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NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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Abstract. In the present paper, new subclasses of bi-univalent functions of complex order associated with hypergeometric functions are introduced and coefficient estimates for functions in these classes are obtained. Several new (or known) consequences of the results are also pointed out.

Keywords: Analytic function; Univalent function; Hypergeometric function Bi-univalent function; Bi-starlike function; Bi-convex function; Complex order; Coefficient bounds and Subordination.

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$.

From Koebe one quarter theorem [17], it is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

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where

$$f^{-1}(w) = g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (2)$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The functions

$$\frac{z}{1-z}, \qquad -\log(1-z), \quad and \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad (3)$$

are in the class Σ (see details in [28]). However, the familiar Koebe function is not bi-univalent. Earlier, Brannan and Taha [3] introduced certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order α denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex functions of order α denoted by $\mathcal{K}_\Sigma(\alpha)$ corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively. Also, they determined non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (see also[30]). Many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (see[2, 11, 14, 28, 29, 31, 32]).

An analytic function F is subordinate to an analytic function G, written $F(z) \prec G(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $F(z) = G(\omega(z))$. Ma and Minda [13] unified various subclasses of starlike and convex functions, $f \in \mathcal{A}$ satisfying the subordination $\frac{z\,f'(z)}{f(z)} \prec \phi(z)$ and convex $1+\frac{z\,f''(z)}{f'(z)} \prec \phi(z)$ respectively. For this purpose, it is assumed that ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\mathbb{U})$ is symmetric with respect to the real axis and has a series expansion of the form

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots , \quad (B_1 > 0). \quad (4)$$

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better. The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_nb_nz^n, \quad (5)$$

where $f(z)$ is given by (1) and $h(z) = z + \sum\limits_{n=2}^{\infty} b_nz^n$.

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ ($\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \;&:=\; \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\; l, m \in \mathbb{N}_0 \;&:=\; \mathbb{N} \cup \{0\}; z \in \mathbb{U}) \end{aligned} \quad (6)$$

where $\mathbb{N}$ denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbb{N}. \end{cases} \quad (7)$$

For positive real values of $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ $(\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m)$, let

$$\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m) : \mathcal{S} \to \mathcal{S}$$

be a linear operator defined by

$$[(\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m))(f)](z) = z \; {}_lF_m(\alpha_1, \alpha_2, \dots \alpha_l; \beta_1, \beta_2 \dots, \beta_m; z) * f(z)$$

$$\mathcal{H}_m^l f(z) = z + \sum_{n=2}^{\infty} \Gamma_n \; a_n \; z^n \tag{8}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \; \frac{1}{(n-1)!} \tag{9}$$

$\alpha_i > 0, (i = 1, 2, ...l), \beta_j > 0, (j = 1, 2, ...m), l \leq m + 1; \; l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For notational simplicity, we use a shorter notation $\mathcal{H}_m^l[\alpha_1]$ for $\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)$ in the sequel. It follows from (8) that

$$\mathcal{H}_1^2[1]f(z) = f(z), \qquad \mathcal{H}_1^2[2]f(z) = zf'(z) \tag{10}$$

The linear operator $\mathcal{H}_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [7]). Further by using the Gaussian hypergeometric function given by (8), Hohlov [8] introduced a generalized convolution operator $H_{a,b,c}$ as

$$H_{a,b,c}f(z) = z \; {}_2F_1(a, b, c; z) * f(z),$$

contains as special cases most of the known linear integral or differential operators. For the suitable choices of l and m in turn the operator $\mathcal{H}_m^l[\alpha_1]$ includes various operators as remarked below:

Remark 1.1. *For* $f \in \mathcal{A}$,

$$\mathcal{H}_1^2(a, 1; c)f(z) = \mathcal{L}(a, c)f(z) = \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \right) * f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n$$

was considered by Carlson and Shaffer [5].

Remark 1.2. *For* $f \in \mathcal{A}$,

$$\mathcal{H}_1^2(\delta + 1, 1; 1)f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = \mathcal{D}^\delta f(z), (\delta > -1)$$

given by

$$\mathcal{D}^\delta f(z) = z + \sum_{n=2}^{\infty} \begin{pmatrix} \delta + n - 1 \\ n - 1 \end{pmatrix} a_n z^n$$

was introduced by Ruscheweyh [18].

Remark 1.3. *Let* $f \in \mathcal{A}$ *and*

$$\mathcal{H}_1^2(c + 1, 1; c + 2)f(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt = \mathcal{J}_c f(z)$$

where $c > -1$. *The operator* $\mathcal{J}_c$ *was introduced by Bernardi* [4]. *In particular, the operator* $\mathcal{J}_1$ *was studied earlier by Libera* [9] *and Livingston*[10].

Remark 1.4. *For $f \in \mathcal{A}$,*

$$\mathcal{H}_1^2(2,1;2-\mu)f(z) = \Gamma(2-\mu)z^{\mu}\mathcal{D}_z^{\mu}f(z) = \Omega^{\mu}f(z), \quad \mu \notin \mathbb{N}\setminus\{1\}$$

is called Owa-Srivastava operator [27] *and Ω^{μ} is also called Srivastava-Owa fractional derivative operator, where $\mathcal{D}_z^{\mu}f(z)$ denotes the fractional derivative of $f(z)$ of order μ, studied by Owa* [16].

In [21], Obradovic et.al gave some criteria for univalence expressing by $\Re(f'(z)) > 0$, for the linear combinations

$$\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\frac{1}{f'(z)} > 0, \qquad (\lambda \geq 1, z \in \mathbb{U}).$$

In [26], Silverman investigated an expression involving the quotient of the analytic representations of convex and starlike functions. Precisely, for $0 < b \leq 1$he considered the class

$$\mathcal{G}_b = f \in A : \Big|\frac{1+(zf''(z)/f'(z))}{zf'(z)/f(z)} - 1\Big| < b, \quad z \in \mathbb{U}$$

and proved that $\mathcal{G}_b \subset \mathcal{S}^*(\frac{2}{1+\sqrt{1+8b}})$. That is the functions in the class $\mathcal{G}_b$ are starlike of order $\frac{2}{1+\sqrt{1+8b}}$. Further, Obradovic and Tuneski improved this result in [20]. In [25], Tuneski introduced the condition for functions in the class $\mathcal{G}_b$ to be Janowski starlike (see[19]also the references cited therein). Based on the above definitions recently, in [12], Lashin introduced and studied the new subclasses of bi-univalent functions.

Motivated by the earlier works of Deniz [6] and Lashin [12], in the present paper we introduce new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C}\backslash\{0\}$, involving Dziok-Srivastava operator $\mathcal{H}_m^l$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of the function class Σ. Several related classes are also considered and connection to earlier known results are made.

Definition 1.1. *A function $f(z) \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma,\phi}^{l,m}(\gamma,\lambda)$ if it satisfies the following conditions :*

$$1+\frac{1}{\gamma}\left(\lambda\big[1+\frac{z(\mathcal{H}_m^l f(z))''}{(\mathcal{H}_m^l f(z))'}\big]+(1-\lambda)\frac{1}{(\mathcal{H}_m^l f(z))'}-1\right) \prec \phi(z) \tag{11}$$

and

$$1+\frac{1}{\gamma}\left(\lambda\big[1+\frac{w(\mathcal{H}_m^l g(z))''}{(\mathcal{H}_m^l g(w))'}\big]+(1-\lambda)\frac{1}{(\mathcal{H}_m^l g(w))'}-1\right) \prec \phi(w) \tag{12}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ $\lambda \geq 1, z, w \in \mathbb{U}$ and the function g is given by (2).

Definition 1.2. *A function $f(z) \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma,\phi}^{l,m}(1,\lambda) \equiv \mathcal{L}_{\Sigma,\phi}^{l,m}(\lambda)$ if it satisfies the following conditions :*

$$\lambda\left(1+\frac{z(\mathcal{H}_m^l f(z))''}{(\mathcal{H}_m^l f(z))'}\right)+(1-\lambda)\frac{1}{(\mathcal{H}_m^l f(z))'} \prec \phi(z) \tag{13}$$

and

$$\lambda\left(1+\frac{w(\mathcal{H}_m^l g(z))''}{(\mathcal{H}_m^l g(w))'}\right)+(1-\lambda)\frac{1}{(\mathcal{H}_m^l g(w))'} \prec \phi(w) \tag{14}$$

where $\lambda \geq 1, z, w \in \mathbb{U}$ the function g is given by (2).

Definition 1.3. *For a function $f(z) \in \Sigma$ given by (1),is said to be in the class $\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma, 1) \equiv \mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma)$ if it satisfies the following conditions :*

$$\left[1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{H}^l_m f(z))''}{(\mathcal{H}^l_m f(z))'}\right)\right] \prec \phi(z) \text{ and } \left[1 + \frac{1}{\gamma}\left(\frac{w(\mathcal{H}^l_m g(w))''}{(\mathcal{H}^l_m g(w))'}\right)\right] \prec \phi(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ $z, w \in \mathbb{U}$ and the function g is given by (2).

Definition 1.4. *A function $f(z) \in \Sigma$ given by (1) is said to be in $\mathcal{M}^{l,m}_{\Sigma}(\phi)$ if it satisfies the following conditions :*

$$\left(\frac{1 + \frac{z(\mathcal{H}^l_m f(z))''}{(\mathcal{H}^l_m f(z))'}}{\frac{z(\mathcal{H}^l_m f(z))'}{\mathcal{H}^l_m f(z)}}\right) \prec \phi(z) \quad \text{and} \quad \left(\frac{1 + \frac{w(\mathcal{H}^l_m g(w))''}{(\mathcal{H}^l_m g(w))'}}{\frac{w(\mathcal{H}^l_m g(w))'}{\mathcal{H}^l_m g(w)}}\right) \prec \phi(w) \tag{15}$$

where $z, w \in \mathbb{U}$ and the function g is given by (2).

We note that by specializing the parameters $\gamma = 1$ and suitably fixing the values for $l = 2; m = 1$ in Definition 1.1, we get the following subclasses of Σ studied by Lashin [12] as listed in the following remark:

Remark 1.5. *Suppose that $f(z) \in \Sigma$. Then we denote*

(1) $\mathcal{S}^{2,1}_{\Sigma,\phi}(1, \lambda) \equiv \mathcal{L}^{2,1}_{\Sigma,\phi}(\lambda) \equiv \mathcal{H}^{\lambda}_{\Sigma}(\phi), \quad \lambda \geq 1$

(2) $\mathcal{K}^{2,1}_{\Sigma,\phi}(1) \equiv \mathcal{K}_{\Sigma}(\phi)$,
and

(3) $\mathcal{M}^{2,1}_{\Sigma}(\phi) = \mathcal{H}_{\Sigma}(\phi)$

Further specializing the parameters l, m one can define the various other interesting new subclasses of Σ involving the differential operators as stated in Remarks 1.1 to 1.4 which have not been discussed in the literature. Note that by taking bi univalent functions given in (3) as $\mathcal{H}^l_m(\frac{z}{1-z})$ or $\mathcal{H}^l_m(-\log(1-z))$ or $\mathcal{H}^l_m(\frac{1}{2}\log\left(\frac{1+z}{1-z}\right))$ one can easily verify the results discussed for the function classes.

Now, in the following section we determine the initial coefficients $|a_2|$ *and* $|a_3|$ for functions f in the subclasses $\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma, \lambda)$, $\mathcal{L}^{l,m}_{\Sigma,\phi}(\lambda)$, $\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma)$ and $\mathcal{M}^{l,m}_{\Sigma}(\phi)$.

2. Coefficient estimates for the function classes $\mathcal{S}^{l,m}_{\Sigma,\phi}(\gamma, \lambda), \mathcal{L}^{l,m}_{\Sigma,\phi}(\lambda)$ *and* $\mathcal{K}^{l,m}_{\Sigma,\phi}(\gamma)$

Define the functions $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$ and suppose that

$$p(z) := p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := q_1 z + q_2 z^2 + \cdots .$$

It is well known that (see [15],p.172)

$$|p_1| \leq 1; \qquad |p_2| \leq 1 - |p_1|^2; \qquad |q_1| \leq 1; \qquad |q_2| \leq 1 - |q_1|^2 \tag{16}$$

It follows that,

$$\phi(p(z)) := 1 + B_1 p_1 z + (B_1 p_2 + B_2 p_1^2) z^2 + \cdots \tag{17}$$

and

$$\phi(q(w)) := 1 + B_1 q_1 w + (B_1 q_2 + B_2 q_1^2) w^2 + \cdots . \tag{18}$$

Theorem 2.1. *Let $f(z)$ be given by (1) be in the class $\mathcal{S}_{\Sigma,\phi}^{l,m}(\gamma,\lambda)$, $\gamma\in\mathbb{C}\backslash\{0\}$ and $\lambda\geq 1$. Then*

$$|a_2|\leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{\left|\gamma(1+\lambda)B_1^2-4(2\lambda-1)^2B_2\right|\Gamma_2^2+4(2\lambda-1)^2\Gamma_2^2B_1}} \tag{19}$$

and

$$|a_3|\leq \begin{cases} \dfrac{|\gamma|B_1}{(1+\lambda)\Gamma_3} & if \quad |B_2|\leq B_1,\\ \dfrac{|\gamma|\{|\gamma(1+\lambda)B_1^2-4(2\lambda-1)^2B_2|B_1+4(2\lambda-1)^2B_1|B_2|\}}{(\lambda+1)\Gamma_3\left\{|\gamma(1+\lambda)B_1^2-4(2\lambda-1)^2B_2|+4(2\lambda-1)^2B_1\right\}} & if \quad |B_2|>|B_1|. \end{cases} \tag{20}$$

Proof. It follows from (11) and (12) that

$$1+\frac{1}{\gamma}\left(\lambda\big[1+\frac{z(\mathcal{H}_m^lf(z))''}{(\mathcal{H}_m^lf(z))'}\big]+(1-\lambda)\frac{1}{(\mathcal{H}_m^lf(z))'}-1\right)=\phi(u(z)) \tag{21}$$

and

$$1+\frac{1}{\gamma}\left(\lambda\big[1+\frac{w(\mathcal{H}_m^lg(z))''}{(\mathcal{H}_m^lg(w))'}\big]+(1-\lambda)\frac{1}{(\mathcal{H}_m^lg(w))'}-1\right)=\phi(v(w)). \tag{22}$$

From (21) and (22), we have

$$\begin{aligned}1+\frac{2}{\gamma}(2\lambda-1)\Gamma_2a_2z+\frac{1}{\gamma}\left[3(3\lambda-1)\Gamma_3a_3+4(1-2\lambda)\Gamma_2^2a_2^2\right]z^2+\cdots\\ =1+B_1p_1z+(B_1p_2+B_2p_1^2)z^2+\cdots\end{aligned}$$

and

$$\begin{aligned}1-\frac{2}{\gamma}(2\lambda-1)\Gamma_2a_2w+\frac{1}{\gamma}\left(2(5\lambda-1)\Gamma_2^2a_2^2-3(3\lambda-1)\Gamma_3a_3\right)w^2-\cdots\\ =1+B_1q_1w+(B_1q_2+B_2q_1^2)w^2+\cdots.\end{aligned}$$

Now, equating the coefficients, we get

$$\frac{2}{\gamma}(2\lambda-1)\Gamma_2a_2=B_1p_1, \tag{23}$$

$$\frac{1}{\gamma}\left[3(3\lambda-1)\Gamma_3a_3+4(1-2\lambda)\Gamma_2^2a_2^2\right]=B_1p_2+B_2p_1^2, \tag{24}$$

$$-\frac{2}{\gamma}(2\lambda-1)\Gamma_2a_2=B_1q_1, \tag{25}$$

and

$$\frac{1}{\gamma}\left(2(5\lambda-1)\Gamma_2^2a_2^2-3(3\lambda-1)\Gamma_3a_3\right)=B_1q_2+B_2q_1^2. \tag{26}$$

From (23) and (25), we get

$$p_1=-q_1 \tag{27}$$

and

$$8(2\lambda-1)^2\Gamma_2^2a_2^2=\gamma^2B_1^2(p_1^2+q_1^2). \tag{28}$$

Now from (24), (26) and (28), we obtain

$$\left(2\gamma(1+\lambda)B_1^2-8(2\lambda-1)^2B_2\right)\Gamma_2^2a_2^2=\gamma^2B_1^3(p_2+q_2). \tag{29}$$

From(27), (29) and by using (16), for the coefficients p_2 and q_2, we have

$$\left|\gamma(1+\lambda)B_1^2-4(2\lambda-1)^2B_2\right|\Gamma_2^2|a_2|^2\leq|\gamma|^2B_1^3|(1-|p_1|^2). \tag{30}$$

From(23) and (30) we obtain

$$|a_2| \leq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{\left|\gamma(1+\lambda)B_1^2 - 4(2\lambda-1)^2B_2\right|\Gamma_2^2 + 4(2\lambda-1)^2\Gamma_2^2 B_1}}.$$

From (24) from (26) and using(27), we get

$$\frac{1}{\gamma}(1+\lambda)(9\lambda-3)\Gamma_3 a_3 = (5\lambda-1)B_1p_2 + 2(2\lambda-1)B_1q_2 + (9\lambda-3)B_2p_1^2.$$

Then by (16), we get

$$(1+\lambda)\Gamma_3|a_3| \leq |\gamma|B_1 + |\gamma|(|B_2| - B_1)|p_1|^2.$$

$$|p_1^2| \leq \frac{4(2\lambda-1)^2\Gamma_2^2}{|\gamma|^2B_1^2}|a_2|^2 \leq \frac{4(2\lambda-1)^2B_1}{\left|\gamma(1+\lambda)B_1^2 - 4(2\lambda-1)^2B_2\right| + 4(2\lambda-1)^2B_1}.$$

Thus we obtain,

$$|a_3| \leq \begin{cases} \dfrac{|\gamma|B_1}{(1+\lambda)\Gamma_3} & if \quad |B_2| \leq B_1, \\ \dfrac{|\gamma|\{|\gamma(1+\lambda)B_1^2 - 4(2\lambda-1)^2B_2|B_1 + 4(2\lambda-1)^2B_1|B_2|\}}{(\lambda+1)\Gamma_3\left\{|\gamma(1+\lambda)B_1^2 - 4(2\lambda-1)^2B_2| + 4(2\lambda-1)^2B_1\right\}} & if \quad |B_2| > |B_1|. \end{cases}$$

□

By taking $\gamma = 1$ we state the following:

Theorem 2.2. *Let $f(z)$ be given by (1) be in the class $\mathcal{L}_{\Sigma,\phi}^{l,m}(\lambda)$, $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+\lambda)B_1^2 - 4(2\lambda-1)^2B_2\right|\Gamma_2^2 + 4(2\lambda-1)^2\Gamma_2^2 B_1}} \tag{31}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{(1+\lambda)\Gamma_3} & if \quad |B_2| \leq B_1, \\ \dfrac{|(1+\lambda)B_1^2 - 4(2\lambda-1)^2B_2|B_1 + 4(2\lambda-1)^2B_1|B_2|}{(\lambda+1)\Gamma_3\left\{|(1+\lambda)B_1^2 - 4(2\lambda-1)^2B_2| + 4(2\lambda-1)^2B_1\right\}} & if \quad |B_2| > |B_1|. \end{cases} \tag{32}$$

By taking $\lambda = 1$ we state the following :

Theorem 2.3. *Let $f(z)$ be given by (1) be in the class $\mathcal{K}_{\Sigma,\phi}^{l,m}(\gamma)$, $\gamma \in \mathbb{C}\backslash\{0\}$. Then*

$$|a_2| \leq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{\left|2\gamma B_1^2 - 4B_2\right|\Gamma_2^2 + 4\Gamma_2^2 B_1}} \tag{33}$$

and

$$|a_3| \leq \begin{cases} \dfrac{|\gamma|B_1}{2\Gamma_3} & if \quad |B_2| \leq B_1, \\ \dfrac{|\gamma|\{|2\gamma B_1^2 - 4B_2|B_1 + 4B_1|B_2|\}}{2\Gamma_3\left\{|2\gamma B_1^2 - 4B_2| + 4B_1\right\}} & if \quad |B_2| > |B_1|. \end{cases} \tag{34}$$

Taking $\gamma = 1$, $l = 2$, $m = 1$ and $\alpha_i = 1$ in Theorem 2.1, we obtain Theorem 1 given by Lashin [12] .

3. Coefficient estimates for the function class $\mathcal{M}_{\Sigma}^{l,m}(\phi)$

Theorem 3.1. *Let $f(z)$ is given by (1) be in the class $\mathcal{M}_{\Sigma}^{l,m}(\phi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\Gamma_2(\sqrt{|B_2|+B_1}} \tag{35}$$

and

$$|a_3| \quad \leq \quad \begin{cases} \dfrac{B_1}{4\Gamma_3} & if \quad |B_1| \leq \frac{1}{4}, \\ \dfrac{B_1}{4\Gamma_3} + (1-\dfrac{1}{4B_1})\dfrac{B_1^3}{\Gamma_3(|B_2|+B_1)} & if \quad |B_1| > \dfrac{1}{4}. \end{cases} \tag{36}$$

Proof. We can write the argument inequalities in (15) equivalently as follows:

$$\left(\frac{1+\frac{z(\mathcal{H}_m^l f(z))''}{(\mathcal{H}_m^l f(z))'}}{\frac{z(\mathcal{H}_m^l f(z))'}{\mathcal{H}_m^l f(z)}} \right) = \phi(u(z)) \tag{37}$$

and

$$\left(\frac{1+\frac{w(\mathcal{H}_m^l g(w))''}{(\mathcal{H}_m^l g(w))'}}{\frac{w(\mathcal{H}_m^l g(w))'}{\mathcal{H}_m^l g(w)}} \right) = \phi(v(w)), \tag{38}$$

and proceeding as in the proof of Theorem 2.1, we can arrive the following relations from (37) and (38)

$$\Gamma_2 a_2 = B_1 p_1, \tag{39}$$

$$4(a_3\Gamma_3 - \Gamma_2^2 a_2^2) = B_1 p_2 + B_2 p_1^2 \tag{40}$$

and

$$-\Gamma_2 a_2 = B_1 q_1, \tag{41}$$

$$-4(a_3\Gamma_3 - \Gamma_2^2 a_2^2) = B_1 q_2 + B_2 q_1^2 \tag{42}$$

From (39) and (41), we get

$$p_1 = -q_1 \tag{43}$$

and

$$2\Gamma_2^2 a_2^2 = B_1^2(p_1^2 + q_1^2). \tag{44}$$

Now from (40), (42) and (44), we obtain

$$2B_2\Gamma_2^2 a_2^2 = -B_1^3(p_2 + q_2). \tag{45}$$

From (43), (45) and by using (16), we get

$$|B_2|\Gamma_2^2|a_2|^2 \leq B_1^3(1-|p_1|^2). \tag{46}$$

From (41)and (46)

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\Gamma_2\sqrt{|B_2|+B_1}}$$

Next, in order to find the bound on $|a_3|$,from (40) (42), and using (43), we get

$$8a_3\Gamma_3 = 8a_2^2\Gamma_2^2 + B_1(p_2 - q_2)$$

Thus we have

$$
\begin{aligned}
|a_3| &\leq \frac{|a_2|^2\Gamma_2^2}{\Gamma_3} + \frac{B_1}{4\Gamma_3}(1-|p_1|^2) \\
&= \frac{B_1}{4\Gamma_3} + \frac{|a_2|^2\Gamma_2^2}{\Gamma_3}(1-\frac{1}{4B_1}) \\
&\leq \begin{cases} \dfrac{B_1}{4\Gamma_3} & if \quad |B_1| \leq \frac{1}{4}, \\ \dfrac{B_1}{4\Gamma_3} + (1-\dfrac{1}{4B_1})\dfrac{B_1^3}{\Gamma_3(|B_2|+B_1)} & if \quad |B_1| > \dfrac{1}{4}. \end{cases}
\end{aligned} \tag{47}
$$

□

Taking $l = 2$, $m = 1$ and $\alpha_i = 1$ in Theorem 2.1, we obtain Theorem 2 given by Lashin [12] .

4. Concluding remarks

(1) For $(0 < \alpha \leq 1)$ and $-1 \leq B < A \leq 1$, taking the function ϕ as

$$\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} = 1 + \alpha(A-B)z - \frac{\alpha}{2}[2B(A-B) + (1-\alpha)(A-B)^2]z^2 + \cdots \tag{48}$$

which gives $B_1 = \alpha(A-B)$ and $B_2 = -\frac{\alpha}{2}[2B(A-B) + (1-\alpha)(A-B)^2]$.

(2) If we take $\alpha = 1$ and $-1 \leq B < A \leq 1$, then we have

$$\phi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z + B(A-B)z^2 + \cdots \tag{49}$$

thus we have $B_1 = A - B$ and $B_2 = B(A-B)$.

(3) By fixing $A = 1$ and $B = -1$ we have

$$\phi(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots \tag{50}$$

thus we have $B_1 = 2$ and $B_2 = 2$

(4) Further for some $c \in (0, 1]$, taking

$$\phi(z) = \sqrt{1+cz} = 1 + \frac{c}{2}z - \frac{c^2}{8}z^2 + \ldots \tag{51}$$

then the class is said to be associated with the right -loop of the Cassinian Ovals [1]. In particular if $c = 1$ then the class is associated with right-half of the lemniscate of Bernoulli [24] is given by

$$\phi(z) = \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \ldots. \tag{52}$$

(5) Taking

$$\phi(z) = z + \sqrt{1+z^2} = 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 + \ldots. \tag{53}$$

then the class is said to be associated with the right crescent [22].

(6) Again by taking

$$\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2 \tag{54}$$

then the class is said to be associated with the cardioid [23].

By specializing the parameters l and m one can define the various other interesting subclasses of Σ involving the differential operators as stated in Remarks 1.1 to 1.4, and results as in Theorems 2.1, 2.2, 2.3 and 3.1.Further , by various choices of ϕ as mentioned above one can state results corresponding to Theorems 2.1, 2.2, 2.3 and 3.1 easily. The details involved may be left as an exercise for the interested reader.

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