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A GALERKIN-LIKE SCHEME TO DETERMINE CURVES OF CONSTANT BREADTH IN EUCLIDEAN 3-SPACE

Ş. YÜZBAŞI[1], M. KARAÇAYIR[1], §

Abstract. The main focus of this study is to obtain the approximate solutions of a first order linear differential equation system characterizing curves of constant breadth in Euclidean 3-space. For this purpose, we outline a polynomial-based method reminiscent of the Galerkin method. Considering the approximate solutions in the form of polynomials, we obtain some relations, which then give rise to a linear system of algebraic equations. The solution of this system gives the approximate solutions of the problem. Additionally, the technique of residual correction, which aims to reduce the error of the approximate solution by estimating this error, is discussed in some detail. The method and the residual correction technique are illustrated with three examples.

Keywords: Curves of constant breadth, systems of linear differential equations, Galerkin method, residual correction, numerical solutions.

AMS Subject Classification: 34K28.

1. Introduction

Differential equations play a prominent role in many disciplines including physics, engineering, economics and biology. Many problems that have arisen from such studies can be considered as mathematical objects which are of interest on their own. Kolmogorov-Petrovskii-Piskunov equation[1], Korteweg–de Vries equation[2, 3], Toda lattice equation[5], Kaup-Kupershmidt equation[6], Boussinesq equation[7], Fitzhugh-Nagumo equation[8] and poplar biomass production[9] are only a few of such problems. It is generally the case that such problems do not have a known exact solution; therefore approximate solutions are sought for. To name a few of such approximate methods, meshless methods[4], soliton structures[6, 7], He's variational iteration method[10], Exp-function method[11, 12] and Frobenius integrable decompositions[13] have been used extensively by many researchers.

Given a curve in Euclidean 3-space, if another curve is at a constant distance to the first curve at all its points, these two curves are said to be of constant breadth(width). In this paper, starting from a curve in $\mathbb{R}^3$, our aim is to determine a second curve with the property that the two curves are of constant breadth. Investigation of such curve pairs began with the study of Euler[14] in the 18th century. Following this, many researchers studied

[1] Akdeniz University, Faculty of Science, Department of Mathematics, TR 07058 Antalya, Turkey.
e-mail: syuzbasi@akdeniz.edu.tr; ORCID: https://orcid.org/0000-0002-5838-7063.
e-mail: mkaracayir@akdeniz.edu.tr; ORCID: https://orcid.org/0000-0001-6230-3638.

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different aspects of space curves of constant breadth[15, 16, 17, 18, 19]. Furthermore, Köse demonstrated in [20] that given a curve C in the 3-space, a corresponding curve C^* could be determined such that the two curves constitute a curve pair of constant breadth having parallel tangents in opposite directions at all their corresponding points.

In [21], Bishop suggested a new way to describe a curve in 3-space, instead of the one provided by the Frenet frame. Given a curve C parametrized by the real variable s in 3-space, its corresponding curve C^* of constant breadth with respect to the Bishop frame has been shown to satisfy the following first order linear differential equation system[23]:

$$\begin{aligned} \frac{\mathrm{d}\lambda_1}{\mathrm{d}s} &= k_1(s)\lambda_2 + k_2(s)\lambda_3 \\ \frac{\mathrm{d}\lambda_2}{\mathrm{d}s} &= -k_1(s)\lambda_1 \\ \frac{\mathrm{d}\lambda_3}{\mathrm{d}s} &= -k_2(s)\lambda_1, \\ \lambda_1(0) &= \alpha_1,\ \lambda_2(0) = \alpha_2,\ \lambda_3(0) = \alpha_3. \end{aligned} \tag{1}$$

Here, k_1 and k_2 are the Bishop curvatures, which were first defined in [21]. Namely, let $\kappa(s)$ and $\tau(s)$ be the curvature and torsion, respectively, of a space curve given by $\alpha(s)$. Then its Bishop curvatures are defined by $k_1(s) = \kappa(s)\cos(\theta)$ and $k_2(s) = \kappa(s)\sin(\theta)$, where $\theta = \int \tau(s)\mathrm{d}s$. Some other related details on Bishop frame as well as the derivation of the above system can be found in [23]. So far, this system has been approximately solved by collocation methods using Taylor and Lucas polynomials[24].

In this paper, our main interest will be in solving the system (1) using a Galerkin-like scheme. The remaining of the paper is designed as follows: In Section 2, the numerical method to be used is presented. The subject of Section 3 is a technique, called *residual correction*, whose aim is to obtain better solutions using an already obtained solution. In Section 4, application of the method to three different example problems is considered. Finally, Section 5 contains comments regarding the results of this paper.

2. Method of Solution

The aim of this section is to describe the numerical scheme that we will use to obtain approximate solutions of the system (1). A similar scheme has also been used by Türkyılmazoğlu[22] in order to solve high-order Fredholm integro-differential equations.

We will seek solutions to the system (1) in the form of polynomials. More explicitly, we start by assuming

$$\lambda_{1,N}(x) = \sum_{k=0}^{N} v_k x^k,\ \lambda_{2,N}(x) = \sum_{k=0}^{N} y_k x^k,\ \lambda_{3,N}(x) = \sum_{k=0}^{N} z_k x^k$$

are the first, second and third coordinates of the approximate solution of the system (1), where we use x as the independent variable and the letters v, y, z in order to denote the coefficients of the approximate solution polynomials. Our aim is to obtain the unknown coefficients v_k, y_k and z_k and hence the approximate solutions $\lambda_{1,N}, \lambda_{2,N}$ and $\lambda_{3,N}$. Let us note that the above equations can be expressed in terms of matrices by collecting the unknown coefficients and variables inside separate vectors as follows:

$$\lambda_{1,N}(x) = \mathbf{X}_N(x)\mathbf{V},\ \lambda_{2,N}(x) = \mathbf{X}_N(x)\mathbf{Y},\ \lambda_{3,N}(x) = \mathbf{X}_N(x)\mathbf{Z}.$$

Here,

$$\begin{aligned}\mathbf{V} &= [\ v_0 \quad v_1 \quad v_2 \quad \dots \quad v_N\ ]^T,\\ \mathbf{Y} &= [\ y_0 \quad y_1 \quad y_2 \quad \dots \quad y_N\ ]^T,\\ \mathbf{Z} &= [\ z_0 \quad z_1 \quad z_2 \quad \dots \quad z_N\ ]^T\end{aligned}$$

are the vectors consisting of the unknowns and $\mathbf{X}_N(x)$ is an auxiliary vector given by

$$\mathbf{X}_N(x) = [\ 1 \quad x \quad x^2 \quad \dots \quad x^N\ ].$$

The derivatives can also be expressed in terms of matrices with the help of the $(N+1)\times(N+1)$ square matrix $\mathbf{B}$ with entries $\mathbf{B}_{i,i+1} = i$ for $i = 1, 2, \dots, N$ and $\mathbf{B}_{i,j} = 0$ otherwise. More explicitly, if $\mathbf{B}$ is the matrix given by

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & N\\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

then the derivatives in (1) satisfy the following:

$$\frac{\mathrm{d}\lambda_{1,N}}{\mathrm{d}x} = \mathbf{X}_N(x)\mathbf{B}\mathbf{V},\ \frac{\mathrm{d}\lambda_{2,N}}{\mathrm{d}x} = \mathbf{X}_N(x)\mathbf{B}\mathbf{Y},\ \frac{\mathrm{d}\lambda_{3,N}}{\mathrm{d}x} = \mathbf{X}_N(x)\mathbf{B}\mathbf{Z}.$$

After these arrangements, substituting the above matrix equalities into the system (1) gives rise to the following expressions:

$$\begin{aligned}\mathbf{X}_N(x)\mathbf{B}\mathbf{V} - k_1(x)\mathbf{X}_N(x)\mathbf{Y} - k_2(x)\mathbf{X}_N(x)\mathbf{Z} &= 0,\\ \mathbf{X}_N(x)\mathbf{B}\mathbf{Y} + k_1(x)\mathbf{X}_N(x)\mathbf{V} &= 0,\\ \mathbf{X}_N(x)\mathbf{B}\mathbf{Z} + k_2(x)\mathbf{X}_N(x)\mathbf{V} &= 0.\end{aligned} \tag{2}$$

Now, we apply inner product to the above equations with the elements of the set $\mathbf{\Phi} = \{1, x, x^2, \dots, x^N\}$, where the inner product is defined by

$$< f, g >= \int_0^a f(t)g(t)\mathrm{d}t.$$

Here, f and g are functions from the Hilbert space $L^2[0, a]$. In order to express the results of these inner products in a simple matrix form, we give left-hand sides of Equations (2) new names as follows:

$$\begin{aligned}\mathbf{G}^1(x) &= \mathbf{X}_N(x)\mathbf{B}\mathbf{V} - k_1(x)\mathbf{X}_N(x)\mathbf{Y} - k_2(x)\mathbf{X}_N(x)\mathbf{Z},\\ \mathbf{G}^2(x) &= \mathbf{X}_N(x)\mathbf{B}\mathbf{Y} + k_1(x)\mathbf{X}_N(x)\mathbf{V},\\ \mathbf{G}^3(x) &= \mathbf{X}_N(x)\mathbf{B}\mathbf{Z} + k_2(x)\mathbf{X}_N(x)\mathbf{V}.\end{aligned}$$

Now, for $k = 0, 1, \dots, N$, we take inner product of Equations (2) with x^k. Since none of the equations contain a nonhomogeneous term, for $i = 1, 2, 3$ we arrive at the equations

$$< \mathbf{G}^i(x), x^k >= 0.$$

All of these $3N+3$ equations are linear having the unknown coefficients v_k, y_k and $z_k, k = 0, 1, \dots, N$, as their unknowns. Let us number these equations such that the inner product of $\mathbf{G}^i(x)$ and x^k corresponds to equation numbered $iN+k+1$, just for convention. Since we have a linear system of equations, we can express it in terms of matrices, writing

$$\mathbf{W}\mathbf{A} = \mathbf{0}. \tag{3}$$

Here, $\mathbf{W}$ is a $(3N+3)\times(3N+3)$ matrix consisting of the coefficients of the linear equations, $\mathbf{A}$ is just the concatenation of the vectors $\mathbf{V}, \mathbf{Y}, \mathbf{Z}$ of the unknown coefficients given by

$$\mathbf{A} = [\ \mathbf{V}^T \ \mid \ \mathbf{Y}^T \ \mid \ \mathbf{Z}^T \ ]^T$$

and $\mathbf{0}$ is the all-zero column matrix of length $3N+3$. Since it is essential that the initial conditions of system (1) are satisfied, we should include them in the above system. We do this by replacing the equations in the system (3) having resulted from the inner products taken with x^N, which are the equations numbered $N+1, 2N+2$ and $3N+3$, by the equations corresponding to the initial conditions $\lambda_1(0)=\alpha_1, \lambda_2(0)=\alpha_2$ and $\lambda_3(0)=\alpha_3$. These equations are $v_0=\alpha_1, y_0=\alpha_2$ and $z_0=\alpha_3$, respectively. Denoting an entire row by a single subscript, incorporation of these equations corresponds to the following alterations in the matrix $\mathbf{W}$ and the right-hand side of the linear system (denote it by $\mathbf{G}$):

$$\mathbf{W}_{N+1} = [\ 1 \quad \mathbf{0}_{3N+2}\ ],\ \mathbf{W}_{2N+2} = [\ \mathbf{0}_{N+1} \quad 1 \quad \mathbf{0}_{2N+1}\ ],$$
$$\mathbf{W}_{3N+3} = [\ \mathbf{0}_{2N+2} \quad 1 \quad \mathbf{0}_{N}\ ],\ \mathbf{G} = [\ \mathbf{0}_N \quad \alpha_1 \quad \mathbf{0}_N \quad \alpha_2 \quad \mathbf{0}_N \quad \alpha_3\ ]^T.$$

Thus, a new linear system $\tilde{\mathbf{W}}\mathbf{A}=\mathbf{G}$ is formed, the solution of which yields the unknown coefficients v_i, y_i and z_i and hence the approximate solutions $\lambda_{1,N}, \lambda_{2,N}$ and $\lambda_{3,N}$.

3. Error Estimation and Residual Correction

In this section, we outline a method commonly known as *residual correction*, aiming to obtain better solutions using the existing ones. This method is based on the observation that substitution of an approximate solution in the original system results in a new system, similar to the original one in structure, in the error of that particular approximate solution.

We start by assuming $\lambda_{1,N}(x), \lambda_{2,N}(x)$ and $\lambda_{3,N}(x)$ are the approximate solutions of the system (1) for some choice of N. We define the error functions for these solutions by

$$e_{i,N}(x) = \lambda_i(x) - \lambda_{i,N}(x),\ i=1,2,3.$$

Since for $i=1,2,3$, $\lambda_i(x)$ denote the exact solutions, they satisfy the system (1) and so the equalities are preserved upon substituting them in the system. Therefore, we have

$$\begin{aligned}
\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} &= \frac{\mathrm{d}(\lambda_{1,N}+e_{1,N})}{\mathrm{d}s} = k_1(s)(\lambda_{2,N}+e_{2,N}) + k_2(s)(\lambda_{3,N}+e_{3,N})\\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}s} &= \frac{\mathrm{d}(\lambda_{2,N}+e_{2,N})}{\mathrm{d}s} = -k_1(s)(\lambda_{1,N}+e_{1,N})\\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}s} &= \frac{\mathrm{d}(\lambda_{3,N}+e_{3,N})}{\mathrm{d}s} = -k_2(s)(\lambda_{1,N}+e_{1,N}).
\end{aligned}$$

Rearrangement of this system yields

$$\begin{aligned}
\frac{\mathrm{d}e_{1,N}}{\mathrm{d}s} &= k_1(s)e_{2,N} + k_2(s)e_{3,N} - R_{1,N}(s),\\
\frac{\mathrm{d}e_{2,N}}{\mathrm{d}s} &= -k_1(s)e_{1,N} - R_{2,N}(s),\\
\frac{\mathrm{d}e_{3,N}}{\mathrm{d}s} &= -k_2(s)e_{1,N} - R_{3,N}(s),
\end{aligned} \tag{4}$$

where the additional terms $R_{i,N}$ are the residuals of the approximate solutions, given by

$$\begin{aligned}
R_{1,N}(s) &= \frac{\mathrm{d}\lambda_{1,N}}{\mathrm{d}s} - k_1(s)\lambda_{2,N} - k_2(s)\lambda_{3,N},\\
R_{2,N}(s) &= \frac{\mathrm{d}\lambda_{2,N}}{\mathrm{d}s} + k_1(s)\lambda_{1,N},\\
R_{3,N}(s) &= \frac{\mathrm{d}\lambda_{3,N}}{\mathrm{d}s} + k_2(s)\lambda_{1,N}.
\end{aligned}$$

Thus, the system (4), which is a linear first order system in the unknowns $e_{i,N}$, is the same as the original system (1) with the exception that this time the residuals $R_{i,N}, i = 1, 2, 3$ are present as nonhomogeneous terms. In addition, since both the exact and approximate solutions satisfy the initial conditions of the system (1), the initial conditions for the system (4) is given by $e_{i,N}(0) = 0, i = 1, 2, 3$. Therefore, we can use the method explained in Section 2 with a choice of the parameter (denote it by M this time) in order to obtain the approximate solutions $e_{i,N,M}$, $i = 1, 2, 3$, of the new system (4). Since these are the approximate solutions for the error functions, they are called the *error estimates* corresponding to $\lambda_{i,N}, i = 1, 2, 3$. One can use these estimates to obtain the *corrected solutions* of system (1) given by

$$\lambda_{i,N,M}(s) = \lambda_{i,N}(s) + e_{i,N,M}(s),\ i = 1, 2, 3.$$

The accuracy of these corrected solutions is directly related to the accuracy of the error estimates $e_{i,N,M}, i = 1, 2, 3$. In order to measure this accuracy, the straightforward way is to consider the residuals of the corrected solutions $\lambda_{i,N,M}, i = 1, 2, 3$. Another way will be explained in the examples studied in the next section.

4. Numerical Applications

In this section, we consider three example systems and solve them by the methods explained in Sections 2 and 3. All the calculations have been carried out in MATLAB.

Example 1: Let us consider the curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ given by

$$\alpha(s) = \Big(\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \frac{\sqrt{3}s}{2}\Big).$$

This curve has been examined in [25]. In order to compute its Bishop curvatures, let us first determine its Frenet apparatus. The unit tangent vector is found to be

$$\mathbf{T}(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|} = \Big(-\frac{1}{2}\sin\left(\frac{s}{2}\right), \frac{1}{2}\cos\left(\frac{s}{2}\right), \frac{\sqrt{3}}{2}\Big)$$

and the curvature is calculated by $\kappa = \|\mathbf{T}'(s)\| = \frac{1}{4}$. In order to calculate the torsion $\tau(s)$, we obtain the principle normal vector and the unit binormal vector by

$$\mathbf{N}(s) = \frac{1}{\kappa}\mathbf{T}'(s) = \Big(-\cos\left(\frac{s}{2}\right), -\sin\left(\frac{s}{2}\right), 0\Big), \mathbf{B} = \mathbf{T}\times\mathbf{N} = \Big(\frac{\sqrt{3}}{2}\sin\left(\frac{s}{2}\right), -\frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right), \frac{1}{2}\Big).$$

Next, the torsion is calculated from $\mathbf{N}$ and $\mathbf{B}$ by $\tau(s) = -\mathbf{N}(s)\cdot\mathbf{B}'(s) = \frac{\sqrt{3}}{4}$. Lastly, the θ in the definition of Bishop curvatures is equal to $\theta = \int \tau(s)\mathrm{d}s = \frac{\sqrt{3}s}{4}$. Thus, the Bishop curvatures of α are found by $k_1(s) = \frac{1}{4}\cos\left(\frac{\sqrt{3}s}{4}\right)$ and $k_2(s) = \frac{1}{4}\sin\left(\frac{\sqrt{3}s}{4}\right)$. Therefore, in order to constitute a curve pair of constant breadth with α, any curve $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ should satisfy the following system:

$$\begin{aligned}
\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} &= \frac{1}{4}\cos\left(\frac{\sqrt{3}s}{4}\right)\lambda_2 + \frac{1}{4}\sin\left(\frac{\sqrt{3}s}{4}\right)\lambda_3,\\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}s} &= -\frac{1}{4}\cos\left(\frac{\sqrt{3}s}{4}\right)\lambda_1,\\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}s} &= -\frac{1}{4}\sin\left(\frac{\sqrt{3}s}{4}\right)\lambda_1.
\end{aligned} \tag{5}$$

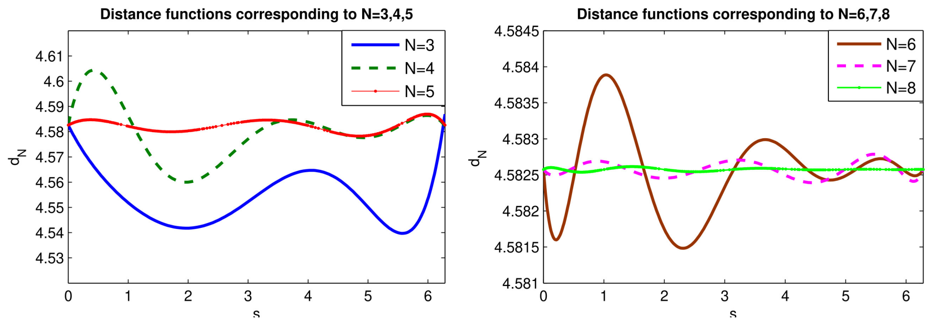

FIGURE 1. Comparison of the absolute distance functions of the approximate solutions of Equation (5) corresponding to $N = 3, 4, 5, 6, 7, 8$.

Let us specify the initial conditions by $\lambda_1(0) = 1, \lambda_2(0) = 4,\ \lambda_3(0) = 2$. Under these conditions, we now obtain several approximate solutions of this system using the method of Section 2. To this end, we choose $N = 3, 4, 5, 6, 7, 8$ as the parameter of the method. For instance, the approximate solution obtained with the choice of $N = 3$ is as follows:

$$\lambda_{1,3}(s) = 1 + 1.1845839076s - 0.0911579966s^2 - 0.0104641343s^3,$$
$$\lambda_{2,3}(s) = 4 - 0.4587131111s - 0.0074404116s^2 + 0.0116612178s^3,$$
$$\lambda_{3,3}(s) = 2 + 0.2161369609s - 0.2910842027s^2 + 0.0268503546s^3.$$

We have also obtained the approximate solutions corresponding to the other N values. In order to assess their accuracy, one way is to consider their residuals as expressed at the end of Section 3. On the other hand, in view of the particular geometric meaning of the problem, one may evaluate the results based on their closeness to constituting a curve pair of constant breadth with the given curve α. More explicitly, we will consider the distance functions d_N corresponding to the approximate solutions $(\lambda_N)_{app} = (\lambda_{1,N}, \lambda_{2,N}, \lambda_{3,N})$ as a criterion for their accuracy. Since these solutions are exact for $s = 0$, the distance of the two curves should be equal to $d(0) = (\lambda_1(0)^2 + \lambda_2(0)^2 + \lambda_3(0)^2)^{1/2} = \sqrt{21} \approx 4.582575695$ for all $s \in [0, 2\pi]$. Therefore, the closer the distance function

$$d_N(s) = (\lambda_{1,N}(s)^2 + \lambda_{2,N}(s)^2 + \lambda_{3,N}(s)^2)^{1/2}$$

is to the constant function $d(s) = \sqrt{21}$, as the more accurate we will accept the approximate solution $(\lambda_N)_{app}$ corresponding to the parameter N.

Figure 1 depicts the distance functions corresponding to the approximate solutions obtained with several N values. The distance functions are seen to approach the constant distance function d as we increase the parameter N. This indicates that our solution method yields more accurate results as we choose bigger values for N. An advantage of this approach is that it enables us to estimate the accuracy of an approximate solution $(\lambda_N)_{app}$ without considering the behaviours of the coordinates $\lambda_{1,N}, \lambda_{2,N}$ and $\lambda_{3,N}$ separately.

Now, we test the usefullness of the technique of residual correction on the approximate solution $(\lambda_3)_{app}$ we have obtained with the choice of $N = 3$. For this purpose we choose $M = 4, 5, 8$ and proceed as explained in Section 3. This gives us the error estimations $e_{3,M}(s) = (e_{1,3,M}(s), e_{2,3,M}(s), e_{3,3,M}(s))$ for these three M values. For instance, the coordinate functions of the error estimation $e_{3,4}$ corresponding to the pair $(N, M) = (3, 4)$ are calculated as

$$e_{1,3,4}(s) = -0.1989460638s + 0.2030087698s^2 - 0.0544947788s^3 + 0.0043230642s^4,$$
$$e_{2,3,4}(s) = 0.3023308928s - 0.2819676161s^2 + 0.0741690112s^3 - 0.0058878547s^4,$$
$$e_{3,3,4}(s) = -0.1271272242s + 0.1246976273s^2 - 0.0346708008s^3 + 0.0028490270s^4.$$

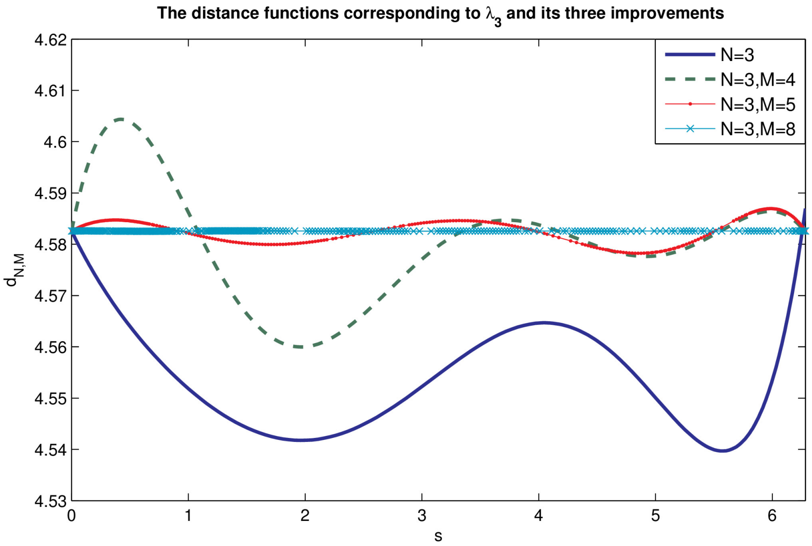

FIGURE 2. Comparison of the distance functions corresponding to the approximate solution $(\lambda_3)_{app}$ and its three corrected versions in Example 1.

TABLE 1. Values of the distance functions obtained by Lucas collocation method (first 3 columns) and the present method (last 3 columns) for some values of s in Example 1.

s	$d_{3,4}(s)$ (LCM)	$d_{3,6}(s)$ (LCM)	$d_{3,8}(s)$ (LCM)	$d_{3,4}(s)$ (PM)	$d_{3,6}(s)$ (PM)	$d_{3,8}(s)$ (PM)
0	4.582575	4.582575	4.582575	4.582575	4.582575	4.582575
$\pi/4$	4.562648	4.585027	4.582442	4.595669	4.583544	4.582547
$\pi/2$	4.550048	4.584816	4.582469	4.564586	4.582842	4.582613
$3\pi/4$	4.557248	4.584571	4.582459	4.563658	4.581488	4.582547
π	4.563080	4.584897	4.582465	4.579692	4.582566	4.582576
$5\pi/4$	4.558316	4.584602	4.582459	4.584196	4.582904	4.582585
$3\pi/2$	4.553611	4.584795	4.582468	4.578165	4.582430	4.582570
$7\pi/4$	4.561324	4.584935	4.582444	4.581842	4.582717	4.582578
2π	4.608106	4.579553	4.582891	4.582254	4.582575	4.582576

The corrected solution $(\lambda_{3,4})_{app}$ is obtained by $\lambda_{i,3,4}(s) = (\lambda_{i,3})_{app}(s) + e_{i,3,4}(s)$ for $i = 1, 2, 3$. The cases $M = 5$ and $M = 8$ are similar. As for the accuracy of these corrected solutions, we again consider their distance functions $d_{N,M}$. These distance functions are demonstrated in Figure 2. Note that the graph of the distance function $d_{3,8}$ corresponding to the corrected solution $(\lambda_{3,8})_{app}$ is almost indistinguishable from the constant function d. Thus, the figure reveals that not only the corrected solutions $(\lambda_{3,M})_{app}$ are more accurate than the original approximate solution $(\lambda_3)_{app}$, but also that increasing M value for a fixed N results in more and more accurate results. We are thus led to comment that residual correction provides a significant improvement over the original approximate solutions.

We also compare the accuracy of the present method with that of Lucas collocation method[25]. In Table 1, distance functions corresponding to three different solutions obtained by Lucas collocation method (LCM) are given together with those corresponding to the solutions obtained by the present method (PM) using the same parameter values. Looking at the values in the table, we can conclude that the present method is more accurate than Lucas collocation method for this example problem.

Example 2: As a second example, let us consider the curve $\alpha : [0, \pi/2] \to \mathbb{R}^3$ given by

$$\alpha(t) = (\cos^3 t, \sin^3 t, \cos(2t)).$$

The curve is not of unit speed, so we first parametrize it with respect to arc length. Since

$$\alpha'(t) = \left(-3\sin t\cos^2 t, 3\sin^2 t\cos t, -4\sin t\cos t\right),$$

we have

$$\begin{aligned}\|\alpha'(t)\| &= \sqrt{(-3\sin t\cos^2 t)^2 + (3\sin^2 t\cos t)^2 + (-4\sin t\cos t)^2}\\ &= \sqrt{9\sin^2 t\cos^4 t + 9\sin^4 t\cos^2 t + 16\sin^2 t\cos^2 t}\\ &= \sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t) + 16\sin^2 t\cos^2 t} = \sqrt{25\sin^2 t\cos^2 t} = 5\cos t\sin t.\end{aligned}$$

Consequently, the arch length s starting from $t = 0$ is calculated by $s = \int_0^t 5\sin u\cos u\, du = \frac{5}{2}\sin^2 t$. From this we can see that $\sin t = \sqrt{\frac{2s}{5}}$ and $\cos t = \sqrt{1 - \frac{2s}{5}}$, giving the arc-length parametrization of the curve by

$$\alpha(s) = \left(\left(1 - \frac{2s}{5}\right)^{3/2}, \left(\frac{2s}{5}\right)^{3/2}, 1 - \frac{4s}{5}\right),$$

where $s \in [0, 5/2]$. Next, we obtain the unit tangent vector as

$$\mathbf{T}(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|} = \left(-\frac{3}{5}\sqrt{1 - \frac{2s}{5}}, \frac{3}{5}\sqrt{\frac{2s}{5}}, -\frac{4}{5}\right).$$

Using this, we can compute the Gaussian curvature by

$$\kappa = \|\mathbf{T}'(s)\| = \frac{3}{5\sqrt{10s - 4s^2}}.$$

Therefore, the principal normal vector and the unit binormal vector can be calculated as

$$\mathbf{N}(s) = \frac{1}{\kappa}\mathbf{T}'(s) = \frac{5\sqrt{2}}{3}\sqrt{5s - 2s^2}\left(\frac{3}{25\sqrt{1 - \frac{2s}{5}}}, \frac{3}{25\sqrt{\frac{2s}{5}}}, 0\right) = \frac{\sqrt{10}}{5}\left(\sqrt{s}, \sqrt{\frac{5}{2} - s}, 0\right),$$

$$\mathbf{B}(s) = \mathbf{T} \times \mathbf{N} = \left(-\frac{4\sqrt{10}}{25}\sqrt{s}, \frac{4\sqrt{10}}{25}\sqrt{\frac{5}{2} - s}, -\frac{3}{5}\right).$$

Then, the torsion is calculated as $\tau(s) = -\mathbf{N}(s)\cdot \mathbf{B}'(s) = \frac{8}{25}$. From this we can calculate θ by $\theta = \int \tau(s)\mathrm{d}s = \frac{8s}{25}$. Thus, the Bishop curvatures of α are obtained as

$$k_1(s) = \kappa\cos(\theta) = \frac{3}{5\sqrt{10s - 4s^2}}\cos\left(\frac{8s}{25}\right),\ k_2(s) = \kappa\sin(\theta) = \frac{3}{5\sqrt{10s - 4s^2}}\sin\left(\frac{8s}{25}\right).$$

Finally, any curve $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ having constant breadth with respect to α should satisfy the following system:

$$\begin{aligned}\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} &= \frac{3}{5\sqrt{10s - 4s^2}}\cos\left(\frac{8s}{25}\right)\lambda_2 + \frac{3}{5\sqrt{10s - 4s^2}}\sin\left(\frac{8s}{25}\right)\lambda_3,\\ \frac{\mathrm{d}\lambda_2}{\mathrm{d}s} &= -\frac{3}{5\sqrt{10s - 4s^2}}\cos\left(\frac{8s}{25}\right)\lambda_1,\\ \frac{\mathrm{d}\lambda_3}{\mathrm{d}s} &= -\frac{3}{5\sqrt{10s - 4s^2}}\sin\left(\frac{8s}{25}\right)\lambda_1.\end{aligned} \tag{6}$$

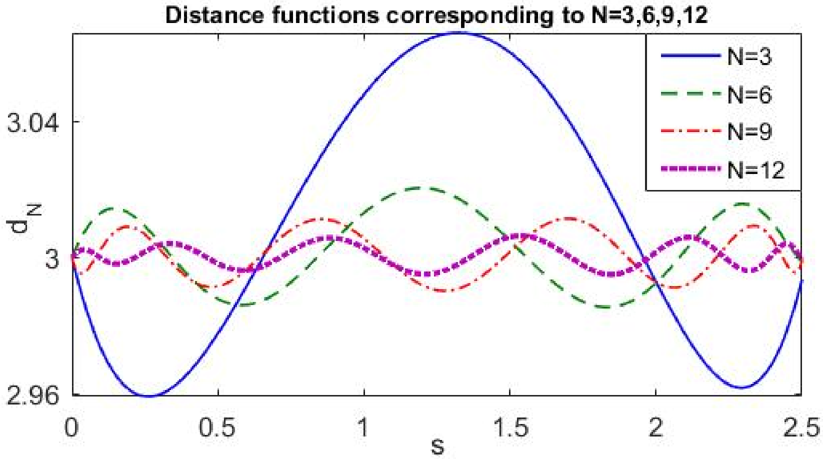

FIGURE 3. Comparison of the distance functions corresponding to the approximate solutions $(\lambda_N)_{app}$ for $N = 3, 6, 9$ and 12 in Example 2.

TABLE 2. Values of the distance functions d_N obtained using $N = 3, 5, 6, 8$ and 12 for some values of s in Example 2.

s	$d_3(s)$	$d_5(s)$	$d_6(s)$	$d_8(s)$	$d_{12}(s)$
0	3	3	3	3	3
0.5	2.977530	3.025712	2.987742	2.999982	2.998146
1	3.047911	2.993232	3.013119	2.998862	3.002700
1.5	3.060202	2.985008	3.004817	3.004493	3.005936
2	2.992693	3.027769	2.992463	2.995213	3.001497
2.5	2.993540	2.997723	2.998412	2.999114	2.999614

As the initial condition, let us take $\lambda_1(0) = 1, \lambda_2(0) = 2, \lambda_3(0) = 2$. As in the first example, we have applied the present method to the system (6) using several N values. For instance, the approximate solution corresponding to $N = 3$ is given by

$$\lambda_{1,3}(s) = 1 + 1.5415112236s - 0.9084444387s^2 + 0.2255402992s^3,$$
$$\lambda_{2,3}(s) = 2 - 1.0532217001s + 0.6886836807s^2 - 0.2049326302s^3,$$
$$\lambda_{3,3}(s) = 2 - 0.2546240471s + 0.3521886824s^2 - 0.1510475994s^3.$$

In order to judge the accuracy of the approximate solutions, let us again consider their distance functions d_N. Since every approximate solution satisfies the initial condition, $d_N(0)$ should be equal to $\sqrt{1^2 + 2^2 + 2^2} = 3$, which means that d_N should be judged based on its closeness to the constant function $y = 3$. Looking at Figure 3, we can see that the graph of the distance function becomes considerably flatter as the parameter N increases. This fact can also be observed from Table 2, where we listed the values of the distance function corresponding to five different N values. Thus we can conclude that increasing N makes the approximate solutions more accurate in this example as well.

Example 3: Lastly, let us consider an example from [24]. Let the curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ be given by

$$\alpha(s) = \left(3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\right).$$

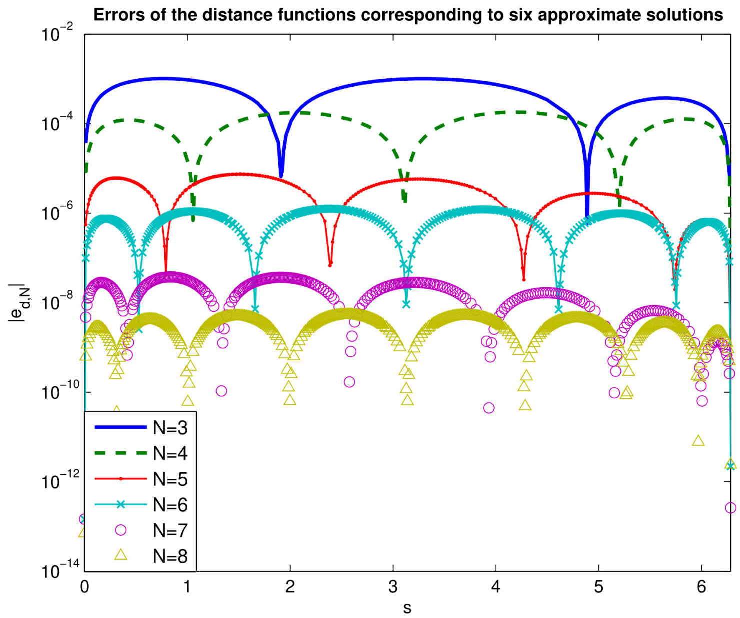

FIGURE 4. Absolute errors of the distance functions corresponding to the approximate solutions $(\lambda_N)_{app}$ for $N = 3, 4, 5, 6, 7, 8$ in Example 3.

The Bishop curvatures are calculated to be $k_1(s) = \frac{3}{25}\cos\left(\frac{4s}{25}\right)$ and $k_2(s) = \frac{3}{25}\sin\left(\frac{4s}{25}\right)$ [24]. Thus, the system characterizing curves of constant breadth with respect to α is given by

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_1}{\mathrm{d}s} &= \frac{3}{25}\cos\left(\frac{4s}{25}\right)\lambda_2 + \frac{3}{25}\sin\left(\frac{4s}{25}\right)\lambda_3, \\
\frac{\mathrm{d}\lambda_2}{\mathrm{d}s} &= -\frac{3}{25}\cos\left(\frac{4s}{25}\right)\lambda_1, \\
\frac{\mathrm{d}\lambda_3}{\mathrm{d}s} &= -\frac{3}{25}\sin\left(\frac{4s}{25}\right)\lambda_1,
\end{aligned}
\tag{7}
$$

where the initial conditions are $\lambda_1(0) = 2, \lambda_2(0) = 1, \lambda_3(0) = 3$. Let us now solve (7) with the same values $N = 3, 4, 5, 6, 7, 8$ as in Example 1. For $N = 3$ we have

$$
\begin{aligned}
\lambda_{1,3}(s) &= 2 + 0.1192403053s + 0.0153936713s^2 - 0.0011643630s^3, \\
\lambda_{2,3}(s) &= 1 - 0.2361209993s - 0.0109991919s^2 + 0.0014691750s^3, \\
\lambda_{3,3}(s) &= 3 + 0.0029397450s - 0.0215332085s^2 - 0.0004009941s^3.
\end{aligned}
$$

The approximate solutions corresponding to the other N values can be obtained similarly. The distance between α and the exact solution λ should be equal to $d = \sqrt{14} \approx 3.7416573867$. In order to evaluate the accuracy of the approximate solutions, let us consider the graph of the absolute errors of their corresponding distance functions instead of the distance functions themselves. More explicitly, this time we consider the graphs of

$$|e_{d,N}(s)| = |d_N(s) - \sqrt{14}|.$$

Figure 4 makes it clear that the absolute errors of the distance functions become closer to zero with increasing N values. The improvement provided by each increment of N is understood clearly looking at the graph, even on a logaritmic scale. Thus, the approximate solutions become more accurate as we increase the value of N for this example problem.

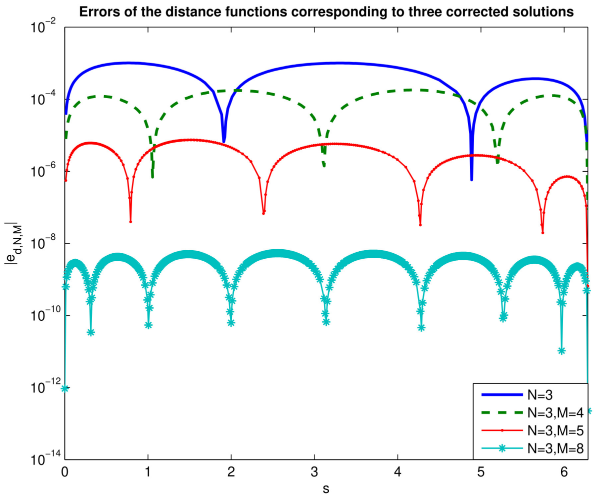

FIGURE 5. Absolute errors of the distance functions d_3 and $d_{3,M}$ corresponding to the approximate solution $(\lambda_3)_{app}$ and three of its corrected versions in Example 3.

TABLE 3. Comparison of the distance functions $d_{N,M}$ obtained by Lucas collocation method and the present method for some values of s in Example 3.

s	$d_{3,4}(s)$ (LCM)	$d_{3,5}(s)$ (LCM)	$d_{3,8}(s)$ (LCM)	$d_{3,4}(s)$ (PM)	$d_{3,5}(s)$ (PM)	$d_{3,8}(s)$ (PM)
0	3.74165739	3.74165739	3.74165739	3.74165739	3.74165739	3.74165739
$\pi/3$	3.74182211	3.74166224	3.74165740	3.74165432	3.74166156	3.74165739
$2\pi/3$	3.74182946	3.74166111	3.74165740	3.74183216	3.74166091	3.74165739
π	3.74174213	3.74166135	3.74165740	3.74164950	3.74165170	3.74165739
$4\pi/3$	3.74187284	3.74166161	3.74165740	3.74147654	3.74165679	3.74165739
$5\pi/3$	3.74179308	3.74166064	3.74165740	3.74166836	3.74165964	3.74165739
2π	3.73954950	3.74165306	3.74165710	3.74165730	3.74165739	3.74165739

As for residual correction, let us improve the approximate solution $(\lambda_3)_{app}$ using the parameters $M = 4, 5, 8$. In order to compare the accuracy of these corrected solutions $(\lambda_{3,M})_{app}$ with that of $(\lambda_3)_{app}$ for $M = 4, 5, 8$, we consider the absolute errors of the distance functions, which are given by $|e_{d,N,M}(s)| = |d_{N,M}(s) - \sqrt{14}|$. These absolute error functions are illustrated in Figure 5. The figure shows that applying residual correction with greater M values yields distance functions with smaller errors, hence more accurate approximate solutions. We also compare the corrected solutions with those obtained by Lucas collocation method[24] in terms of their distance functions in Table 3. The values reveal that the two methods are almost equally accurate for this problem.

5. CONCLUSIONS

In this paper, we outlined a numerical method to solve a first order linear model characterizing curves of constant breadth according to Bishop Frame in Euclidean 3-space. The scheme relies on transforming the given problem to a system of linear equations, whose solution yields three polynomials of degree N as the approximate solutions. We also explained a technique known as residual correction, aiming to obtain better solutions from the already obtained solutions by means of estimating their error. We then applied the method to three different example problems. The numerical results revealed that increasing the parameter N improves the accuracy of the approximate solutions. In addition, it turned out that the accuracy of the approximate solution can be considerably improved by applying residual correction. On the whole, the presented scheme is an easy-to-implement method that can be used to solve models of similar type with a remarkably good accuracy.

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Şuayip Yüzbaşı is an associate professor in the Department of Mathematics, Faculty of Sciences, Akdeniz University, Antalya, Turkey. His research interests are numerical and analytical methods for ordinary, functional and partial differential, integral, integro-differential and fractional differential equations and mathematical modelling. He has over 100 publications in high-impact international journals, over 60 papers in national and international conferences and 6 national research projects. Besides being the editor-in-chief of the journal "Research and Communications in Mathematics and Mathematical Sciences", he is a member of editor board in 11 international journals.

Murat Karaçayır received his PhD Degree in Mathematics in 2018 from Akdeniz University, Antalya, Turkey and he is currently a research assistant in the Department of Mathematics. His research interests focus on numerical solutions of differential and delay-differential equations. He has co-authored numerous research papers in high-impact international journals.