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ON THE METRIC CHARACTRIZATION FOR WELL-POSEDNESS OF K-SPLIT HEMIVARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, we extend the concept of well-posedness for a class of k-split hemivariational like inequalities and characterize some conditions for the well-posedness of it. Then we show that Tykhonov well-posedness for this family of k-split hemivariational like inequalities is equivalent to the existence and uniqueness of solution.

Keywords: Split hemivariational inequality, Clarke's generalized gradient, Well-posedness.

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The study of vector optimization problems has become an important research direction of vector variational inequalities . In particular, minimization problem is closely related to a variational inequality [10, 15]. So, the concept of the well-posedness which this plays a crucial role in the theory of optimization problems can be generalized to variational inequalities. For example, Tykhonov well-posedness, which was introduced by Tykhonov [20] in 1966 for a global minimization problem, requires the existence and uniqueness of solution to the global minimization problem and the convergence of every minimizing sequence toward the unique solution. Hence nowadays, the concept of well-posedness variational inequalities have attracted much attention of mathematical researchers.

Throughout this investigations, in recent years, the concept of well-posedness for variational inequalities has been generalized to the other types of variational inequalities such as variational-like inequalities [3], quasi variational inequalities [12], mixed variational inequalities [13], etc. One of the useful generalizations for variational inequalities is the concept of hemivariational inequalities, which first was introduced by Panagiotopoulos [19]. Recently, hemivariational inequalities have drawn much attention of mathematical researchers. See for example [21] and the references therein. Another generalization of the concept of variational inequalities is the concept of split variational inequalities [2]. The split variational inequalities have been proved very efficient to describe a wide range of practical problems such as image processing and signal recovery. See for example [8]. Recently, Hu and Fang in [9] studied two kinds of well-posedness for various split variational inequalities. For other more work on the well-posedness of variational inequalities, we can refer to [3, 7, 11, 15] and the references therein.

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In this paper , first, we introduce a new class of k-split hemivariational like inequalities as a special case of the classical hemivariational inequalities below.

We suppose that $[0,T]$ is a real interval and for $i=1,\dots,k$, $X_i \subseteq C^0[0,T]$ is a reflexive Banach space with topological dual X_i^* and with norm $\|x\| = \|x\|_\infty$ for all $x \in X_i$. Also, $A_i : [0,T] \times X_i \to X_i^*$ is a single-valued function and each function $t \to A_i(t,x_i)$ is integrable for all $x_i \in X_i$, f_i is some given element in X_i^* and $\langle \cdot,\cdot \rangle_{X_i^* \times X_i}$ denotes the dual pair between X_i, X_i^*. Let $\eta_i : X_i \times X_i \to X_i$ is a vector valued function, $J_i : [0,T] \times X_i \to \mathbb{R}$ is a locally Lipschitz functional and $J_i^\circ\big(t, x_i, \eta_i(y,x_i)\big)$ denotes the Clarke's generalized directional derivative of function $J_i(t,.,.)$ at x_i in direction $\eta_i(y,x_i)$ and $S : X_1 \times \cdots \times X_{k-1} \to X_k$ is a continuous mapping. By this assumption, we introduce a new class of k-split hemivariational-like inequality (SHVI)$(A_i, f_i, J_i)_{i=1,\dots,k}$ as find $(x_1,\dots,x_k) \in X_1 \times \cdots \times X_k$ which

$$x_k = S(x_1,\dots,x_{k-1})$$

$$\int_0^T [\langle A_i(t,x_i) - f_i, \eta_i(y,x_i)\rangle_{X_i \times X_i^*} + J_i^\circ\big(t, x_i, \eta_i(y,x_i)\big)]dt \geq 0, \ \forall y \in X_i,$$

for $i=1,\dots,k$.

Then we aim to extend the concept of the well-posedness for this special case of the classical hemivariational inequality and develop the abstract results and investigate the metric characterization of well-posedness of this case.

1. Preliminaries

Let X be a Banach space with dual space X^*. Then the functional $\theta : X \to \mathbb{R}$ is called a Lipschitz continuous function on X if there exists a constant $M > 0$ such that

$$|\theta(x_1) - \theta(x_2)| \leq M\|x_1 - x_2\|_X \quad \forall x_1, x_2 \in X,$$

and if, for each $x \in X$ there exists a neighborhood V_x of x and a constant $M_x > 0$ such that

$$|\theta(x_1) - \theta(x_2)| \leq M_x\|x_1 - x_2\|_X \quad \forall x_1, x_2 \in V_x,$$

then θ is called a locally Lipschitz continuous functional on X. When θ is nondifferentiable and locally Lipschitz, $\theta^\circ(x_0,h)$; the Clarke's generalized directional derivative of θ at point of $x_0 \in X$ in direction $h \in X$, is defined by

$$\theta^\circ(x_0,h) = \limsup_{x \to x_0, \lambda \downarrow 0} \frac{\theta(x+\lambda h) - \theta(x)}{\lambda}.$$

It is easy to show that the function $h \to \theta^\circ(x_0,h)$ is finite, subadditive and positively homogeneous, namely, for all $h_1, h_2 \in X$ and $\lambda \geq 0$

$$\theta^\circ(x_0, h_1 + h_2) \leq \theta^\circ(x_0,h_1) + \theta^\circ(x_0,h_2) \text{ and } \theta^\circ(x_0,\lambda h) = \lambda\theta^\circ(x_0,h),$$

see([5]). Also, $\theta^\circ(.,.)$ is upper semicontinuous, i.e., for each $x_1, x_2 \in X$ and $\{x_1^n\}, \{x_2^n\} \subset X$ that $x_1^n \to x_1$ and $x_2^n \to x_2$, we have

$$\limsup_{x \to \infty} \theta^\circ(x_1^n, x_2^n) \leq \theta^\circ(x_1,x_2).$$

$\partial_C\theta(x_0)$; the generalized gradient in the sense of Clarke of θ at $x_0 \in X$, is defined by

$$\partial_C\theta(x_0) = \{x^* \in X^* : \langle x^*, h\rangle \leq \theta^\circ(x_0;h), \forall h \in X\}.$$

One can show that for the locally Lipschitz functional θ on X and each $x_0 \in X$, $\partial_C\theta(x_0)$ is a nonempty, bounded and convex set in X^* and with respect to weak* topology is compact. Also, for each $h \in X$ we have

$$\theta^\circ(x_0;h) = \max\{\langle x^*, h\rangle : x^* \in \partial_C\theta(x_0)\}.$$

The vector valued function $\eta : X \times X \to X$ is said to be skew if

$$\eta(x, y) + \eta(y, x) = 0 \quad \forall x, y \in X$$

and it satisfies condition C if for each $x, y \in X$ and $\lambda \in [0, 1]$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y), \quad \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

One can easily show that if $\eta : X \times X \to X$ satisfies condition C, then for each $x, y \in X$

$$\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y).$$

Definition 1.1. *Let X be a Banach space and X^* be it's dual space. Then a mapping $A : [0, T] \times X \to X^*$ is said to be*
(1) inavariant monoton with respect to η, if for each $t \in [0, T]$ and $x, y \in X$,

$$\int_0^T \big(\langle A(t, x), \eta(y, x)\rangle + \langle A(t, y), \eta(x, y)\big)dt \leq 0,$$

(2) hemicontinuous, if for each $t \in [0, T]$ and $x, y \in X$, the function $\lambda \to \langle A(t, x + \lambda y), y\rangle$ is continuous on $[0, 1]$.

Definition 1.2. *A sequence $\{(x_1^n, \ldots, x_k^n)\} \subseteq X_1 \times \cdots \times X_k$ is called an approximating sequence for the* $\mathrm{SHVI}(A_i, f_i, J_i)_{i=1,\ldots,k}$ *if there exists $0 < \epsilon_n \to 0$ such that*

$$\|x_k^n - S(x_1^n, \ldots, x_{k-1}^n)\|_{X_k} \leq \epsilon_n,$$

$$\int_0^T [\langle A_i(t, x_i) - f_i, \eta_i(y, x_i)\rangle_{X_i \times X_i^*} + J_i^\circ\big(t, x_i, \eta_i(y, x_i)\big)]dt \geq -\epsilon_n\|\eta_i(y, x_i^n)\|, \ \forall y \in X_i,$$

for $i = 1, \ldots, k$.

Definition 1.3. *The* $\mathrm{SHVI}(A_i, f_i, J_i)_{i=1,\ldots,k,}$ *is said to be strongly (resp., weakly) well-posed if it has a unique solution on $X_1 \times \cdots \times X_k$ and every approximating sequence for* $\mathrm{SHVI}(A_i, f_i, J_i)_{i=1,\ldots,k,}$ *converges strongly (resp., weakly) to the unique solution.*

By similar argument as [18], [22], we can deduce following results:

Proposition 1.1. *Let X be a Banach space with X^* being it's dual space and $A : [0, T] \times X \to X^*$ be an operator. If A is continuous with respect to the second component, then it is weakly* continuous with respect to the second component, which, in turn, implies that it is hemicontinuous with respect to the second component. Moreover, if A is invariant monotone with respect to η, then the notions of weak* continuity and hemicontinuity coincid.*

Proposition 1.2. *Let X be a Banach space with X^* being its dual space and suppose that $A : [0, T] \times X \to X^*$ be an operator. Then the following statement holds: If $\{x_n\} \subset X$, $x_n \to x \in X$, $A(t, x_n) \to A(t, x) \in X^*$ in w^*-topology, then $\langle A((t, x_n), \eta(y, x_n)\rangle \to \langle A((t, x), \eta(y, x)\rangle$, $\forall y \in X$.*

2. Metric characterizations for well-posedness

In this section, we first present an equivalent formulation of the $\mathrm{SHVI}(A_i, f_i, J_i)_{i=1,\ldots,k}$, under the assumption of the invaiant monotonicity for two single-valued operators $A_i, i = 1, \ldots, k$. Also, we are ready to under the assumption of the invariant monotonicity of two single-valued map A_i investigate the metric characterization of the well-posedness of k-split hemivariational like-inequality $\mathrm{SHVI}(A_i, f_i, J_i)_{i=1,\ldots,k}$, which it is introduced in the section 1 and we extend the well-posedness of it.

Theorem 2.1. *Let $A_i : [0,T] \times X_i \to X_i^*$ be an operator such that $A_i(t,.)$ is hemicontinuous and $\eta_i : X_i \times X_i \to X_i$ satisfies condition C. Assume that either one of the following conditions is satisfied:*

(1) *$(x_1, \dots x_k) \in X_1 \times \cdots \times X_k$ is a solution to the following associated k-split hemivariational inequality:*
ASHVI$(A_i, f_i, J_i)_{i=1,\dots,k}$: *Find $(x_1, \dots x_k) \in X_1 \times \cdots \times X_k$, such that*
$$x_k = S(x_1, \dots, x_{k_1}),$$
$$\int_0^T [\langle A_i(t,y) - f_i, \eta_i(y,x_i)\rangle + J_i^\circ(t,y;\eta_i(y,x_i)]dt \geq 0, \ \forall y \in X_i, \tag{1}$$
for $i = 1, \dots, k$.

(2) *$(x_1, \dots x_k) \in X_1 \times \cdots \times X_k$ is a solution to the following k-split multi-valued variational inequality:*
$SMVI(A_i, f_i, J_i)_{i=1,\dots,k}$: Find $(x_1, \dots x_k) \in X_1 \times \cdots \times X_k$ such that, for all $y \in X_i$, there exists $\gamma_i \in \partial_C J_i(t,y)$ satisfying
$$\textstyle\int_0^T \langle A_i(t,y) + \gamma_i - f_i, \eta_i(y,x_i)\rangle dt \geq 0,$$
for $i = 1, \dots, k$.

Then $(x_1, \dots x_k) \in X_1 \times \cdots \times X_k$ is solution to the k-split hemivariational like inequality SHVI$(A_i, f_i, J_i)_{i=1,\dots,k}$.

Proof. Let $(x_1, \dots x_k) \in X_1 \times \cdots \times X_k$ be a solution of (ASHVI)$(A_i, f_i, J_i)_{i=1,\dots,k}$. In 1, for $i = 1, \dots, k$, set $y = x_i + \lambda\eta_i(z_i, x_i) \in X_i$ in which $z_i \in X_i$ and $\lambda \in [0,1]$ are arbitrary. Then we have
$$\int_0^T [\langle A_i(t, x_i + \lambda\eta_i(z_i,x_i) - f_i, \lambda\eta_i(z_i,x_i))\rangle + J_i^\circ(t, x_i + \lambda\eta_i(z_i,x_i); \lambda\eta_i(z_i,x_i))]dt$$
$$\geq 0, \quad \forall z_i \in X_i.$$
By using the condition C we have $\eta_i(x_i + \lambda\eta_i(z_i,x_i), x_i) = \lambda\eta_i(z_i,x_i)$. Also, note that $J_i^\circ(t,x;.)$ is positively homogeneous. So, by this fact, upper semicontinuity of $J_i^\circ(t,.;.)$ and hemicontinuity of A_i, passing to the upper limit when $\lambda \to 0$, we get
$$\textstyle\int_0^T [\langle A_i(t,x_i) - f_i, \eta_i(z_i,x_i))\rangle + J_i^\circ(t,x_i;\eta_i(z_i,x_i))]dt \geq 0, \quad \forall z_i \in X_,$$
for $i = 1, \dots, k$.
The similar argument can be applied to prove the second assertion. □

Note that if for $i = 1, \dots, k$, $\eta_i : X_i \times X_i \to X_i$ satisfies condition C and A_i is invariant monotone with respect to η_i, then we can easily deduce that $(x_1, \dots, x_k) \in X_1 \times \cdots \times X_k$ is solution to the ASHVI$(A_i, f_i, J_i)_{i=1,\dots,k}$, when it solves SHVI$(A_i, f_i, J_i)_{i=1,\dots,k}$.

Example 2.1. *Assume that S be the identitiy function and for $i = 1, 2$, $f_i \equiv 0$, $\eta_i(x,y) = y - x$ and $J_i : [0,1] \times X_i \to \mathbb{R}$ be defined as*
$$J_1(t,x_1) = \begin{cases} tx_1 & \text{if } x_1 \geq 0, \\ -x_1 & \text{if } x_1 < 0 \end{cases}$$
and
$$J_2(t,x_2) = \begin{cases} x_2^2 + 2tx_2 & \text{if } x_2 > 0, \\ x_2^2 - 2x_2 & \text{if } x_2 \leq 0. \end{cases}$$
We can see that
$$\partial_C J_1(t,x_1) = \begin{cases} t & \text{if } x_1 > 0, \\ [-1,t] & \text{if } x_1 = 0, \\ -1 & \text{if } x_1 < 0 \end{cases}$$

and

$$\partial_C J_2(t,x_2)=\begin{cases} 2x_2+2t & \text{if } x_2>0,\\ [-2,2t] & \text{if } x_2=0,\\ 2x_2-2 & \text{if } x_2<0\end{cases}$$

and $J_i^\circ\big(t,y_i;y_i-x_i\big)\geq 0$, *for all* $x_i,y_i\in X_i$. *Now, consider the function* $A_i:[0,1]\times X_i\to X_i^*$ *be an operator, defined by* $A_1(t,x_1)=e^t x_1$ *and* $A_2(t,x_2)=t^2x_2$. *We can easily show that*

$$\textstyle\int_0^1[\langle A_1(t,y_1),y_1\rangle+J^\circ\big(t,y_1;y_1\big)]dt\geq 0,\quad \forall y_1\in X_1.$$

With the same argument, we have

$$\textstyle\int_0^1[\langle A_2(t,y_2),y_2\rangle+J^\circ\big(t,y_2;y_2\big)]dt\geq 0,\quad \forall y_2\in X_2.$$

So, $x=(0,0)$ *solves* $\mathrm{ASHVI}(A_i,f_i,J_i)_{i=1,2}$ *and from Theorem 2.1, we can deduce that* $x=(0,0)$ *is a solution of* $\mathrm{SHVI}(A_i,f_i,J_i)_{i=1,2}$. *In fact, we have*

$$\textstyle\int_0^1[\langle A_i(t,0),y_i\rangle+J_i^\circ\big(t,x_i;y_i\big)]dt=\int_0^1 J_i^\circ\big(t,x_i;y_i\big)]dt\geq 0,\quad \forall y_i\in X_i.$$

Also, It can be easily shown that A_i *is monotone with respect to* η_i. *Therefore the solution sets of* $\mathrm{SHVI}(A_i,f_i,J_i)_{i=1,2}$ *and* $\mathrm{ASHVI}(A_i,f_i,J_i)_{i=1,2}$ *are equivalent.*

Now, we define two sets as

$$\begin{aligned}F(\epsilon)=\Big\{&(x_1,\dots,x_k)\in X_1\times\dots,\times X_k;\|x_k-S(x_1,\dots,x_{k-1})\|_{X_k}\leq\epsilon,\\ &\int_0^T[\langle A_i(t,x_i)-f_i,\eta_i(y,x_i)\rangle_{X_i\times X_i^*}+J_i^\circ\big(t,x_i,\eta_i(y,x_i)\big)]dt\geq-\epsilon\|\eta_i(y,x_i)\|,\\ &\forall y\in X_i, i=1,\dots,k\Big\}\end{aligned}$$

and

$$\begin{aligned}G(\epsilon)=\Big\{&(x_1,\dots,x_k)\in X_1\times\dots,\times X_k;\|x_k-S(x_1,\dots,x_{k-1})\|_{X_k}\leq\epsilon,\\ &\int_0^T[\langle A_i(t,y)-f_i,\eta_i(y,x_i)\rangle_{X_i\times X_i^*}+J_i^\circ\big(t,x_i,\eta_i(y,x_i)\big)]dt\geq-\epsilon\|\eta_i(y,x_i)\|,\\ &\forall y\in X_i, i=1,\dots,k\Big\}\end{aligned}$$

Lemma 2.1. *For* $i=1,\dots,k$, *let* X_i *be a Banach spaces with* X_i^* *being their dual spaces,* $A_i:[0,T]\times X_i\to X_i^*$ *is hemicontinuous and invariant monotone with respect to* η_i, *where* η_i *satisfies the condition C and* $J_i:X_i\to\mathbb{R}$ *is a locally Lipschitz functional. Then for each* $\epsilon>0$ *we have* $F(\epsilon)=G(\epsilon)$.

Proof. Let $\epsilon>0$ be given and $(x_1,\dots,x_k)\in F(\epsilon)$. Then for $i=1,\dots,k$, we have

$$\int_0^T[\langle A_i(t,x_i)-f_i,\eta_i(y,x_i)\rangle+J_i^\circ\big(t,x_i,\eta_i(y,x_i)\big)]dt\geq-\epsilon\|\eta_i(y,x_i)\|_{X_i},\quad\forall y\in X_i.\tag{2}$$

Since A_i is monotone with respect to η_i, so for $x_i\in X_i$ we have

$$\int_0^T[\langle A_i(t,x_i)-A_i(t,y),\eta_i(y,x_i)\rangle]dt\leq 0,\quad y\in X_i.\tag{3}$$

Hence by (2), (3), we can write

$$\int_0^T[\langle A_i(t,y)-f_i,\eta_i(y,x_i)\rangle+J_i^\circ\big(t,x_i,\eta_i(y,x_i)\big)]dt\geq-\epsilon\|\eta_i(y,x_i)\|_{X_i},\quad\forall y\in X_i.\tag{4}$$

Now, the inequalities $\|x_k-S(x_1,\dots,x_{k-1})\|_{X_k}\leq\epsilon$ and (4) showes that $(x_1,\dots,x_k)\in G(\epsilon)$ and thus $F(\epsilon)\subseteq G(\epsilon)$.

For the revers, let $(x_1,\dots,x_k)\in G(\epsilon)$ and $(z_1,\dots,z_k)$ is any point in $X_1\times\cdots\times X_k$ and $\lambda\in[0,1]$. So, for $i=1,\dots,k$, we have

$$\int_0^T [\langle A_i(t,y)-f_i,\eta_i(y,x_i)\rangle + J_i^\circ\big(t,x_i,\eta_i(y,x_i)\big)]dt \geq -\epsilon\|\eta_i(y,x_i)\|_{X_i}, \quad \forall y\in X_i. \tag{5}$$

Now, we substitut $y_i=x_i+\lambda\eta_i(z_i,x_i)$ in (5). Since $J_i^\circ(t,x_i,\cdot)$ is positively homogeneous, so for $i=1,\dots,k$, one may write

$$\int_0^T [\langle A_i(t,x_i+\lambda\eta_i(z_i,x_i))-f_i,\eta_i(z_i,x_i))\rangle + J_i^\circ\big(t,x_i,\eta_i(z_i,x_i)\big)]dt \geq -\epsilon\|\eta_i(z_i,x_i)\|_{X_i} \tag{6}$$

and if $\lambda\to 0^+$ in (6), then from the hemicontiuity of the mapping A_i, it follows

$$\|x_k-S(x_1,\dots,x_{k-1})\|_{X_k}\leq\epsilon,$$

$$\int_0^T [\langle A_i(t,x_i)-f_i,\eta_i(z_i-x_i)\rangle + J_1^\circ\big(t,x_i,\eta_i(z_i-x_i)\big)]dt \geq -\epsilon\|\eta_i(z_i-x_i)\|_{X_i},$$

Since $(z_1,\dots,z_k)\in X_1\times\cdots\times X_k$ is arbitrary, we conclude that $(x_1,\dots,x_k)\in F(\epsilon)$ and thus $G(\epsilon)\subseteq F(\epsilon)$. This completes the proof. □

Lemma 2.2. *Let X_i be a reflexive Banach spaces with X_i^* as it's dual space, $J_i: X_i\to\mathbb{R}$ be a locally Lipschitz functional for $i=1,\dots,k$, and $S: X_1\times\cdots\times X_{k-1}\to X_k$ be a continuous map. Then $G(\epsilon)$ in $X_1\times\cdots\times X_k$ is closed for all $\epsilon>0$.*

Proof. Let $\{(x_1^n,\dots,x_k^n)\}\subset G(\epsilon)$ and $(x_1^n,\dots,x_k^n)\to(x_1,\dots,x_k)$ in $X_1\times\cdots\times X_k$. Then we have

$$\|x_k^n-S(x_1^n,\dots x_{k-1}^n)\|_{X_k}\leq\epsilon,$$

$$\int_0^T [\langle A_i(t,y)-f_i,\eta_i(y,x_i^n)\rangle + J_i^\circ\big(t,x_i^n,\eta_i(y,x_i^n)\big)]dt \geq -\epsilon\|\eta_i(y,x_i^n)\|_{X_i}, \quad \forall y\in X_i \tag{7}$$

and due to upper semicontinuity of $J_i^\circ(t,.;.)$, we can write

$$\limsup_{n\to\infty} J_i^\circ(t,x_i^n,\eta_i(y,x_i^n)))\leq J_i^\circ(t,x_i;\eta_i(y,x_i)), \quad \forall y\in X_i, \tag{8}$$

for $i=1,\dots,k$.

Now by taking lim sup from the both sides of the last inequality in (7) and using (8), for $i=1,\dots,k$, we obtain

$$\int_0^T [\langle A_i(t,y)-f_i,\eta_i(y,x_i)\rangle + J_i^\circ\big(t,x_i,\eta_i(y,x_i))\big)]dt$$

$$\geq \limsup \int_0^T [\langle A_i(t,y)-f_i,\eta_i(y,x_i^n)\rangle + J_i^\circ\big(t,x_i^n,\eta_i(y,x_i^n)\big)]dt$$

$$\geq -\epsilon\|\eta_i(y,x_i)\|_{X_i}, \quad \forall y\in X_i.$$

On the other hand, for each $n\in\mathbb{N}$, we have $(x_1^n,\dots,x_k^n)\in G(\epsilon)$, so

$$\|x_k^n-S(x_1^n,\dots,x_{k-1}^n)\|_{X_k}\leq\epsilon.$$

Hence by the continuity of mapping S and the norm function,

$$\|x_k-S(x_1,\dots,x_{k-1})\|_{X_k}\leq\epsilon,$$

so $(x_1,\dots,x_k)\in G(\epsilon)$ and this implies that $G(\epsilon)$ is closed and the proof is completed. □

Lemma 2.3. *For $i = 1, \dots, k$, let X_i be a reflexive Banach spaces with X_i^* as it's dual space, $J_i : X_i \to \mathbb{R}$ be a locally Lipschitz functional, $A_i : [0, T] \times X_i \to X_i^*$ is monotone with respect to η_i and hemicontinuous and $S : X_1 \times \cdots \times X_{k-1} \to X_k$ be a continuous map. Then $F(\epsilon)$ in $X_1 \times \cdots \times X_k$ is closed for all $\epsilon > 0$.*

Proof. Let $\{(x_1^n, \dots, x_k^n)\} \subset F(\epsilon)$ and $(x_1^n, \dots, x_k^n) \to (x_1, \dots, x_k)$ in $X_1 \times \cdots \times X_k$. Then for $i = 1, \dots, k$, we have

$$\|x_k^n - S(x_1^n, \dots x_{k-1}^n)\|_{X_k} \leq \epsilon,$$

$$\int_0^T [\langle A_i(t, x_i^n) - f_i, \eta_i(y, x_i^n)\rangle + J_i^\circ\big(t, x_i^n, \eta_i(y, x_i^n)\big)]dt \geq -\epsilon \|\eta_i(y, x_i^n)\|_{X_i}, \quad \forall y \in X_i, \tag{9}$$

Since $A_i : [0, T] \times X_i \to X_i^*$ is monoton with respect to η_i and hemicontinuous, so from Propostion 1.1, it is weakly* continuous, i.e., $A_i x_i^n \to A_i x_i$ in weak* topology when $n \to \infty$. Hence, by Proposition 1.2, we can write

$$\lim_{n\to\infty} \int_0^T [\langle A_i(t, x_i^n) - f_i, \eta_i(y, x_i^n)\rangle dt = \int_0^T [\langle A_i(t, x_i) - f_i, \eta_i(y, x_i)\rangle dt$$

$$\text{for } i = 1, \dots, k. \tag{10}$$

On the other hand $J_i^\circ(t, ., .)$ is upper semicontinuous, so by (9) and (10) we have

$$\begin{aligned}
&\int_0^T [\langle A_i(t, x_i) - f_i, \eta_i(y, x_i)\rangle + J_i^\circ\big(t, x_i, \eta_i(y, x_i)\big)]dt \\
&\geq \limsup \int_0^T [\langle A_i(t, x_i) - f_i, \eta_i(y, x_i)\rangle + J_i^\circ\big(t, x_i, \eta_i(y, x_i)\big)]dt \\
&\geq \limsup \big(-\epsilon \|\eta_i(y, x_i^n)\|_{X_i}\big) \\
&= -\epsilon \|\eta_i(y, x_i)\|_{X_i}, \quad \forall y \in X_i,
\end{aligned} \tag{11}$$

for $i = 1, \dots, k$.
Also it is easy to see that $\|x_k^n - S(x_1^n, \dots x_{k-1}^n)\|_{X_k} \leq \epsilon$, which this and (11) garantee $(x_1, \dots, x_k) \in F(\epsilon)$. Hence $F(\epsilon)$ is closed in $X_1 \times \cdots \times X_k$ and the proof is completed. □

Now, by the properties of $G(\epsilon)$, we study the metric characterization of k-split hemivariational inequality (SHVI)$(A_i, f_i, J_i)_{i=1,\dots,k}$ as below.

Theorem 2.2. *Let for $i = 1, \dots, k$, X_i be a Banach space with X_i^* as it's dual space, $A_i : [0, T] \times X_i \to X_i^*$ is a map, $J_i : X_i \to \mathbb{R}$ be a locally Lipschitz functional and $S : X_1 \times \cdots \times X_{k-1} \to X_k$ be a continuous map. Then the k-split hemivariational like inequality* (SHVI)$(A_i, f_i, J_i)_{i=1,\dots,k}$, *is strongly well-posed if and only if it's solution set K is nonempty and* $\operatorname{diam} F(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Proof. Let (SHVI)$(A_i, f_i, J_i)_{i=1,\dots,k}$, is well-posed. So by definition, it's solution set is nonempty, i.e., $K \neq \emptyset$. Now if $\operatorname{diam} F(\epsilon) \nrightarrow 0$ when $\epsilon \to 0$ we can find $\delta > 0$ and a sequence ϵ_n convergence to 0 and the members $(x_1^n, \dots, x_k^n), (y_1^n, \dots, y_k^n) \in F(\epsilon_n)$ such that

$$\|(x_1^n, \dots, x_k^n) - (y_1^n, \dots, y_k^n)\|_{X_1 \times \cdots \times X_k} \geq \delta, \quad \forall n \in \mathbb{N}. \tag{12}$$

Since $(x_1^n, \dots, x_k^n), (y_1^n, \dots, y_k^n) \in F(\epsilon_n)$, then $\{(x_1^n, \dots, x_k^n)\}$ and $\{(y_1^n, \dots, y_k^n)\}$ are approximating sequences for (SHVI)$(A_i, f_i, J_i)_{i=1,\dots,k}$. Hence the well-posedness of (SHVI) $(A_i, f_i, J_i)_{i=1,\dots,k}$, implies that $\{(x_1^n, \dots, x_k^n)\}$ and $\{(y_1^n, \dots, y_k^n)\}$ are converge to the uniqe solution of (SHVI)$(A_i, f_i, J_i)_{i=1,\dots,k,}$, which this is contradiction with (12). So, $\operatorname{diam} F(\epsilon) \to 0$ as $\epsilon \to 0$.
For the revers, let the solution set K of (SHVI)$(A_i, f_i, J_i)_{i=1,\dots,k}$, is nonempty and $\operatorname{diam} F(\epsilon) \to$

0 as $\epsilon \to 0$ and $\{(x_1^n, \ldots, x_k^n)\}$ is an approximating sequences for (SHVI)$(A_i, f_i, J_i)_{i=1,\ldots,k}$. Then there is $0 < \epsilon_n \to 0$ such that

$$\|x_k^n - S(x_1^n, \ldots, x_{k-1}^n)\|_{X_k} \le \epsilon_n, \tag{13}$$

$$\int_0^T [\langle A_i(t, x_i^n) - f_i, \eta_i(y, x_i^n)\rangle + J_1^\circ\big(t, x_i^n, \eta_i(y, x_i^n)\big)]dt \ge -\epsilon_n \|\eta_i(y, x_i^n)\|_{X_i}, \quad \forall y \in X_i,$$

for $i = 1, \ldots, k$. Then $(x_1^n, \ldots, x_k^n) \in F(\epsilon_n)$ with $0 < \epsilon_n \to 0$. Now , if we show that the solution set K of (SHVI)$(A_i, f_i, J_i)_{i=1,\ldots,k}$, is singleton, then the proof is completed. For this end, suppose that $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are two diference solution for (SHVI)$(A_i, f_i, J_i)_{i=1,\ldots,k}$. It is obvious that for each $\epsilon > 0$, $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in F(\epsilon)$ and

$$\|(x_1, \ldots, x_k) - (y_1, \ldots, y_k)\|_{X_1\times\cdots\times X_k} \le \operatorname{diam} G(\epsilon) \to 0, \quad \epsilon \to 0.$$

So, $(x_1, \ldots, x_k) = (y_1, \ldots, y_k)$. Note that for each $n \in \mathbb{N}$ and $0 < \epsilon_n \to 0$ we have $(x_1^n, \ldots, x_k^n), (x_1, \ldots, x_k) \in F(\epsilon_n)$, which this confirm that $(x_1^n, \ldots, x_k^n) \to (x_1, \ldots, x_k)$ and the proof is completed . □

Theorem 2.3. *Let for $i = 1, \ldots, k$, X_i be a Banach space with X_i^* as it's dual space, $A_i : [0, T] \times X_i \to X_i^*$ is a hemicontinuous mapping and monotone with respect to η_i, $J_i : X_i \to \mathbb{R}$ be a locally Lipschitz functional and also $S : X_1 \times \cdots \times X_{k-1} \to X_k$ be a continuous map. Then the k-split hemivariational like inequality* (SHVI)$(A_i, f_i, J_i)_{i=1,\ldots,k}$ *is strongly well-posed if and only if for all $\epsilon > 0$, $F(\epsilon) \ne \emptyset$ and* $\operatorname{diam} F(\epsilon) \to 0$ *as $\epsilon \to 0$.*

Proof. When the (SHVI)$(A_i, f_i, J_i)_{i=1,\ldots,k}$ is strongly well-posed, we conclude that it has a singleton solution set K and for each $0 < \epsilon$, $K \subseteq F(\epsilon)$, i.e., $F(\epsilon) \ne \emptyset$. Then we can follow the remain of the proof by a similar argument in Theorem 2.3 to obtain the necessity part. For the sufficiency part, letting $(x_1^n, \ldots, x_k^n)$ in $X_1\times\cdots\times X_k$, be an approximating sequence for (SHVI)$(A_i, f_i, J_i)_{i=1,\ldots,k}$, we can deduce that there is a sequence $0 < \epsilon_n \to 0$ such that

$$\|x_k^n - S(x_1^n, \ldots, x_{k-1}^n)\|_{X_k} \le \epsilon_n,$$

$$\int_0^T [\langle A_i(t, x_i^n) - f_i, \eta_i(y, x_i^n)\rangle + J_i^\circ\big(t, x_i^n, \eta_i(y, x_i^n)\big)]dt \ge -\epsilon_n \|\eta_i(y, x_i^n)\|_{X_i}, \quad \forall y \in X_i,$$

for $i = 1, \ldots, k$.
So, $(x_1^n, \ldots, x_k^n) \in F(\epsilon_n)$ and according to the assumption $\operatorname{diam} F(\epsilon_n) \to 0$ as $\epsilon_n \to 0$, we can say $(x_1^n, \ldots, x_k^n)$ is a cauchy sequence. So it converges strongly to a some point $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$. Now, we must show that $(x_1, \ldots, x_k)$, is a unique solution of the (SHVI)$(A_i, f_i, J_i)_{i=1,\ldots,k}$. Since the mapping A_i is monotone with respect to η_i, the Clarke's generalized directional derivative $J_i^\circ(t, ., .)$ is upper semicontinuous, we can write

$$\begin{aligned}
&\int_0^T [\langle A_i(t, y) - f_i, \eta_i(y, x_i)\rangle + J_i^\circ\big(t, x_i, \eta_i(y, x_i)\big)]dt \\
&\ge \limsup_{n\to\infty} \int_0^T [\langle A_i(t, y) - f_i, \eta_i(y, x_i^n)\rangle + J_i^\circ\big(t, x_i^n, \eta_i(y, x_i^n)\big)]dt \\
&\ge \limsup_{n\to\infty} \int_0^T [\langle A_i(t, x_i^n) - f_i, \eta_i(y, x_i^n)\rangle + J_i^\circ\big(t, x_i^n, \eta_i(y, x_i^n)\big)]dt \\
&\ge \limsup_{n\to\infty} \big(- \epsilon_n \|\eta_i(y, x_i^n)\|_{X_i}\big) = 0, \quad \forall y \in X_i.
\end{aligned} \tag{14}$$

If in (14) for each $\lambda \in [0, 1]$ and $z \in X_i$ we take $y = x_i + \lambda\eta_i(z, x_i)$, then

$$\int_0^T [\langle A_i(t, x_i + \lambda\eta_i(z, x_i) - f_i, \lambda\eta_i(z, x_i)\rangle + J_i^\circ(t, x_i; \lambda\eta_i(z, x_i))]dt \ge 0.$$

On the other hand, the positive homogeneousness of $J_i^\circ(t, x_i; .)$ showes that

$$\int_0^T [\langle A_i(t, x_i + \lambda(z - x_i) - f_i, z - x_i\rangle + J_i^\circ(t, x_i; z - x_i)]dt \geq 0. \qquad (15)$$

Now, if $\lambda \to 0^+$, then by ussing hemicontinuity of mapping A_i we obtain

$$\int_0^T [\langle A_i(t, x_i) - f_i, \eta_i(z, x_i)\rangle + J_i^\circ(t, x_i; \eta_i(z, x_i))]dt \geq 0, \quad \forall z \in X_i,$$

and this confirm that $(x_1, \ldots, x_k)$ is a solution for the $(\mathrm{SHVI})(A_i, f_i, J_i)_{i=1,\ldots,k}$. At the end, the assumption $\operatorname{diam} F(\epsilon) \to 0$ as $\epsilon \to 0$, implies that this solution is unique for $(\mathrm{SHVI})(A_i, f_i, J_i)_{i=1,\ldots,k}$, and the proof is completed. □

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