpha \\ &\leq \theta_2^2 d(x_{n-2}, x_{n-1}) + 2\theta_2^n\alpha \\ &\vdots \\ &\leq \theta_2^n d(x_0, x_1) + n\theta_2^n\alpha \to 0, \ when\, n \to +\infty. \end{aligned} \tag{4}$$

On the other hand, we prove that (x_n) is a Cauchy sequence in A. One suppose that there exists $\epsilon > 0$ and for all $k \in \mathbb{N}$, there exist $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and $d(x_{m_k}, x_{n_k}) > \epsilon$. We have

$$\begin{aligned} d(A, B) &\leq d(y_{n_k-1}, x_{m_k}) \\ &\leq d(y_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) \\ &\leq d(y_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) \\ &\leq d(y_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + ... + d(x_{m_k-1}, x_{m_k}). \end{aligned} \tag{5}$$

Taking the limit, we get

$$\lim_{k\to\infty} d(y_{n_k-1}, x_{n_k}) + \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = d(A, B).$$

Which gives $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = 0$. We get (x_n) a Cauchy sequence in a complete metric space and so the sequence (x_n) is convergent in X. Let $\overline{x_2} = \lim_n x_n$. Since A is closed, we get $\overline{x_2} \in A$.

Since $d(x_n, x_{n+1}) = d(y_{n-1}, y_n)$, the sequence (y_n) converges in X. Let $\overline{y} = \lim_n y_n$. Since B is closed, we get $\overline{x_2} \in B$ and

$$\begin{aligned} 0 &\leq d(y_n, F_2(\overline{x_2})) \\ &\leq D(F_2(x_n), F_2(\overline{x_2})) \\ &\leq \theta_2 d(x_n, \overline{x_2}). \end{aligned}$$

Taking the limit, $n \to +\infty$, we get $d(\overline{y}, F_2(\overline{x_2})) = 0$. We obtain $\overline{x_2}$ as the best proximity point in A which satisfies $d(\overline{x_2}, F_2(\overline{x_2})) = d(A, B)$. Then we get

$$\begin{aligned} d(x_0, \overline{x_2}) &\leq \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=0}^{\infty} (\theta_2^n d(x_0, x_1) + n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2} d(x_0, x_1) + \sum_{n=1}^{\infty} n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2} (d(x_0, x_1) + \varepsilon) \\ &\leq \frac{1}{1-\theta_2} (D(F_1(x_0), F_2(x_0)) + 2\varepsilon) \end{aligned} \tag{6}$$

We obtain for all $\varepsilon > 0$,

$$d(x_{0,1}, \overline{x_2}) \leq \frac{1}{1-\theta_2} (D(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon).$$

As previously, let $x_{0,2} \in PF_2$, there exists $\overline{x_1} \in PF_1$ such that

$$d(x_{0,2}, \overline{x_1}) \leq \frac{1}{1-\theta_1} (D(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon), \ \forall \varepsilon > 0.$$

□

Theorem 2.3. *Let (X, p) a complete partial metric space and A, B two closed nonempty subsets of X such that $A_0 \neq \emptyset$ and the pair (A, B) satisfies the P-property. Let $F_i : A \to 2^B, i = 1, 2$ two multifunctions with compact values. If there exist two constants $\theta_1, \theta_2 \in (0, 1)$ such that*

$$D_p(F_1(x), F_1(y)) \leq \theta_1 p(x, y) \ \forall x \in A\,, \forall y \in A,$$
$$D_p(F_2(x), F_2(y)) \leq \theta_2 p(x, y) \ \forall x \in A\,, \forall y \in A,$$

and $F_i(x) \subseteq B_0, i = 1, 2,$ for each $x \in A_0$. Then

$$D_p(PF_1, PF_2) \leq \frac{1}{1 - max\{\theta_1, \theta_2\}} [\sup_{x \in A} D_p(F_1(x), F_2(y))].$$

Proof. Let $\varepsilon > 0$, we choose $\beta > 0$ such that $\beta \sum n\theta_2^n < 1$ and $\alpha = \dfrac{\beta\varepsilon}{(1-\theta_2)}$. Let $x_{0,1} = x_0 \in PF_1$, there exist $y_{0,1} \in F_1(x_0) \subseteq B_0$ such that $d(x_0, y_{0,1}) = d(A, B)$. Otherwise, there exist $y_0 \in F_2(x_0) \subseteq B_0$ and $x_1 \in A_0$ such that $p(x_1, y_0) = p(A, B)$. By the P-property, we get $p(x_0, x_1) = p(y_{0,1}, y_0)$, which gives

$$p(x_0, x_1) \leq D_p(F_1(x_0), F_2(x_0)) + \varepsilon.$$

We take $y_1 \in F(x_1) \subseteq B_0$ such that

$$p(y_0, y_1) \leq D_p(F_2(x_0), F_2(x_1)) + \theta_2.$$

Else, there exists $x_1 \in A_0$, such that $p(x_1, y_1) = p(A, B)$, and there exists $y_2 \in F_2(x_2) \subseteq B_0$

such that

$$p(y_1, y_2) \leq D_p(F_2(x_1), F_2(x_2)) + \theta_2^2.$$

We continue to find $x_n \in A_0$, such that $p(x_n, y_{n-1}) = p(A, B)$, and $y_n \in F_2(x_n) \subseteq B_0$ such that

$$p(y_{n-1}, y_n) \leq D_p(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n.$$

Since $p(x_{n+1}, y_n) = p(A, B)$ and $p(x_n, y_{n-1}) = p(A, B)$, by the P-property, we get $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$. Which gives

$$\begin{aligned} p(x_n, x_{n+1}) = p(y_{n-1}, y_n) &\leq D_p(F_2(x_{n-1}), F_2(x_n)) + \theta_2^n \\ &\leq \theta_2 p(x_{n-1}, x_n) + \theta_2^n \\ &\leq \theta_2(D_p(F_2(y_{n-2}), F_2(y_{n-1})) + \theta_2^{n-1}) \\ &\leq \theta_2(D_p(F_2(x_{n-2}), F_2(x_{n-1})) + \theta_2^{n-1}) + \theta_2^n \\ &\leq \theta_2^2 p(x_{n-2}, x_{n-1}) + 2\theta_2^n \\ &\vdots \\ &\leq \theta_2^n p(x_0, x_1) + n\theta_2^n \to 0, \ when\, n \to +\infty. \end{aligned}$$

By definition of p^s, we get $p^s(x_n, x_{n+1}) \leq 2p(x_n, x_{n+p}) \to 0$, when $n \to +\infty$. On the other hand, we prove that (x_n) is a Cauchy in (X, p^s). One suppose that there exist $\epsilon > 0$ and for all $k \in \mathbb{N}$, there exists $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and $p(x_{m_k}, x_{n_k}) > \epsilon$. We have

$$\begin{aligned} p(A, B) &\leq p(y_{n_k - 1}, x_{m_k}) \\ &\leq p(y_{n_k - 1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) - p(x_{n_k}, x_{n_k}) \\ &\leq p(y_{n_k - 1}, x_{n_k}) + p(x_{n_k}, x_{m_k}) \\ &\leq p(y_{n_k - 1}, x_{n_k}) + p(x_{n_k}, x_{n_k + 1}) + ... + p(x_{m_k - 1}, x_{m_k}). \end{aligned}$$

Taking the limit we get

$$\lim_{k \to \infty} p(y_{n_k - 1}, x_{n_k}) + \lim_{k \to \infty} p(x_{n_k}, x_{m_k}) = p(A, B).$$

Which gives $\lim_{k \to \infty} p(x_{n_k}, x_{m_k}) = 0$. Since $p^s(x_{n_k}, x_{m_k}) < 2p(x_{n_k}, x_{m_k}) \to 0$, we get (x_n) a Cauchy sequence in (X, p^s), and since (X, p) is complete then by lemma 1.1, (X, p^s) is a complete metric space and the sequence (x_n) converges in X. Let $\overline{x_2} = \lim_n x_n$. Since A is closed we have $\overline{x_2} \in A$. Next, since $p(x_n, x_{n+1}) = p(y_{n-1}, y_n)$, the sequence (y_n) converges in X. Let $\overline{y} = \lim_n y_n$. Since B is closed we have $\overline{y} \in B$.
Otherwise, by lemma 1.1, we have

$$p(\overline{x_2}, \overline{x_2}) = \lim_{n \to +\infty} p(x_n, \overline{x_2}) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0,$$

$$p(\overline{y},\overline{y}) = \lim_{n\to+\infty} p(y_n,\overline{y}) = \lim_{n,m\to+\infty} p(y_n,y_m) = 0,$$

and $p(\overline{x_2},\overline{y}) = p(A,B)$. As result,

$$\begin{aligned} 0 &\leq p(y_n, F_2(\overline{x_2})) \\ &\leq D_p(F_2(x_n), F_2(\overline{x_2})) \\ &\leq \theta_2 p(x_n, \overline{x_2}). \end{aligned}$$

Taking the limit $n \to +\infty$, we get $p(\overline{y}, F_2(\overline{x_2})) = 0$ which gives $p(\overline{y}, F_2(\overline{x_2})) = p(\overline{y},\overline{y})$, using remark 1.1, to get $\overline{y} \in \overline{F_2(\overline{x_2})} = F_2(\overline{x_2})$. Then we obtain $\overline{x_2}$ as a best proximity point in A which satisfies $p(\overline{x_2}, F_2(\overline{x_2})) = p(A,B)$. Then we get

$$\begin{aligned} p(x_0,\overline{x_2}) &\leq \sum_{n=0}^{\infty} p(x_n,x_{n+1}) - \sum_{n=1}^{\infty} p(x_n,x_n) \\ &\leq \sum_{n=0}^{\infty} p(x_n,x_{n+1}) \\ &\leq \sum_{n=0}^{\infty} (\theta_2^n p(x_0,x_1) + n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2} p(x_0,x_1) + \sum_{n=1}^{\infty} n\theta_2^n \alpha) \\ &\leq \frac{1}{1-\theta_2} (p(x_0,x_1) + \varepsilon) \\ &\leq \frac{1}{1-\theta_2} (D_p(F_1(x_0), F_2(x_0)) + 2\varepsilon). \end{aligned} \tag{7}$$

We obtain for all $\varepsilon > 0$,

$$p(x_{0,1},\overline{x_2}) \leq \frac{1}{1-\theta_2}(D_p(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon).$$

As previously, let $x_{0,2} \in PF_2$, there exists $\overline{x_1} \in PF_1$ such that

$$p(x_{0,2},\overline{x_1}) \leq \frac{1}{1-\theta_1}(D_p(F_1(x_{0,1}), F_2(x_{0,1})) + \varepsilon), \ \forall \varepsilon > 0.$$

□

3. Conclusions

In this paper, we have proved some results of best points of proximity for a multifunctional non-self-mapping in a partial metric space and some approximations on the sets of the best points of proximity.

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