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EDGE H-DOMINATION IN GRAPH

B. M. KAKRECHA[1], §

Abstract. This paper is about edge H-domination of the graph. The edge H-dominating set is defined and the characterization of a minimal edge H-dominating set of the graph with minimum degree 2 is given. The upper bound on the edge H-domination number of the graph is discussed. The changes in the edge H-domination number are observed under vertex(edge) removal operation on graph. The concepts called weak isolated edge and edge H-private neighborhood of the graph are defined.

Keywords: edge dominating set, edge domination number, edge H-dominating set, edge H-domination number, weak isolated edge, edge H-private neighborhood.

AMS Subject Classification: 05C69

1. Introduction

The *hypergraph* H is an order pair (V, E). The nonempty set V contains the elements $\{v_1, v_2, \ldots, v_n\}$ and $E = \{E_1, E_2, \ldots, E_m\}$ is a family of subsets of V such that $\bigcup_{i=1}^{m} E_i = V$ and each E_i is nonempty. The elements of V and E are called vertices and edges of the hypergraph H, respectively [3]. Two vertices x and y of the hypergraph H are *adjacent* if there is an edge E_i of H such that $\{x, y\} \subseteq E_i$. A set of vertices $S \subseteq V(H)$ of hypergraph H is a *dominating set* of H if for each vertex $v \in V(H) - S$, there is a vertex $u \in S$ such that the vertices u and v are adjacent in H [1]. A set of vertices $S \subseteq V(H)$ is an *H-dominating set* of hypergraph H if for each vertex $v \in V(H) - S$, there is an edge F containing v such that $F - \{v\}$ is a nonempty subset of S [6].

Let $G = (V(G), E(G))$ be a graph without isolated vertices. The *dual hypergraph* G^* of the graph G is a hypergraph with vertex set $V(G^*) = E(G)$ and edge set $E(G^*) = \{\overline{v} \mid v \in V(G)\}$ where $\overline{v} = \{e \in E(G) \mid v$ is an end vertex of $e\}$ [3]. The dual hypergraph is one type of transformation of the graph. From the H-domination of hypergraph, the transformation inspire to define a new edge variant of the graph, called edge H-domination of the graph. The interesting results about H-domination in hypergraph are stated and proved in [6]. In this paper, We observe various properties of edge H-domination, bounds on edge H-domination number and the effects of vertex(edge) removal operation on edge H-domination number of the graph. For any edge $e = uv \in E(G)$, e is an isolated edge of

[1] Department of Mathematics, L. E. College, Morbi−363642, Gujarat, India.
e-mail: kakrecha.bhavesh2011@gmail.com; ORCID: https://orcid.org/0000-0002-5825-3831.
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the graph G if both end vertices u and v are pendant vertices, that is, $deg(u) = deg(v) = 1$. e is pendant edge if degree of exactly one end vertex of e is one. We define edge H-domination in graph as follow.

Definition 1.1. A set $F \subseteq E(G)$ is said to be an *edge H-dominating set* of G if the following conditions are satisfied by any edge $e = uv$ in $E(G)$.
(1) If e is an isolated edge then $e \in F$.
(2) If e is a pendant edge with v as a pendant vertex and u is not a pendant vertex. If $uv \notin F$ then all the edges incident at u (except e) are in F.
(3) If e is a pendant edge with u as a pendant vertex and v is not a pendant vertex. If $uv \notin F$ then all the edges incident at v (except e) are in F.
(4) If neither u nor v is a pendant vertex and $uv \notin F$ then all the edges incident at u (except e) are in F or all the edges incident at v (except e) are in F.

We recall the definition of edge domination in graph. Let G be a graph. A subset F of an edge set $E(G)$ is said to be an *edge dominating set* of G if for every edge e not in F is adjacent to some edge in F. An edge dominating set F of G is a *minimal* edge dominating set if F does not have a proper subset which is an edge dominating set. An edge dominating set with minimum cardinality is a *minimum* edge dominating set. The cardinality of a minimum edge dominating set is the *edge domination number* (denoted by $\gamma^{'}(G)$) of the graph G [4].

Example 1.1. Consider the following graph with vertices 1, 2, 3, 4, 5, 6 and edges 12, 23, 34, 45, 51 and 56.

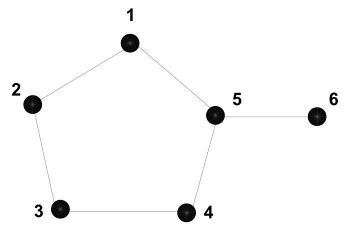

FIGURE 1. Graph with six vertices

The edge sets $\{23, 45, 15\}$, $\{23, 45, 56\}$ are edge H-dominating sets of the graph. The edge set. $F = \{23, 45\}$ is an edge dominating set but F is not an edge H-dominating set.

Remark 1.1. (**1**) If an edge set F of the graph G is an edge H-dominating set then F contains all isolated edges of G.
(**2**) Let G be a graph. Every edge H-dominating set of G is edge dominating set but the converse need not be true.
(**3**) If G is a graph with $\triangle(G) \leq 2$ then every edge dominating set of G is an edge H-dominating set, where $\triangle(G)$ = maximum degree of a vertex in G. In particular, if the graph G is a cycle graph or path graph then every edge dominating set of G is an edge H-dominating set.

Definition 1.2. Let G be a graph and F be an edge H-dominating set of G. The set F is said to be a minimal edge H-dominating set if $F - \{e\}$ is not an edge H-dominating set for every edge $e \in F$.

Example 1.2. The edge sets $\{12, 45, 56\}$, $\{34, 15, 56\}$ are minimal edge H-dominating sets of the graph given in figure 1.

The following theorem characterizes a minimal edge H-dominating set of the graph G with $\delta(G) \geq 2$, where $\delta(G) =$ minimum degree of a vertex in G.

Theorem 1.1. Let G be a graph with $\delta(G) \geq 2$ and F be an edge H-dominating set of G then F is a minimal edge H-dominating set if and only if for every edge $e = uv$ in F, one of the following holds.
(**1**) There is an edge h incident at u such that $h \neq uv$ and $h \notin F$ also there is an edge $h^{'}$ incident at v such that $h^{'} \neq uv$ and $h^{'} \notin F$.
(**2**) There is an edge $f = uy$ such that $f \notin F$ and the following two conditions hold.
(i) all the edges incident at u (except uy) are in F.
(ii) there is an edge g incident at y ($g \neq uy$) such that $g \notin F$.

or (of (**2**))

There is an edge $vx \notin F$ such that all the edges incident at v (except vx) are in F and there is an edge $g^{'}$ incident at x ($g^{'} \neq vx$) such that $g^{'} \notin F$.

Proof. Suppose F is minimal and let $e = uv \in F$. Consider the set $F_1 = F - \{e\}$ then F_1 is not an edge H-dominating set of G. Therefore there is an edge l not in F_1 such that $l = xy$ and some edge incident at x is not in F_1 and some edge incident at y is not in F_1.
Case 1. $l = e$ then as stated above, there is an edge incident at u, which is not in F_1 and there is an edge incident at v, which is not in F_1. Therefore these two edges incident at u and v respectively can not be in F also. Thus, condition (1) holds.
Case 2. $l \neq e$ then $l \notin F$. Note that there is an edge incident at x, which is not in F_1 and also there is an edge incident at y, which is not in F_1. However, $l = xy \notin F$ therefore all the edges incident at x are in F or all the edges incident at y are in F. Suppose all the edges incident at x are in F. However, as mentioned above, there is an edge h incident at x which is not in F_1. Therefore this edge must be $h = uv$. Therefore $u = x$ or $v = x$. We may assume that $u = x$. Since $l \neq e$ and since $x = u$, $y \neq v$. Again as mentioned above, there is an edge g incident at y such that $g \notin F_1$. Infect $g \notin F$, because otherwise $g = uv$ and it will imply that $v = y$ which is not true. By symmetric argument, if all the edges incident at y are in F, then (2) is proved with different notations.

Conversely, suppose one of the conditions (1) or (2) is satisfied. Suppose (1) is satisfied. Consider the set $F_1 = F - \{e\}$. Since there is an edge incident at u which is not in F, it is also not in F_1. Similarly, there is an edge incident at v which is not in F, it is also not in F_1. Therefore F_1 is not an edge H-dominating set of G. Suppose (2) is satisfied. First suppose an edge $l = xy = uy$ satisfies condition (2). We may note that all the edges incident at u are in F but there is an edge namely uv incident at u which is not in $F - \{uv\} = F_1$. Also there is an edge g incident at y which is not in F. Since $F_1 \subseteq F$, $g \notin F_1$. Therefore F_1 fails to edge H-dominate uy. Therefore F is a minimal edge H-dominating set. □

Theorem 1.2. Let G be a graph without isolated edges. If an edge set F is a minimal edge dominating set of G then $E(G) - F$ is an edge H-dominating set of G.

Proof. Let F be a minimal edge dominating set of G and $e = uv \in F$. Since F is minimal edge dominating set, one of the following two conditions holds.
(1) e is not adjacent with any other edge of F.
(2) There is an edge $f \notin F$ such that f is adjacent with only one member of F namely e.

Suppose (1) holds for $e = uv$. Since e is not an isolated edge of G, there is an edge h incident at u or there is an edge $h^{'}$ incident at v. Since e is not adjacent with any edge in F, all the edges incident at u (except e) are in $E(G) - F$ and all the edges incident at v (except e) are in $E(G) - F$.

Suppose (2) holds. Since f is adjacent to e. f can be written as $f = uy$ ($y \neq v$) or $f = vx$ ($x \neq u$). We may assume that $f = uy$. Also f is not adjacent with any other edge of F. Therefore any edge h incident at u (except uv) can not be in F. Therefore h must be in $E(G) - F$. Thus, all the edges incident at u (except uv) are in $E(G) - F$.

Similarly, if $f = vx$ then all the edges incident at v (except uv) are in $E(G) - F$. Thus, $E(G) - F$ is an edge H-dominating set of G. □

Theorem 1.3. Let G be a graph with $\delta(G) \geq 2$. If an edge set F is a minimal edge H-dominating set of G then $E(G) - F$ is an edge dominating set of G.

Proof. Let F be a minimal edge H-dominating set of G and $e = uv \in F$. Since F is minimal edge H-dominating set, condition (1) or (2) of theorem 1.1 holds.

Suppose (1) holds for $e = uv$ then there is an edge h incident at u such that $h \notin F$ and there is an edge $h^{'}$ incident at v such that $h^{'} \notin F$. Therefore $h \in E(G) - F$ and $h^{'} \in E(G) - F$ and both h and $h^{'}$ are adjacent to e. Thus, we have proved that if $e \notin E(G) - F$ and if condition (1) holds then e is adjacent to some member of $E(G) - F$.

Suppose (2) holds then there is an edge f which is either in the form uy or vx such that $f \notin E(G) - F$. Thus e is adjacent to f which is a member of $E(G) - F$. Here also we have proved that if $e \notin E(G) - F$ then e is adjacent to some f which is in $E(G) - F$. Therefore $E(G) - F$ is an edge dominating set. □

Definition 1.3. An edge H-dominating set with minimum cardinality is called *minimum* edge H-dominating set of the graph G. The cardinality of a minimum edge H-dominating set is called *edge H-domination number* of G, denoted by $\gamma_H^{'}(G)$.

2. Upper bound of number

Consider the graph G with $|V(G)| = n$ and $|E(G)| = m$.

Proposition 2.1. Let G be a graph without isolated edges then $\gamma^{'}(G) \leq \frac{m}{2}$.

Proof. Omitted. □

Proposition 2.2. Let G be a graph with $\delta(G) \geq 2$ then $\gamma_H^{'}(G) \leq m - \triangle(G)$.

Proof. Let G be a graph with $\triangle(G) = k$. Let v be a vertex such that $deg(v) = k$. Suppose $u_1, u_2, \ldots, u_k$ are all the neighbors of v also $deg(u_i) \geq 2$ for $i = 1, 2, 3, \ldots, k$, then the edge set $E(G) - \{vu_1, vu_2, \ldots, vu_k\}$ is an edge H-dominating set.
It implies that, $\gamma_H^{'}(G) \leq |E(G) - \{vu_1, vu_2, \ldots, vu_k\}|$.
That is, $\gamma_H^{'}(G) \leq |E(G)| - |\{vu_1, vu_2, \ldots, vu_k\}|$.
It implies that, $\gamma_H^{'}(G) \leq m - \triangle(G)$. □

Theorem 2.1. Let G be a graph without isolated edges then $\gamma^{'}(G) + \gamma_H^{'}(G) \leq m$.

Proof. Let G be a graph without isolated edges. Let F be a minimum edge dominating set of G. Obviously, F is a minimal edge dominating set of G then by theorem 1.2, $E(G) - F$ is an edge H-dominating set of G. Therefore,
$\gamma_H^{'}(G) \leq |E(G) - F| = |E(G)| - |F| = |E(G)| - \gamma^{'}(G)$.
Therefore, $\gamma_H^{'}(G) \leq |E(G)| - \gamma^{'}(G)$
It implies that, $\gamma^{'}(G) + \gamma_H^{'}(G) \leq m$. □

Corollary 2.1. Let G be a graph without isolated edges. If $\gamma^{'}(G) + \gamma_H^{'}(G) < m$ then $\gamma^{'}(G) < \frac{m}{2}$.

Corollary 2.2. Let G be a graph without isolated edges. If $\gamma'(G) = \frac{m}{2}$ then $\gamma'_H(G) = \frac{m}{2}$.

Proof. Let G be a graph without isolated edges. By theorem 2.1, $\gamma'(G) + \gamma'_H(G) \leq m$.
It implies that, $\frac{m}{2} + \gamma'_H(G) \leq m$.
It implies that, $\gamma'_H(G) \leq \frac{m}{2}$.
However, by Remark 1.1(2), $\gamma'(G) \leq \gamma'_H(G)$.
Therefore, $\frac{m}{2} \leq \gamma'_H(G) \leq \frac{m}{2}$.
It implies that, $\gamma'_H(G) = \frac{m}{2}$. □

Corollary 2.3. Let G be a graph without isolated edges. If $\gamma'(G) = \frac{m}{2}$ then the complement of every minimum edge dominating set is a minimum edge H-dominating set and vice versa.

3. Vertex removal from the graph

We consider vertex removal operation on graph and observe the effect of this operation on edge H-domination number of the graph.

Example 3.1. Consider the graph G of row 2 in the following figure 2. The graphs after removing a vertex from G are given in row 1.

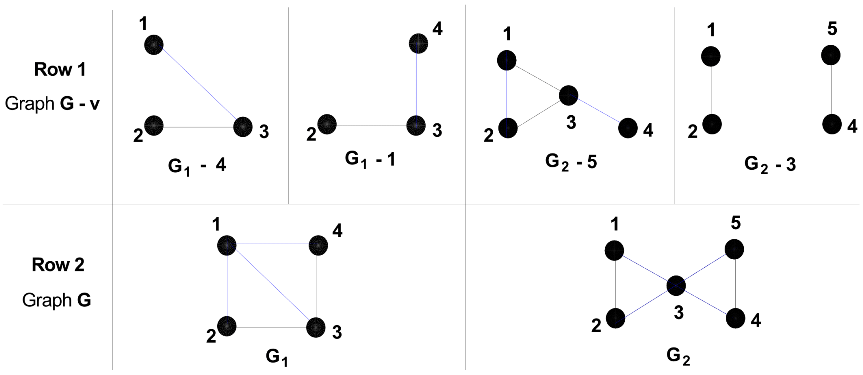

FIGURE 2. Graphs indicating vertex removal effect on edge H-domination

The edge H-domination number of G_1 is 2. If any vertex is removed from G_1 then the edge H-domination number of $G_1 - v$ (for any $v = 1, 2, 3, 4$) is 1. Thus, edge H-domination number of G_1 decreases when a vertex is removed from G_1. The edge H-domination number of G_2 is 2. If any vertex is removed from G_2 then the edge H-domination number of $G_2 - v$ (for any $v = 1, 2, 3, 4, 5$) is 2. Thus, the edge H-domination number does not change.

Proposition 3.1. Let G be a graph and a vertex $v \in V(G)$ be an isolated vertex then $\gamma'_H(G - v) = \gamma'_H(G)$.

Proof. Let F be a minimum edge H-dominating set of G and let $e = xy$ be any edge of $G - v$.
Case 1. Suppose $e = xy$ is an isolated edge of $G - v$ then e is also an isolated edge of G because v is isolated vertex of G. Therefore $e \in F$.
Case 2. Suppose y is a pendant vertex and $deg(x) \geq 2$ in $G - v$ then the same is true in G also because v is isolated vertex of G. Since all the edges (in G) incident at x are in F if $xy \notin F$, all the edges (in $G - v$) incident at x are in F if $xy \notin F$.

Case 3. Suppose $deg(x) \geq 2$ and $deg(y) \geq 2$ in $G - v$ then the same is true in G also because v is isolated vertex of G. Therefore if $xy \notin F$ then all the edges (in $G - v$) incident at x (except xy) are in F or all the edges (in $G - v$) incident at y (except xy) are in F. Thus, F is an edge H-dominating set of $G - v$. Therefore, $\gamma'_H(G - v) \leq \gamma'_H(G)$.

Conversely, by similar arguments, it can be proved that $\gamma'_H(G) \leq \gamma'_H(G - v)$. Therefore, $\gamma'_H(G - v) = \gamma'_H(G)$. □

Theorem 3.1. Let G be a graph and $e = uv$ be an isolated edge of G then $\gamma'_H(G - v) < \gamma'_H(G)$ and $\gamma'_H(G - u) < \gamma'_H(G)$.

Proof. Consider the subgraph $G - v$. Let F be any edge H-dominating set of G. Consider the set $F_1 = F - \{e\}$. Let $f = xy$ be any edge of $G - v$.

Case 1. If xy is an isolated edge of $G - v$ then xy is also an isolated edge in G because $e = uv$ is an isolated edge of G. Therefore, $xy \in F$. Hence, $xy \in F_1$.

Case 2. If xy is a pendant edge of $G - v$ with $deg(x) = 1$ and $deg(y) \geq 2$, then xy is also a pendant edge in G because $e = uv$ is an isolated edge of G. Therefore all the edges incident at y are in F if $xy \notin F$. Hence, all the edges incident at y are in F_1 if $xy \notin F_1$. If xy is a pendant edge of $G - v$ with $deg(y) = 1$ and $deg(x) \geq 2$ then also all the edges incident at x are in F_1 if $xy \notin F_1$.

Case 3. If xy is an edge of $G - v$ with $deg(x) \geq 2$ and $deg(y) \geq 2$ then the same is true in G also because $e = uv$ is an isolated edge of G. Thus, all the edges incident at x are in F or all the edges incident at y are in F if $xy \notin F$. Therefore, all the edges incident at x are in F_1 or all the edges incident at y are in F_1 if $xy \notin F_1$.

Thus, from all the cases, F_1 is an edge H-dominating set of $G - v$ and therefore $\gamma'_H(G - v) \leq |F_1| < |F| = \gamma'_H(G)$. By similar argument, we can prove $\gamma'_H(G - u) < \gamma'_H(G)$. □

Theorem 3.2. Let G be a graph and $v \in V(G)$ with $deg(v) \geq 1$ and suppose $\delta(G - v) \geq 2$ then $\gamma'_H(G - v) \leq \gamma'_H(G)$.

Proof. Let G be a graph and $v \in V(G)$ with $deg(v) \geq 1$.

Case 1. Let F be any minimum edge H-dominating set of G such that no edge incident at v is in F. Consider the subgraph $G - v$. Let xy be an edge of $G - v$ such that $xy \notin F$. Suppose $x = u$ for some u where uv is an edge of G. Now $uv \notin F$ therefore all the edges incident at u are in F or all the edges incident at v are in F. But all the edges incident at v are not in F and the edge $xy = uy$ is also not in F. This contradicts the fact that F is an edge H-dominating set of G. Thus, it follows that if xy is an edge of $G - v$ such that $xy \notin F$ then $x \neq u$. Now xy is an edge of G and $xy \notin F$. Therefore, all the edges incident at x are in F or all the edges incident at y are in F. Thus, F is an edge H-dominating set of $G - v$.

Case 2. Suppose for every minimum edge H-dominating set F of G, some edge incident at v is in F. Let F be any minimum edge H-dominating set of G and let $u_1v, u_2v, \ldots, u_kv$ are all the edges incident at v which are in F. Let $F_1 = F - \{u_1v, u_2v, \ldots, u_kv\}$ and consider the subgraph $G - v$. Let xy be any edge of $G - v$ such that $xy \notin F_1$. Suppose $x = u_i$ for some i. Since $u_iv \in F$, it may happen that there is some edge incident at u_i which is not in F. But $xy = u_iy$ is an edge of G which is not in F_1. Therefore, all the edges incident at y are in F_1. Similarly, if $x \neq u_i$ for any i then all the edges incident at x are in F_1 or all the edges incident at y are in F_1. Thus, F_1 is an edge H-dominating set of $G - v$.

Thus, from both the cases, $\gamma'_H(G - v) \leq \gamma'_H(G)$. □

4. Edge removal from the graph

We consider edge removal operation on graph and observe the effect of this operation on edge H-domination number of the graph.

Example 4.1. Consider the graph G of row 2 in the following figure 3. The graphs after removing an edge from G are given in row 1.

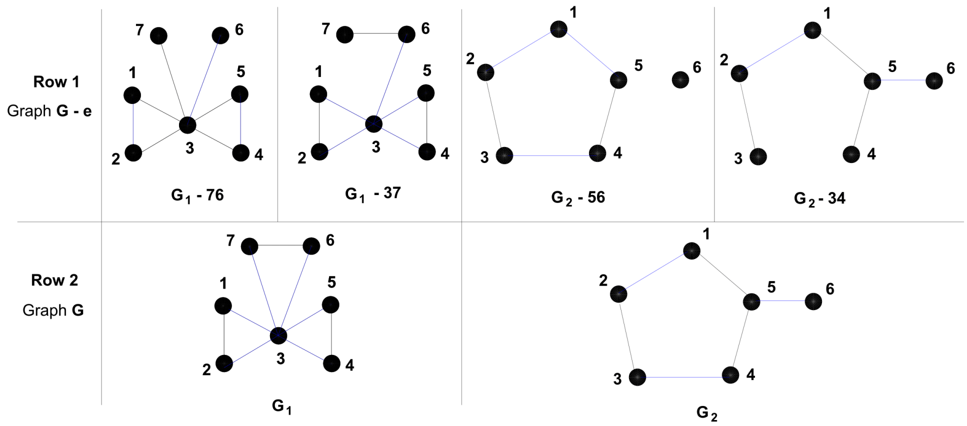

Figure 3. Graphs indicating edge removal effect on edge H-domination

The minimum edge H-dominating set of G_1 is $\{12, 45, 76\}$. The edge H-domination number of G_1 is 3. If the edge 76 is removed from G_1 then the minimum edge H-dominating sets of $G_1 - 76$ are $\{13, 23, 34, 35, 37\}$ and $\{13, 23, 34, 35, 36\}$. Therefore, the edge H-domination number of $G_1 - 76$ is 5. Thus, the edge H-domination number of G_1 increases when 76 edge is removed from G_1. If edge 37 is removed from G_1 then the edge H-domination number of $G_1 - 37$ is 3. Thus, the edge H-domination number of G_1 does not change. The minimum edge H-dominating set of G_2 contains three edges. The minimum sets are $\{15, 45, 23\}$, $\{15, 45, 12\}$, $\{15, 45, 34\}$ etc. The edge H-domination number of G_2 is 3. If edge 56 is removed from G_2 then $\gamma_H^{'}(G_2 - 56) = 2$. Also $\gamma_H^{'}(G_2 - 34) = 3$. It follows that, the removal of edge from the graph may increase, decrease or does not change the edge H-domination number of the graph. We try to identify the circumstances, under which, the number $\gamma_H^{'}$ increases or decreases.

Let G be a graph and $e = uv$ be an isolated edge, pendant edge or an edge with $deg(u) \geq 2$ and $deg(v) \geq 2$. If $e \notin F$, for a minimum edge H-dominating set F of G then we prove that the edge H-domination number of G does not increase when an edge e is removed from the graph.

Theorem 4.1. Let $e = uv$ be an isolated edge of G then $\gamma_H^{'}(G - e) < \gamma_H^{'}(G)$.

Proof. Let F be a minimum edge H-dominating set of G and $e = uv$ is an isolated edge of G. Therefore $e \in F$. Consider the set $F_1 = F - \{e\}$. Let $f = xy$ be any edge of $G - e$.
Case 1. If $f = xy$ is an isolated edge of $G - e$ then f is also an isolated edge of G because e is an isolated edge of G. Therefore $f \in F$ implies that $f \in F_1$.
Case 2. If $f = xy$ is a pendant edge of $G - e$. Suppose y is a pendant vertex of $G - e$ and $deg(x) \geq 2$ in $G - e$. Therefore y is a pendant vertex of G and $deg(x) \geq 2$ in G also because e is an isolated edge of G. Therefore all the edges (in G) incident at x are in F if $xy \notin F$. Since e is an isolated edge of G, none of these edges equal to e. Therefore, all the edges (in $G - e$) incident at x are in F_1, if $xy \notin F_1$.

Similarly, If x is a pendant vertex of $G - e$ and $deg(y) \geq 2$ in $G - e$ then also by similar arguments, all the edges (in $G - e$) incident at y are in F_1, if $xy \notin F_1$.
Case 3. If $f = xy$ is an edge of $G - e$ such that $deg(x) \geq 2$ and $deg(y) \geq 2$ in $G - e$. By similar arguments it can be proved that all the edges (in $G - e$) incident at x are in F_1, if $xy \notin F_1$ or all the edges (in $G - e$) incident at y are in F_1, if $xy \notin F_1$.

Thus from all cases, F_1 is an edge H-dominating set of $G - e$. Hence, $\gamma_H^{'}(G - e) \leq |F_1| < |F| = \gamma_H^{'}(G)$. □

Theorem 4.2. Let $e = uv$ be a pendant edge of G and suppose there is a minimum edge H-dominating set F of G such that $e \notin F$ then $\gamma_H^{'}(G - e) \leq \gamma_H^{'}(G)$.

Proof. Let F be a minimum edge H-dominating set of G and $e = uv$ is a pendant edge of G with $e \notin F$. Since $e = uv$ is a pendant edge, the following two cases are possible.
Case a. $deg(u) \geq 2$ and $deg(v) = 1$ in G. Since F is an edge H-dominating set of G and $e \notin F$, all the edges incident at u (except uv) are in F.
Case b. $deg(v) \geq 2$ and $deg(u) = 1$ in G. Since F is an edge H-dominating set of G and $e \notin F$, all the edges incident at v (except uv) are in F.

Consider **Case a** and let $xy \in E(G - e)$.
(**1**) xy is an isolated edge of $G - e$.
(**1**)(**i**) xy is also an isolated edge of G, that is xy is not adjacent with an edge $e = uv$ in G. Therefore $xy \in F$.
(**1**)(**ii**) xy is not an isolated edge of G, that is xy is adjacent with an edge $e = uv$ in G. The edge xy is incident with a vertex u in G because $deg(v) = 1$ in G. Therefore either $x = u$ or $y = u$. Suppose $x = u$. Since F is an edge H-dominating set of G with $e = uv \notin F$, all the edges adjacent to uv are in F. Therefore, $xy \in F$. If $y = u$ then also by similar argument, we prove that $xy \in F$.
(**2**) xy is a pendant edge of $G - e$.
(**2**)(**i**) xy is also a pendant edge of G then
(**i1**) xy is adjacent to uv in G.

In this case, xy must incident with u in G since $e = uv$ is a pendant edge in G with $deg(v) = 1$. Therefore, either $x = u$ and $deg(y) = 1$ in G or $y = u$ and $deg(x) = 1$ in G because xy is also a pendant edge of G. Suppose $x = u$ and $deg(y) = 1$ in G. Since F is an edge H-dominating set of G with $e = uv \notin F$, all the edges adjacent to uv are in F. Therefore, $xy \in F$. If we consider $y = u$ and $deg(x) = 1$ in G then also $xy \in F$.
(**i2**) xy is not adjacent to uv in G.

In this case, F remains an edge H-dominating set of $G - e$ because xy is not adjacent to $e = uv$ in G and $e \notin F$.
(**2**)(**ii**) xy is not a pendant edge of G.

In this case, xy is adjacent with uv in G. Since $e = uv$ is a pendant edge in G with $deg(v) = 1$, either $x = u$ (with $deg(x) = 2$) and $deg(y) \geq 2$ in G or $y = u$ (with $deg(y) = 2$) and $deg(x) \geq 2$ in G because xy is also a pendant edge of $G - e$. Suppose $x = u$ and $deg(y) \geq 2$ in G. Since F is an edge H-dominating set of G with $e = uv \notin F$, all the edges incident at u are in F. Therefore, $xy \in F$. If we consider $y = u$ (with $deg(y) = 2$) and $deg(x) \geq 2$ in G then also $xy \in F$.
(**3**) xy is an edge of $G - e$ with $deg(x) \geq 2$ and $deg(y) \geq 2$. Therefore, xy is also an edge of G with $deg(x) \geq 2$ and $deg(y) \geq 2$. Since F is an edge H-dominating set of G, all the edges incident at x are in F or all the edges incident at y are in F if $xy \notin F$.

Thus, we have proved that if F is an edge H-dominating set of G and for an edge $e \in E(G)$ with $e \notin F$, F is also an edge H-dominating set of $G - e$.

If we consider **Case b** then also, by similar arguments, we prove that F is an edge H-dominating set of $G - e$. Therefore, $\gamma'_H(G - e) \leq |F| = \gamma'_H(G)$. □

Corollary 4.1. Let $e = uv$ be an edge of G with $deg(u) \geq 2$ and $deg(v) \geq 2$ and there is a minimum edge H-dominating set F of G such that $e \notin F$ then $\gamma'_H(G - e) \leq \gamma'_H(G)$.

Theorem 4.3. Let G be a graph with $\delta(G) \geq 3$ and $e = uv$ be an edge of G then $\gamma'_H(G - e) \leq \gamma'_H(G)$.

Proof. Let G be a graph with $\delta(G) \geq 3$ and $e = uv$ be an edge of G.
Case 1. Suppose for every minimum edge H-dominating set F of G, $e \in F$. Let F be a minimum edge H-dominating set of G then $e \in F$ also $F_1 = F - \{e\}$.

Consider the subgraph $G - e$ and the set F_1. Let $f = xy$ be any edge of $G - e$ which is not in F_1 then $f \notin F$. Also note that $deg(x) \geq 3$ and $deg(y) \geq 3$ in G. Therefore, all the edges incident at x are in F or all the edges incident at y are in F in G. This implies that, all the edges incident at x are in F_1 or all the edges incident at y are in F_1 in $G - e$ also. Therefore, F_1 is an edge H-dominating set of $G - e$.
Case 2. Suppose there is a minimum edge H-dominating set F of G such that $e \notin F$ then by corollary 4.1, F is an edge H-dominating set of $G - e$.

Thus, from both cases, $\gamma'_H(G - e) \leq \gamma'_H(G)$. □

Remark 4.1. In the above theorem 4.3, we have assumed that $\delta(G) \geq 3$ in order that $\gamma'_H(G - e) \leq \gamma'_H(G)$. However the theorem is not true if $\delta(G) < 3$. This fact is verified by considering the graph G_1 of figure 3. We consider the following example.

Example 4.2. Consider the graph G. The vertices of G are 1, 2, 3, 4, 5 and edges are 12, 13, 23, 34, 35, 45. Also $\delta(G) = 2$.

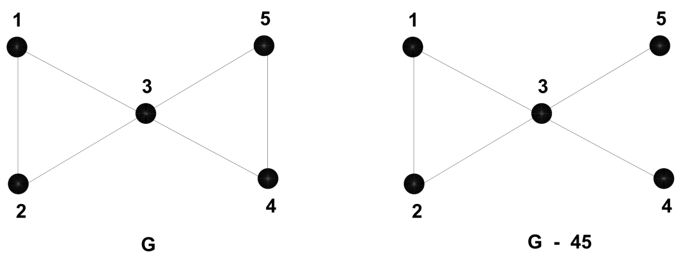

Figure 4

The minimum edge H-dominating set of G is $F = \{12, 45\}$. Therefore, $\gamma'_H(G) = 2$. The edge $e = 45 \in F$. The minimum edge H-dominating sets of $G - 45$ are $\{13, 23, 34\}$, $\{13, 23, 35\}$, $\{13, 34, 35\}$, $\{23, 34, 35\}$. Therefore, $\gamma'_H(G - 45) = 3$. Thus, $\gamma'_H(G - e) > \gamma'_H(G)$.

Remark 4.2. We summarize the above theorems as follows.
(1) If uv is an isolated edge of G then $\gamma'_H(G - e) < \gamma'_H(G)$.
(2) If uv is not an isolated edge of G and if there is a minimum edge H-dominating set of G which does not contain uv then $\gamma'_H(G - e) \leq \gamma'_H(G)$.
(3) If $e = uv$ is any edge of G with $\delta(G) \geq 3$ then $\gamma'_H(G - e) \leq \gamma'_H(G)$.

Definition 4.1. Let G be a graph and $F \subseteq E(G)$. An edge $e = uv \in F$ is said to be a *weak isolated edge* of F if one of the following two conditions holds.
(1) e is an isolated edge of F.

(2) There is an edge h incident at u such that $h \neq uv$ and $h \notin F$ also there is an edge $h^{'}$ incident at v such that $h^{'} \neq uv$ and $h^{'} \notin F$.

Definition 4.2. Let G be a graph and $F \subseteq E(G)$. An edge $e = uv \in F$. An *edge H-private neighborhood* of e with respect to F is denoted by $pn_H[e, F]$. An edge $f \in pn_H[e, F]$ if the following conditions are satisfied.
(1) If $f = e$ then f is weak isolated edge of G.
(2) If $f \neq e$ then $f = uy \notin F$ and all the edges incident at u (except uy) are in F also there is an edge g incident at y ($g \neq uy$) such that $g \notin F$.

or (of (2))

If $f \neq e$ then $f = vx \notin F$ and all the edges incident at v (except vx) are in F and there is an edge $g^{'}$ incident at x ($g^{'} \neq vx$) such that $g^{'} \notin F$.

Theorem 4.4. Let G be a graph with $\delta(G) = 2$ and $e = uv$ be an edge of G such that $deg(u) = 2$ and $deg(v) = 2$ then $\gamma_H^{'}(G - e) < \gamma_H^{'}(G)$ if and only if there is a minimum edge H-dominating set F of G such that $e \in F$ and $pn_H[e, F] = \{e\}$.

Proof. Suppose $e = uv$ be an edge of G such that $deg(u) = 2$ and $deg(v) = 2$. Suppose $\gamma_H^{'}(G - e) < \gamma_H^{'}(G)$. Let F_1 be a minimum edge H-dominating set of $G - e$. Now F_1 is not an edge H-dominating set of G and $e \notin F_1$. Therefore, there is an edge h incident at u such that $h \neq uv$ and $h \notin F_1$ also there is an edge $h^{'}$ incident at v such that $h^{'} \neq uv$ and $h^{'} \notin F_1$.

Let $F = F_1 \cup \{e\}$ then F is a minimum edge H-dominating set of G and $e \in F$. Also e is a weak isolated edge of F therefore $e \in pn_H[e, F]$. Suppose $f \neq e$ and $f \in pn_H[e, F]$ then $f \notin F$. Let $f = uy$. Since $f \in pn_H[e, F]$, all the edges incident at u (except uy) must be in F. Since $deg(u) = 2$ in G and $f = uy$ therefore u is a pendant vertex in $G - e$. Since F_1 is an edge H-dominating set of $G - e$ and u is a pendant vertex in $G - e$, all the edges incident at y must be in F_1. However, there is an edge g incident at y which is not in F and therefore g is not in F_1. Which is a contradiction. Thus, it is impossible that there is an edge $f \neq e$ with $f \in pn_H[e, F]$.

Similarly, if $f = vx$ then also we get a contradiction. Thus, it follows that there is no edge outside F such that it belongs to $pn_H[e, F]$. Therefore, $pn_H[e, F] = \{e\}$.

Conversely, suppose there is a minimum edge H-dominating set F of G containing e such that $pn_H[e, F] = \{e\}$. Let $F_1 = F - \{e\}$. Now we prove that F_1 is an edge H-dominating set of $G - e$. Let $f = xy$ be an edge of $G - e$ which is not in F_1 then $f \notin F$. Now all the edges incident at x are in F or all the edges incident at y are in F.

Suppose all the edges incident at x are in F and e is one of these edges then we can write $xy = uy$. Since $deg(u) = 2$ in G, $deg(x)$ is also 2 in G. Now x is a pendant vertex in $G - e$. If there is an edge h incident at y which is not in F then $uy \in pn_H[e, F]$. This is contradiction because there is only one edge which belongs to the $pn_H[e, F]$ namely e. Therefore all the edges incident at y must be in F. Note that none of these edges can be e because e is incident at x. Therefore all the edges incident at y are in F_1. Thus, we have proved that in this case, all the edges incident at y are in F_1 if x is a pendant vertex in $G - e$.

Similarly, if e is incident at y then y is a pendant vertex in $G - e$ and all the edges incident at x are in F_1. Suppose e is not incident at x and e is not incident at y. If all the edges incident at x are in F then they are in F_1 also. Similarly, if all the edges incident at y are in F then they are in F_1 also. Thus F_1 is an edge H-dominating set of $G - e$. Therefore, $\gamma_H^{'}(G - e) < \gamma_H^{'}(G)$. □

5. Concluding remarks

The edge H-domination of the graph is a new variant and 'superhereditary' property which explain a particular edge set of the graph. In this paper, the characterization of a minimal edge H-dominating set of G with $\delta(G) \geq 2$ is given. What is the characterization of minimal edge set when the condition $\delta(G) \geq 2$ is removed? is an open problem. The upper bound of an edge H-domination number of the graph is indicated. what about lower bound? and 'Is it possible to make the upper bound more sharp?' are arising questions. The edge(vertex) removal operation on graph is considered and the change in edge H-domination number is observed. When an edge is removed from the graph, exactly how many units the number $\gamma_H^{'}$ increases(or decreases) is further research exercise.

References

[1] Acharya, B. D., (2007), Domination in hypergraphs, AKCE J. Graphs. Combin., 4(2), pp. 117-126.
[2] Arumugam, S. and Velammal, S., (1998), Edge Domination in Graphs, Taiwanese Journal of Mathematics, 2(2), pp. 173-179.
[3] Berge, C., (1989), Hypergraphs: Combinatorics of Finite Sets, North-Holland, Amsterdam.
[4] Haynes, T. W., Hedetniemi, S. T., Slater, P. J., (1998), Fundamental of Domination in Graphs, Marcel Dekker, Inc., New York.
[5] Kakrecha, B. M., (2017), Ph. D. Thesis, Saurashtra University, Rajkot.
[6] Kalariya, K. N., (2016), Ph. D. Thesis, Saurashtra University, Rajkot.
[7] Mitchell, S., Hedetniemi, S. T., (1977), Edge domination in trees, Congr. Numer., 19, pp. 489−509.
[8] Thakkar, D. K., Kakrecha, B. M., (2016), About the Edge Domination Number of the Graphs, Advances in Theoretical and Applied Mathematics−RIP, 11(2), pp. 93−98.
[9] Thakkar, D. K., Kakrecha, B. M., (2016), Vertex Removal and the Edge Domination Number of Graphs, International Journal of Mathematics and its Applications, 4(2A) pp. 1−5.

Bhavesh M. Kakrecha obtained his M. Sc. Degree and M. Phil. Degree in Mathematics from Department of Mathematics, Saurashtra University, Rajkot, Gujarat (India) in 2009 and 2010 respectively. At present, he is working as an assistant professor in mathematics at Lukhdhirji Engineering College, Morbi, Gujarat (India). He obtained his Ph.D. degree in mathematics from the Department of Mathematics, Saurashtra University, Rajkot in 2017 under the supervision of Dr. D. K. Thakkar.