TWMS J. App. and Eng. Math. V.11, N.2, 2021, pp. 480-489

DIRECTED PATHOS MIDDLE DIGRAPH OF AN ARBORESCENCE

H. M. NAGESH[1], §

ABSTRACT. A *directed pathos middle digraph* of an arborescence A_r, written $Q = DPM(A_r)$, is the digraph whose vertex set $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$, where $V(A_r)$ is the vertex set, $A(A_r)$ is the arc set, and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; for every $v \in V(A_r)$, all arcs a_1v, va_2; for which v is a head of the arc a_1 and tail of the arc a_2 in A_r; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j. The problem of reconstructing an arborescence from its directed pathos middle digraph is presented. The characterization of digraphs whose $DPM(A_r)$ are planar; outerplanar; maximal outerplanar; and minimally non-outerplanar is studied.

Keywords: Line digraph, directed path number, crossing number, inner vertex number.

AMS Subject Classification: 05C20

1. Introduction

Notations and definitions not introduced here can be found in [1,4]. There are many (di)graph valued functions (or (di)graph operators) with which one can construct a new (di)graph from a given (di)graph, such as the line (di)graphs, the middle (di)graphs, and their generalizations.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common. This concept was originated with Whitney [9].

Harary and Norman [3] extended the concept of line graph of a graph and introduced the concept of line digraph of a directed graph. The *line digraph* $L(D)$ of a digraph $D = (V, A)$ has the arcs of D as vertices. There is an arc from D-arc pq towards D-arc uv if and only if $q = u$.

Hamada and Yoshimura [5] defined a graph $M(G)$ as an intersection graph $\Omega(G)$ on the vertex set $V(G)$ of any graph G. Let $E(G)$ be the edge set of G and $F = V^{'}(G) \cup E(G)$,

[1] Department of Science and Humanities, PES University - Electronic City Campus, Bangalore, India. e-mail: nageshhm@pes.edu; ORCID: https://orcid.org/0000-0001-9864-8937.

§ Manuscript received: May 9, 2019; accepted: July 11, 2019.

where $V^{'}(G)$ indicates the family of one-vertex subsets of the set $V(G)$. Let $M(G) = \Omega(G)$. $M(G)$ is called the middle graph of G.

Zamfirescu [10] extended the concept of middle graph of a graph to the directed case there by introducing a digraph operator called the middle digraph. The *middle digraph* $M(D)$ of a directed graph $D = (V, A)$ is obtained from $L(D)$ by adding all vertices of V, and, for every $v \in V(D)$, all arcs $a_1 v, va_2$; for which v is a head of the arc a_1 and tail of the arc a_2 in D.

The concept of pathos of a graph G was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The *path number* of a tree T equals k, where $2k$ is the number of odd degree vertices of T. Stanton and Cowan [8] calculated the path number of certain classes of graphs like trees and complete graphs.

Muddebihal and Chandrasekhar [7] extended the concept of pathos of a graph and introduced the concept of pathos line graph of a tree. A *pathos line graph* of a tree T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T have a vertex in common or the edge lies on the corresponding path of the pathos.

Note that there is freedom in marking the paths of a pathos of a tree T in different ways, provided that the path number k of T is fixed. For example, consider the marking of the paths of pathos of trees in Figure.1 (where $k = 3$). Since the order of marking the paths of a pathos of a tree is not unique, the corresponding pathos line graph is also not unique.

See Figure.1 for an example of a tree (with pathos) and its pathos line graph.

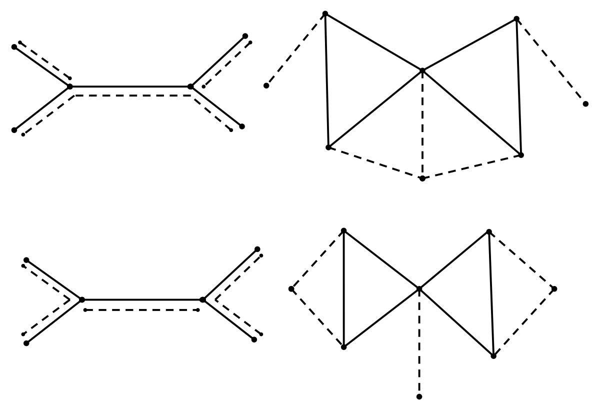

Figure.1

The object of this paper is to extend the concept of pathos of a tree to the directed case by introducing a digraph operator called a directed pathos middle digraph of an arborescence and to develop some of its properties.

2. Preliminaries

A *graph* $G = (V, E)$ is a pair, consisting of some set V, the so-called vertex set, and some subset E of the set of all 2-element subsets of V, the edge set. If a path starts at one vertex and ends at a different vertex, then it is called an open path.

A graph G is planar if it has a drawing without crossings. For a planar graph G, the inner vertex number $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*, i.e., $i(G) = 0$. An outerplanar graph G is maximal outerplanar if no edge can be added without losing outerplanarity. A graph G is said to be minimally non-outerplanar if $i(G)$=1.

A directed graph (or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. Here $V(D)$ is the vertex set and $A(D)$ is the arc set of D. For an arc (u, v) or uv in D, the first vertex u is its *tail* and the second vertex v is its *head*. For an arc $e = (u, v)$, u is a *neighbor* of v; and u is adjacent to e and e is adjacent to v. A vertex u is adjacent to v if the arc uv is in D; u is adjacent from v if vu is in D. The total degree $td(v)$ of a vertex v is the number of arcs incident with v, i.e., $td(v) = d^-(v) + d^+(v)$. A *source* is any vertex of in-degree zero and a *sink* is a vertex of out-degree zero. A *block* B of a digraph D is a maximal weak subdigraph of D, which has no cut-vertex v such that $B - v$ is disconnected. An entire digraph is a block if it has only one block.

Digraphs that can be drawn without crossings between arcs (except at end vertices) are called planar digraphs. Clearly this property does not depend on the orientation of the arcs and hence the orientation of the arcs is ignored while defining the planarity; outerplanarity; maximal outerplanarity; and minimally non-outerplanarity of a digraph. Furthermore, since most of the results and definitions of undirected graphs are valid for planar digraphs as far as their underlying graphs are concerned, the following definitions hold good for planar digraphs. A digraph D is said to be *outerplanar* if $i(D) = 0$ and minimally non-outerplanar if $i(D) = 1$.

3. Definition of $DPM(A_r)$

Definition 3.1. An arborescence A_r is a directed graph in which, for a vertex u called the *root* and any other vertex v, there is exactly one directed path from u to v.

Definition 3.2. A root arc of an arborescence A_r is an arc which is directed out of the root of A_r, i.e., a root arc of A_r is an arc whose tail is the root of A_r.

Definition 3.3. *If a directed path $\vec{P}_n$ on $n \geq 2$ vertices starts at one vertex and ends at a different vertex, then $\vec{P}_n$ is called an open directed path.*

Definition 3.4. The directed pathos of an arborescence A_r is defined as a collection of minimum number of arc disjoint open directed paths whose union is A_r.

Definition 3.5. The directed path number $k^{'}$ of an arborescence A_r is the number of directed paths in any directed pathos of A_r, and is equal to the number of sinks in A_r, i.e., $k^{'}$ equals the number of sinks in A_r.

Note that the directed path number $k^{'}$ of an arborescence A_r is "minimum" only when the out-degree of the root of A_r is exactly one. Therefore, unless otherwise specified, the out-degree of the root of every arborescence is exactly one. Finally, the direction of the directed pathos is along the direction of the arcs in A_r.

Definition 3.6. A directed pathos middle digraph of an arborescence A_r, written $Q = DPM(A_r)$, is the digraph whose vertex set $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$, where $V(A_r)$ is the vertex set, $A(A_r)$ is the arc set, and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; for every $v \in V(A_r)$, all arcs a_1v, va_2; for which v is a head of the arc a_1 and tail of the arc a_2 in A_r; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.

See Figure.2 and Figure.3 for an example of an arborescence (with directed pathos) and its directed pathos middle digraph.

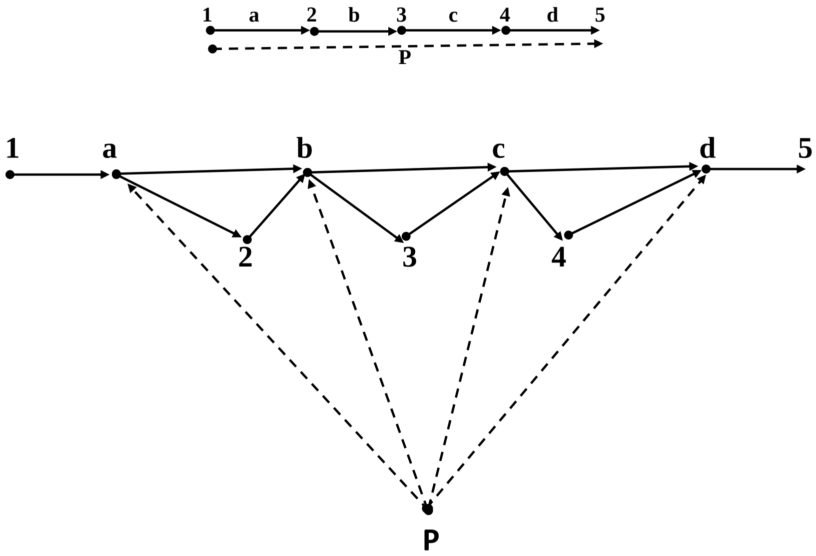

Figure.2

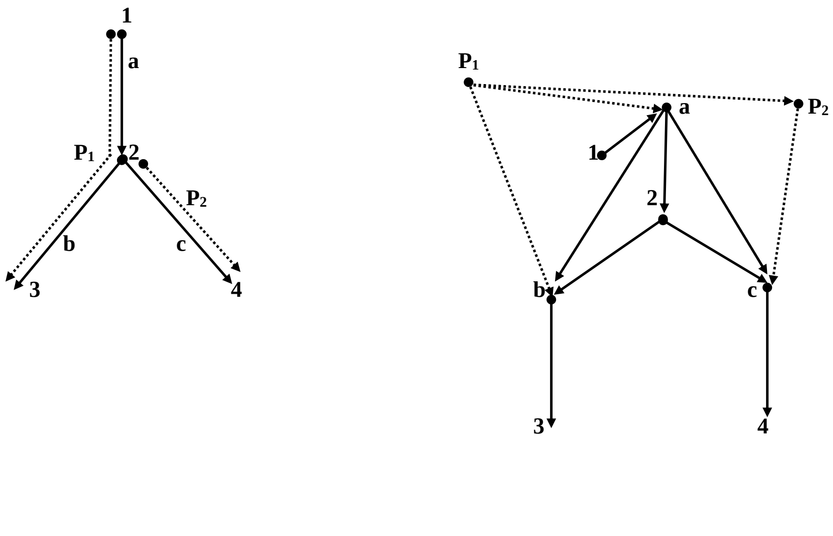

Figure.3

Note that there is freedom in marking the directed paths of a directed pathos of an arborescence A_r in different ways, provided that the directed path number $k^{'}$ of A_r is fixed. Since the order of marking the directed paths of a directed pathos of an arborescence is not unique, the corresponding directed pathos middle digraph is also not unique. This obviously raises the question of the existence of "unique" directed pathos middle digraph.

It is known that the middle digraph of a directed graph is unique. One can easily check that if the directed path number of an arborescence is exactly one (i.e., $k^{'} = 1$), then the corresponding directed pathos middle digraph is unique. For example, the directed path number of a directed path $\vec{P_n}$ on $n \geq 2$ vertices is exactly one (since $\vec{P_n}$ has exactly one sink). Thus the directed pathos middle digraph of a directed path is unique. Furthermore, one can also observe easily that, for different ways of marking of the directed paths of a directed pathos of an arborescence whose underlying graph is a star graph $K_{1,n}$ on $n \geq 3$ vertices, the corresponding directed pathos middle digraphs are isomorphic.

4. A criterion for directed pathos middle digraphs

The main objective is to determine a necessary and sufficient condition that a digraph be a directed pathos middle digraph.

A complete bipartite digraph is a directed graph D whose vertices can be partitioned into non-empty disjoint sets A and B such that each vertex of A has exactly one arc directed towards each vertex of B and such that D contains no other arc.

Let A_r be an arborescence with vertex set $V(A_r) = \{v_1, v_2, \ldots, v_n\}$ and a directed pathos set $P(A_r) = \{P_1, P_2, \ldots, P_t\}$.

Case 1: Let v be a vertex of A_r with $d^-(v) = \alpha$ and $d^+(v) = \beta$. Then α arcs coming into v and the β arcs going out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head. This is the decomposition of $L(A_r)$ (i.e., the line digraph of A_r) into mutually arc disjoint complete bipartite subdigraphs.

Case 2: An arc $e = (u, v)$ with $d^+(u) = d^-(v) = 1$ give rise to a complete bipartite subdigraph with u as the tail and e head. This contributes $(n-1)$ arcs to $DPM(A_r)$.

Case 3: An arc $e = (u, v)$ with $d^+(u) = d^-(v) = 1$ give rise to a complete bipartite subdigraph with e as the tail and v head. This also contributes $(n-1)$ arcs to $DPM(A_r)$.

Case 4: Let P_j be a directed path which lies on $\alpha^{'}$ arcs in A_r. Then $\alpha^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail P_j and $\alpha^{'}$ heads and $\alpha^{'}$ arcs joining P_j with each head. This again contributes $(n-1)$ arcs to $DPM(A_r)$.

Case 5: Let P_j be a directed path and let $\beta^{'}$ be the number of directed paths whose head is reachable from the tail of P_j through a common vertex in A_r. Then $\beta^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail P_j and $\beta^{'}$ heads and $\beta^{'}$ arcs joining P_j with each head. This contributes $(k^{'}-1)$ arcs to $DPM(A_r)$.

Hence by all the cases above, $Q = DPM(A_r)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with $V(Q) = V(A_r) \cup A(A_r) \cup P(A_r)$ and arc sets, (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively; (ii) thrice the size of A_r, i.e., $3(n-1)$; and (iii) $k^{'}-1$ (see Figure.4).

Conversely, let $A_r^{'}$ be a digraph of the type described above. Let $t_1, t_2, \ldots, t_l$ be the vertices corresponding to complete bipartite subdigraphs $T_1, T_2, \ldots, T_l$ of Case 1, respectively; and let $w^1, w^2, \ldots, w^t$ be the vertices corresponding to complete bipartite subdigraphs $P_1^{'}, P_2^{'}, \ldots, P_t^{'}$ of Case 4, respectively. Finally, let t_0 be a vertex chosen arbitrarily.

For each vertex v of the complete bipartite subdigraphs $T_1, T_2, \ldots, T_l$, an arc a_v is drawn as follows:

(a) If $d^+(v) > 0$, $d^-(v) = 0$, then $a_v := (t_0, t_i)$, where i is the base (or index) of T_i such that $v \in Y_i$.
(b) If $d^+(v) > 0$, $d^-(v) > 0$, then $a_v := (t_i, t_j)$, where i and j are the indices of T_i and T_j such that $v \in X_j \cap Y_i$.
(c) If $d^+(v) = 0$, $d^-(v) = 1$, then $a_v := (t_j, w^n)$ for $1 \leq n \leq t$, where j is the base of T_j such that $v \in X_j$.

Note that, in (t_j, w^n) no matter what the value of j is, n varies from 1 to t such that the number of arcs of the form (t_j, w^n) is exactly t.

Now the directed pathos are marked as follows. It is easy to observe that the directed path number $k^{'}$ equals the number of subdigraphs of Case 4. Let $\psi_1, \psi_2, \ldots, \psi_t$ be the number of heads of subdigraphs $P_1^{'}, P_2^{'}, \ldots, P_t^{'}$, respectively. Suppose the directed path P_1 is marked. For this any ψ_1 number of arcs can be chosen and mark P_1 on ψ_1 arcs. Similarly, ψ_2 number of arcs can be chosen and mark P_2 on ψ_2 arcs. This process is repeated until all the directed paths are marked. The digraph A_r with directed pathos thus constructed apparently has $A_r^{'}$ as directed pathos middle digraph. Therefore,

Theorem 4.1. *A digraph $A_r^{'}$ is a directed pathos middle digraph of an arborescence A_r if and only if $V(A_r^{'}) = V(A_r) \cup A(A_r) \cup P(A_r)$ and arc sets, (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively; (ii) thrice the size of A_r, i.e., $3(n-1)$; and (iii) $k^{'} - 1$.*

Given a directed pathos middle digraph Q, the proof of the sufficiency of Theorem 4.1 shows how to find an arborescence A_r such that $DPM(A_r) = Q$. This obviously raises the question of whether Q determines A_r "uniquely". Although the answer to this in general is no, the extent to which A_r is determined is given as follows.

One can check easily that using the reconstruction procedure of the sufficiency of Theorem 4.1, any arborescence A_r (without directed pathos) is uniquely reconstructed from its directed pathos middle digraph, but if the directed path number is more than one (i.e., $k^{'} > 1$), then it is not possible to mark the directed paths of directed pathos of A_r uniquely. This clearly indicates the fact that $k^{'}$ must be exactly one for the unique reconstruction of A_r together with its directed pathos. It is known that a directed path is a special case of an arborescence. Since the directed path number of $\vec{P_n}$ is one, it follows that only $\vec{P_n}$ can be reconstructed uniquely from its $DPM(\vec{P_n})$.

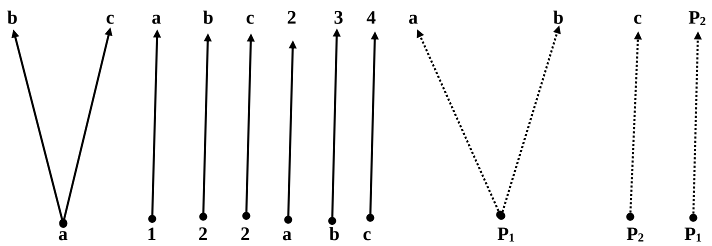

Figure.4: Decomposition of $DPM(A_r)$ (showed in Figure.3) into complete bipartite subdigraphs.

5. Properties of $DPM(A_r)$

Property 5.1. *For an arborescence A_r, $L(A_r) \subseteq M(A_r) \subseteq DPM(A_r)$, where $\subseteq$ is the subdigraph notation.*

Property 5.2. *Let v be vertex of $DPM(A_r)$ corresponding to the root arc of A_r. Then the removal of v from $DPM(A_r)$ makes it a disconnected. Therefore, $DPM(A_r)$ is not a block.*

Property 5.3. If v is the root of A_r, then in $DPM(A_r)$, $d^-(v) = 0$ and $d^+(v) = 1$. Therefore, $DPM(A_r)$ is non-Eulerian (since $d^-(v) \neq d^+(v)$).

Property 5.4. If the total degree of a vertex v in A_r is n, then the total degree of the corresponding vertex v in $DPM(A_r)$ is also n.

Property 5.5. Let v be the vertex of $DPM(A_r)$ corresponding to the root arc (a pendant arc) of A_r. Then the in-degree of v is always two.

The following Theorem and definitions are used in order to prove the next property.

Theorem 5.1. (***Gutin*** [6]) : *Let D be an acyclic digraph with precisely one source x in D. Then for every $v \in V(D)$, there is an $(x, v)-$directed path in D.*

A *transmitter* is a vertex v whose out-degree is positive and whose in-degree is zero, i.e., $d^+(v) > 0$ and $d^-(v) = 0$. A *carrier* is a vertex v whose out-degree and in-degree are both one, i.e., $d^+(v) = d^-(v) = 1$. A *receiver* is a vertex v whose out-degree is zero and whose in-degree is positive, i.e., $d^+(v) = 0$ and $d^-(v) > 0$. A vertex v is said to be *ordinary* if $d^+(v) > 0$ and $d^-(v) > 0$.

Definition 5.1. A *directed pathos vertex* of $DPM(A_r)$ is a vertex corresponding to the directed path of a directed pathos of A_r.

Proposition 5.1. *Let A_r be an arborescence of order n $(n \geq 2)$ with v_1 and $e_1 = (v_1, v_2)$ as the root and root arc of A_r, respectively. Then there exists exactly one vertex v with $d^+(v) > 0$ and $d^-(v) = 0$ (i.e., transmitter), and for every vertex $x \in DPM(A_r)$ (except for the vertex v_1), there is an $(v, x)-$ directed path in $DPM(A_r)$.*

Proof. Let A_r be an arborescence with vertex set $V(A_r) = \{v_1, v_1, \ldots, v_n\}$ and arc set $A(A_r) = \{e_1, e_2, \ldots, e_{n-1}\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively. By definition of $L(A_r)$, the vertices $e_2, e_3, \ldots, e_{n-1}$ are reachable from e_1 by a unique directed path. Let $P(A_r) = \{P_1, P_2, \ldots, P_{k'}\}$ be a directed pathos set of A_r such that P_1 lies on the arc e_1. Since the direction of the directed pathos is along the direction of the arcs in A_r, in $DPM(A_r)$, $d^+(v_1) = 1, d^-(v_1) = 0$; $d^+(P_1) > 0$, $d^-(P_1) = 0$; and the remaining vertices are either receiver or carrier or ordinary. Clearly, $DPM(A_r)$ is acyclic. By Theorem 5.6, for every (except v_1) vertex $x \in DPM(A_r)$, there is an $(P_1, x)-$ directed path in $DPM(A_r)$. This completes the proof. $\square$

While defining any class of digraphs, it is desirable to know the order and size of each; it is easy to determine for $DPM(A_r)$.

Proposition 5.2. *Let A_r be an arborescence with n vertices $v_1, v_2, \ldots, v_n$ and k' sinks. Then the order and size of $DPM(A_r)$ are $2n + k' - 1$ and $3n + \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + k' - 4$, respectively.*

Proof. If A_r has n vertices and $k^{'}$ sinks, then it follows immediately that $DPM(A_r)$ contains $n + n - 1 + k^{'} = 2n + k^{'} - 1$ vertices. Furthermore, every arc of $DPM(A_r)$ corresponds to an arc in A_r (there are $n - 1$ arcs); adjacent arcs in A_r (this is given by $\sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i)$); an arc adjacent to a vertex in A_r (there are $n - 1$ of these); a vertex adjacent to an arc in A_r (there are $n - 1$ of these); and the arcs whose end-vertices are the directed pathos vertices (this is given by $k^{'} - 1$). Therefore, $DPM(A_r)$ has $(n-1) + \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + 2(n-1) + k^{'} - 1 = 3n + \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + k^{'} - 4$ arcs. □

6. Characterization of $DPM(A_r)$

In this section the characterization of digraphs whose $DPM(A_r)$ are planar; outerplanar; maximal outerplanar; and minimally non-outerplanar is studied.

Theorem 6.1. *A directed pathos middle digraph $DPM(A_r)$ of an arborescence A_r is planar if A_r is either a directed path $\vec{P_n}$ on $n \geq 2$ vertices or the underlying graph of A_r is a star graph $K_{1,n}$ on $n \geq 3$ vertices.*

Proof. Suppose that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. Let $V(\vec{P_n}) = \{v_1, v_2, \ldots, v_n\}$ and the arcs of $\vec{P_n}$ be $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n-1$. By definition, (v_i, e_i); (e_i, v_{i+1}) for $1 \leq i \leq n-1$; and (e_i, e_{i+1}) for $1 \leq i \leq n-2$ are the arcs of $M(A_r)$. The directed path number of $\vec{P_n}$ is one, say P, and the corresponding directed pathos vertex P is a neighbor of every vertex (i.e., e_i for $1 \leq i \leq n-1$) of $M(A_r)$. This shows that the crossing number of $DPM(A_r)$ is zero. Thus $DPM(A_r)$ is planar.

On the other hand, suppose that the underlying graph of A_r is a star graph $K_{1,n}$ on $n \geq 3$ vertices. Let $V(A_r) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set and $A(A_r) = \{e_1, e_2, \ldots, e_{n-1}\}$ be the arc set of A_r such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively; and $e_i = (v_2, v_{i+1})$ for $2 \leq i \leq n$. By definition, (v_i, e_i) for $1 \leq i \leq 2$; (v_2, e_i) for $3 \leq i \leq n$; (e_i, v_{i+1}) for $1 \leq i \leq n$; and (e_1, e_i) for $2 \leq i \leq n$ are the arcs of $M(A_r)$. Let $P(A_r) = \{P_1, P_2, \ldots, P_{n-1}\}$ be a directed pathos set of A_r such that P_1 lies on the arcs e_1, e_2; and P_i lies on the arcs e_{i+1} for $2 \leq i \leq n-1$. Then the directed pathos vertex P_1 is a neighbor of the vertices e_1, e_2, P_i; and P_i is a neighbor of e_{i+1} for $2 \leq i \leq n-1$. This again shows that the crossing number of $DPM(A_r)$ is zero. Thus $DPM(A_r)$ is planar. □

Theorem 6.2. *A directed pathos middle digraph $DPM(A_r)$ of an arborescence A_r is outerplanar if and only if A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices.*

Proof. Suppose that $DPM(A_r)$ is outerplanar. Assume that there exist a vertex of total degree three in A_r, i.e., the underlying graph of A_r is $K_{1,3}$. Let $V(A_r) = \{v_1, v_2, v_3, v_4\}$ and $A(A_r) = \{e_1, e_2, e_3\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively; and $e_i = (v_2, v_{i+1})$ for $2 \leq i \leq 3$. By definition, (v_i, e_i) for $1 \leq i \leq 2$; (v_2, e_3); (e_i, v_{i+1}) for $1 \leq i \leq 3$; and (e_1, e_i) for $2 \leq i \leq 3$ are the arcs of $M(A_r)$. Let $P(A_r) = \{P_1, P_2\}$ be a directed pathos set of A_r such that P_1 lies on the arcs e_1, e_2; and P_2 lies on the arc e_3. Then the directed pathos vertex P_1 is a neighbor of the vertices e_1, e_2, P_2; and P_2 is a neighbor of the vertex e_2. This shows that the inner vertex number of $DPM(A_r)$ is more than one, i.e., $i(DPM(A_r)) > 1$ (see Figure.3), a contradiction.

Conversely, suppose that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. By Theorem 6.1, $\mathrm{cr}(DPM(A_r)) = 0$, and also the inner vertex number of $DPM(A_r)$ is zero, i.e., $i(DPM(A_r)) = 0$ (see Figure.2). Thus $DPM(A_r)$ is outerplanar. □

Theorem 6.3. *For any arborescence A_r, $DPM(A_r)$ is not maximal outerplanar.*

Proof. Suppose that $DPM(A_r)$ is maximal outerplanar. Assume that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. By Theorem 6.2, $DPM(A_r)$ is outerplanar. Furthermore, the addition of an arc does not alter the outerplanarity of $DPM(A_r)$. Therefore, $DPM(A_r)$ is not maximal outerplanar, a contradiction. This completes the proof. □

Theorem 6.4. *For any arborescence A_r, $DPM(A_r)$ is not minimally non-outerplanar.*

Proof. Suppose that $DPM(A_r)$ is minimally non-outerplanar, i.e., $i(DPM(A_r)) = 1$.
Case 1. Suppose that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. By Theorem 6.2, $DPM(A_r)$ is outerplanar, a contradiction.
Case 2. Suppose there exist a vertex of total degree three in A_r. By necessity of Theorem 6.2, $i(DPM(A_r)) > 1$, again a contradiction. Hence by all the cases above, $DPM(A_r)$ is not minimally non-outerplanar. □

7. Conclusions

In this paper, the digraph operator called a *directed pathos middle digraph* of an arborescence is defined and the problem of reconstructing an arborescence from its directed pathos middle digraph is presented. The characterization of digraphs whose directed pathos middle digraphs are planar; outerplanar; maximal outerplanar; and minimally non-outerplanar is studied. Harary [2] and Stanton [8] calculated the path number of certain classes of graphs like trees and complete graphs. The path number of a tree was defined by Muddebihal [7]. The directed path number of a digraph D is not known yet. Finding the directed path number of a digraph seems to be interesting one and it leads to the study of many digraph operators. What one can say about the properties of these digraph operators?

Acknowledgement. The author is grateful to the anonymous reviewers for giving valuable comments and suggestions.

References

[1] Harary, F., (1969). Graph Theory, Addison-Wesley, Reading, Mass.
[2] Harary, F., (1970). Converging and packing in graphs-I, Annals of New York Academy of Science, 175, pp. 198-205.
[3] Harary, F. and Norman, R. Z., (1987). Some properties of line digraphs, Rend. Circ. Mat. Palermo., (9), pp. 161-168.
[4] Harary, F. and Norman, R. Z., and Cartwright, D., (1965). Structural models: an introduction the theory of directed graphs, New York.
[5] Hamada, T. and Yoshimura, I., (1976). Traversability and connectivity of the middle graph of a graph, Discrete Mathematics, 14, pp. 247-256.
[6] Jorgen Bang-Jensen., and Gregory Gutin., (2009). Digraphs Theory, Algorithms and applications, Springer-Verlag London Limited.
[7] Muddebihal, M. H., and Chandrasekhar, R., (2001). On pathos line graph of a tree, National Academy of Science Letters, 24, pp. 116-123.
[8] Stanton, R. G., and Cowan, D. D., (1970). Some results on path numbers, Proc. Louisiana Conf. on Combinatorics, Graph Theory and Computing., pp. 198-205.
[9] Whitney, H., (1932). Congruent graphs and the connectivity of graph, Amer. J. Math., 54, pp. 150-168.
[10] Zamfirescu, C., (1981). Local and global characterization of middle digraphs, Theory and Applications of Graphs., pp. 595-607.

H. M. Nagesh is working as an assistant professor in the department of Science & Humanities, PES University - Electronic City Campus, Bangalore. His research interests are in the area of Topological Graph Theory. He has published 15 research papers in reputed international journal of mathematical sciences.