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A NOVEL FINITE DIFFERENCE SCHEME FOR TIME FRACTIONAL DIFFUSION-WAVE EQUATION WITH SINGULAR KERNEL

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Abstract. In this paper, we suggest a novel numerical approximation of the Caputo-Fabrizio fractional derivative of order α $(1 < \alpha < 2)$. Our novel discretization is found by using discret fractional derivative at $t = t_k$ with new coefficients $e^{-\frac{(\alpha-1)(t_k - t_{m+\frac{1}{2}})}{2-\alpha}} - e^{-\frac{(\alpha-1)(t_k - t_{m-\frac{1}{2}})}{2-\alpha}}$. Also, we prove that the difference scheme is unconditionally stable.

Keywords: Caputo-Fabrizio, diffusion-wave equation, novel finite difference, stability.

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1. Introduction

In recent years, fractional differential equations have interested in real-life phenomena. It describes diverse phenomena in the sciences and engineering fields. They appear naturally in viscoelasticity, porous media, chemistry, electromagnetism physics[6], mechanics and biology[3]. So more applications have been found. The solution of non -integer order partial differential equations (PDE) has important property, it describes future and present states. But in many cases, it is difficult to find the solution. Therefore, several researchers have suggested numerical methods for studying PDE with fractional order: finite element methods[5, 7], mixed finite element methods[9, 10], finite difference methods[13, 14], finite volume methods[4]. In 2015, Caputo and Fabrizio[1] proposed a new derivative. This derivative is a product of convolution of $f'(t)$ (derivative of function $f(t)$) and exponential function $(e^{\frac{-\alpha}{1-\alpha}t})$ where $0 < \alpha < 1$.

The fractional diffusion -wave equation plays an important role to modeling the diffusion and wave in fluid flow, oil strata..ect. In recent years, many eminent researchers innovated some numerical methods to study this kind of equations. In 2005, Sun and Wu[15] showed a novel finite difference discret scheme for a diffusion-wave system. They proved the stability and L_∞ convergence by using the energy method.

Our target is to extend diffusion-wave equation to the scope of fractional calculus using

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Caputo-Fabrizio derivative with fractional order and we give a novel discretization for this new equation.

The paper is organized as follows. In section 2, we recall some definitions of fractional calculus. Section 3 is concerned to study the existence and uniqueness of solution. section 4, we give the novel finite difference discretization scheme. Section 5, we discuss the stability of the fractional numerical scheme and we give some numerical examples.

2. Preliminary definitions

In this section, we present certain relevant definitions of fractional derivatives and anti-derivatives. For more details, we refer to [1, 2, 11].

Definition 2.1. *Let $f \in L^1(0,\infty)$, and $\alpha \in (0,1)$ then, the Caputo derivative is defined as*

$$\boldsymbol{D}^{\alpha}_{0|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial f(\tau)}{\partial \tau} d\tau, \tag{1}$$

where Γ is the gamma function.

The anti- derivative of Caputo derivative is given in the following definition.

Definition 2.2. *Consider a function $f : [0,\infty) \to \mathbb{R}$. The Riemann-Liouville fractional integral is defined by*

$$^{C}I^{\alpha}_{0|t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{2}$$

for $t > 0$ and $0 < \alpha < 1$.

We recall the new definition of the Caputo fractional derivative.

Definition 2.3. *Let $f \in H^1(0,\infty)$ and $\alpha \in (0,1)$ then, the definition of the new Caputo derivative (Caputo -Fabrizio) is given as*

$$\mathbb{D}^{\alpha}_{0|t} f(t) = \frac{1}{1-\alpha} \int_0^t \mathrm{e}^{-\frac{\alpha(t-\tau)}{1-\alpha}} \frac{\partial f(\tau)}{\partial \tau} d\tau. \tag{3}$$

The anti- derivative of the Caputo- Fabrizio derivative is recalled as

Definition 2.4. *Let $0 < \alpha < 1$. The fractional integral of a function f is given as*

$$^{CF}I^{\alpha}_{0|t} f(t) = (1-\alpha) f(t) + \alpha \int_0^t f(\tau) d\tau. \tag{4}$$

Lemma 2.1 ([1]). *Let $0 < \alpha < 1$, then*

(1) $^{CF}I^{\alpha}_{0|t} \mathbb{D}^{\alpha}_{0|t} f(t) = f(t) - f(0)$,
(2) $\mathbb{D}^{\alpha}_{0|t} D^n f(t) = \mathbb{D}^{\alpha+n}_{0|t} f(t)$.

3. Diffusion wave equation with Caputo-Fabrizio fractional derivative

In this section, we apply Picard-Lindelof method to prove the existence and the uniqueness of the solution.

We consider the following time-fractional diffusion-wave equation

$$\mathbb{D}^{\alpha}_{0|t} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t). \tag{5}$$

Over region $\Omega = [0, L] \times [0, T]$, $1 < \alpha < 2$ with the initial conditions

$$u(x,0) = f(x), \quad D_t u(x,t)\Big|_{t=0} = 0, \tag{6}$$

and homogeneous boundary conditions

$$u(0,t) = u(L,t) = 0. \tag{7}$$

Obviously, the Caputo-Fabrizio operator $\mathbb{D}^{\alpha}_{0|t}$ is the composition of $\mathbb{D}^{\alpha-1}_{0|t}$ and D_t, i.e.

$$\mathbb{D}^{\alpha}_{0|t} u(x,t) = \mathbb{D}^{\alpha-1}_{0|t} D_t u(x,t).$$

Setting $v = D_t u$, we have the following formulation

$$\begin{cases} \mathbb{D}^{\alpha-1}_{0|t} v(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t), \\ v(x,t) = D_t u(x,t), \quad v(x,0) = 0. \end{cases} \tag{8}$$

By applying the anti- derivative operator ${}^{CF}I^{\alpha}_{0|t}$ on the both side of Eq.(8), we get

$$v(x,t) = (2-\alpha) \times \left\{ \frac{\partial^2 u(x,t)}{\partial x^2} + q(x,t) \right\} + (\alpha - 1) \times \int_0^t \left\{ \frac{\partial^2 u(x,y)}{\partial x^2} + q(x,y) \right\} dy. \tag{9}$$

For simplicity, let us put

$$u(x,t) = F(t).$$

Then, equation (9) can be re-write as

$$v(x,t) = (2-\alpha) \times \left\{ \frac{\partial^2 F(t)}{\partial x^2} + q(x,t) \right\} + (\alpha - 1) \times \int_0^t \left\{ \frac{\partial^2 F(y)}{\partial x^2} + q(x,y) \right\} dy. \tag{10}$$

For more simplicity, we defined the operator H as following

$$H(F,t) = \frac{\partial^2 F(t)}{\partial x^2} + q(x,t).$$

Let

$$\begin{aligned} &C[c,v] = [t_0 - c, t_0 + c] \times [F_0 - v, F_0 + v], \quad L = \sup \|H(F,t)\|_{C[c,v]} \\ &\|F(t)\|_\infty = \sup_{t \in [t_0 - c, t_0 + c]} |F(t)|. \end{aligned} \tag{11}$$

We define the Picard's operator $P : C[c,v] \to C[c,v]$ as

$$P(D_t F, t) = (2-\alpha) H(F,t) + (\alpha - 1) \int_0^t H(F,y) dy.$$

First, we prove P is well posed. By using (11) we have

$$\begin{aligned} \|P(D_t F(t))\| &\leqslant (2-\alpha)\|H(F,t)\| + (\alpha-1) \int_0^t \|H(F,y)\| dy \\ &\leqslant (2-\alpha)L + (\alpha-1)cL. \end{aligned}$$

We choose c small enough such that

$$(2-\alpha)L + (\alpha-1)cL \leqslant L.$$

Second, we show that P is a contraction map. For $D_tF, D_tG \in C[c,v]$, we have

$$\begin{aligned}\big\|P(D_tF(t)) - P(D_tG(t))\big\| &= \left\|(2-\alpha)\{H(F,t)-H(G,t)\} + (\alpha-1)\int_0^t \{H(F,y)-H(G,y)\}dy\right\| \\ &\leqslant (2-\alpha)\left\|H(F,t)-H(G,t)\| + (\alpha-1)\int_0^t \|H(F,y)-H(G,y)\right\|dy \\ &\leqslant M\big\{(2-\alpha)+(\alpha-1)c\big\}\|F-G\|.\end{aligned}$$

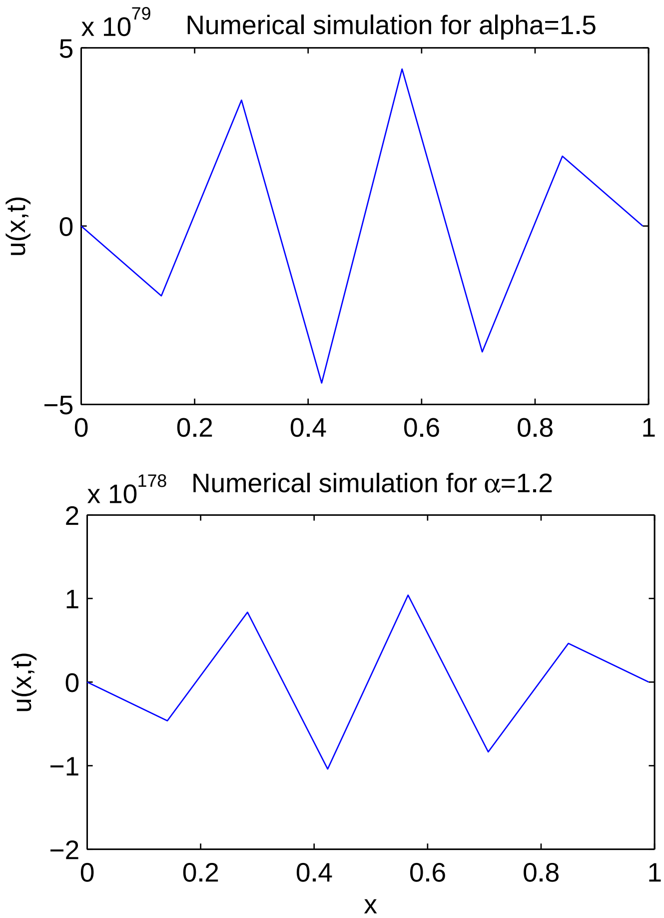
Due to the following inequality

$$\|H(F,t)-H(G,t)\| \leqslant M\|F-G\|.$$

We choose c such that

$$M\big\{(2-\alpha)+(\alpha-1)c\big\} < 1.$$

Therefore, P is a strict contraction on $C[c,v]$. According to the Banach fixed point theorem, then problem $(5)-(7)$ admits a unique solution.



4. A novel finite difference scheme

In this section, we investigate the approximate numerical solution of problem (5), using implicit finite differences. To achieve this aim, we need to numerically approximate to the Caputo-Fabrizio derivative.

For some positive integers N, M, the gird sizes in time for finite difference technique is defined by $K = \frac{1}{M}$, the grid points in the time interval $[0,T]$ are labeled $t_j = jK, j =$

$0...TM$, while the grid points in the space interval $[0,L]$ are numbers $x_i = ih$ where $h = \frac{1}{N}$ it is grid sizes in the space. Denotes u_i^j the approximate value of $u(x_i, t_j)$ and f^i is the value of $f(x_i)$. Define

$$\delta_x u_{i-\frac{1}{2}} = \frac{u_i - u_{i-1}}{h}, \quad \delta_x^2 u_i = \frac{\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}}{h} \quad \text{and} \quad \delta_t u^n = \frac{u^n - u^{n-1}}{K}.$$

The standard central difference scheme

$$v_i^{k+\frac{1}{2}} = \frac{u_i^{k+1} - u_i^k}{K} + O(K^2). \tag{12}$$

The approximate numerical of Caputo-Fabrizio derivative $\mathbb{D}_{0|t}^{\alpha-1} v(x,t)$ obtained by the following formula

$$\begin{aligned}
\mathbb{D}_{0|t}^{\alpha-1} v(x_i, t_k) &= \frac{1}{2-\alpha} \int_0^{t_k} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \frac{\partial v(x_i,\tau)}{\partial \tau} d\tau \\
&= \frac{1}{2-\alpha} \left[\int_{t_{k-\frac{1}{2}}}^{t_k} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \frac{\partial v(x_i,\tau)}{\partial \tau} d\tau + \int_{t_{\frac{1}{2}}}^{t_{k-\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \frac{\partial v(x_i,\tau)}{\partial \tau} d\tau \right. \\
&\left. + \int_0^{t_{\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \frac{\partial v(x_i,\tau)}{\partial \tau} d\tau \right] \\
&= \frac{1}{2-\alpha} \left[\int_{t_{k-\frac{1}{2}}}^{t_k} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \frac{\partial v(x_i,\tau)}{\partial \tau} d\tau + \sum_{m=0}^{k-1} \int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \frac{\partial v(x_i,\tau)}{\partial \tau} d\tau \right. \\
&\left. - \int_{t_{\frac{-1}{2}}}^{0} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \frac{\partial v(x_i,\tau)}{\partial \tau} d\tau \right] \\
&= \frac{1}{2-\alpha} \left[\int_{t_{k-\frac{1}{2}}}^{t_k} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \left[\frac{v_i^{k-\frac{1}{2}} - v^{k-\frac{3}{2}}}{K} + O(K) \right] d\tau \right. \\
&\left. - \int_{t_{\frac{-1}{2}}}^{0} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \left[\frac{v_i^0 - v_i^{-\frac{1}{2}}}{K} + O(K) \right] d\tau \right] \\
&+ \frac{1}{2-\alpha} \sum_{m=0}^{k-1} \int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}} \left[\frac{v_i^{m+\frac{1}{2}} - v_i^{m-\frac{1}{2}}}{K} + \left(\frac{\partial v(x_i,\tau)}{\partial \tau} - \frac{v_i^{m+\frac{1}{2}} - v_i^{m-\frac{1}{2}}}{K} \right) d\tau \right].
\end{aligned} \tag{13}$$

Denote that $u_i^{-1} = u_i^0 - K v_i^0$ for $i \geqslant 0$. Then

$$v_i^{\frac{-1}{2}} = \frac{u_i^0 - u_i^{-1}}{K} + O(K^2) = v_i^0 + O(K^2). \tag{14}$$

Substituting (12) and (14) into (13), we get

$$\mathbb{D}^{\alpha}_{0|t}u(x_i,t_k) = \frac{1}{2-\alpha}\left[\frac{u_i^k - 2u_i^{k-1} + u_i^{k-2}}{K^2}\right]\int_{t_{k-\frac{1}{2}}}^{t_k} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}d\tau - \frac{1}{2-\alpha}\int_{t_{\frac{-1}{2}}}^{0} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}O(K)d\tau$$
$$+\frac{1}{2-\alpha}\sum_{m=0}^{k-1}\left(\frac{u_i^{m+1} - 2u_i^m + u^{m-1}}{K^2}\right)\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}d\tau$$
$$+\frac{1}{2-\alpha}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\left(\frac{\partial v(x_i,\tau)}{\partial\tau} - \frac{v_i^{m+\frac{1}{2}} - v_i^{m-\frac{1}{2}}}{K}\right)d\tau. \tag{15}$$

Setting

$$\varsigma_K = \frac{1}{(\alpha-1)K^2},\quad w_{k,\alpha} = \left(1 - \mathrm{e}^{-\frac{(\alpha-1)(t_k - t_{k-\frac{1}{2}})}{2-\alpha}}\right),\quad d_{m,\alpha} = \left(\mathrm{e}^{-\frac{(\alpha-1)(t_k - t_{m+\frac{1}{2}})}{2-\alpha}} - \mathrm{e}^{-\frac{(\alpha-1)(t_k - t_{m-\frac{1}{2}})}{2-\alpha}}\right),$$

$$R = \frac{1}{2-\alpha}\sum_{m=0}^{k-1}\int_{t_{m-\frac{1}{2}}}^{t_{m+\frac{1}{2}}} \mathrm{e}^{-\frac{(\alpha-1)(t_k-\tau)}{2-\alpha}}\left(\frac{\partial v(x_i,\tau)}{\partial\tau} - \frac{v_i^{m+\frac{1}{2}} - v_i^{m-\frac{1}{2}}}{K}\right)d\tau.$$

Therefore

$$\mathbb{D}^{\alpha}_{0|t}u(x_i,t_k) = \varsigma_K\left(u_i^k - 2u_i^{k-1} + u_i^{k-2}\right)w_{k,\alpha} + \varsigma_K\sum_{m=0}^{k-1}\left(u_i^{m+1} - 2u_i^m + u^{m-1}\right)d_{m,\alpha}$$
$$+ O(K^{3-\alpha}) + R. \tag{16}$$

Also, the second partial derivative with respect to x at the grid point (i,k) given as

$$\frac{\partial^2 u(x_i,t_k)}{\partial t^2} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} + O(h^2). \tag{17}$$

Using (16) and (17) to discretize problem (5) at point (x_i,t_k) as

$$\varsigma_K\left(u_i^k - 2u_i^{k-1} + u_i^{k-2}\right)w_{k,\alpha} + \varsigma_K\sum_{m=0}^{k-1}\left(u_i^{m+1} - 2u_i^m + u_i^{m-1}\right)d_{m,\alpha} + R$$
$$= \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} + q_i^k + O(K^{3-\alpha} + h^2). \tag{18}$$

The first initial condition, can be written as

$$u(x_i,0) = f(x_i) = f_i \quad i = 0...N. \tag{19}$$

Approximating the second initial condition, we obtain

$$D_t u(x_i,t_k)\Big|_{t_0=0} \simeq \frac{u_i^0 - u_i^{-1}}{K} = 0, \qquad i = 0...N. \tag{20}$$

Similarly method used in [8]. We denote $|R| = O(K^{3-\alpha})$.

5. Stability analysis

In this section, we etablish the stability of the numerical method by using Fourier method.

Let $\beta_k^i = u_k^i - U_k^i$ where U_k^i is the approximate of u_k^i. Assume that the β_k^i is written as follows

$$\beta_k^i = \xi_k \mathrm{e}^{qiwh}, \tag{21}$$

where $\xi_k = |\beta_k^i|$, w real number and $q = \sqrt{-1}$. We will prove the following result.

Theorem 5.1. *Let* $1 < \alpha < 2$*, the numerical method described in* (18) *to address the solvability of problem* $(5) - (6)$*, is unconditionally stable.*

Proof. Substituting (21) into (18), we obtain

$$w_{k,\alpha}(\xi_k - 2\xi_{k-1} + \xi_{k-2}) + \sum_{m=0}^{k-1}(\xi_{m+1} - 2\xi_m + \xi_{m-1})d_{m,\alpha} = \frac{-4}{h^2 \varsigma_K}\xi_k \sin^2\left(\frac{wh}{2}\right).$$

By a simple calculation, we get

$$\begin{aligned}&\xi_k\left[\frac{4}{h^2\varsigma_K}\sin^2\left(\frac{wh}{2}\right) + (w_{k,\alpha} + d_{k-1,\alpha})\right]\\ &= (\xi_{k-2} - 2\xi_{k-1})(w_{k,\alpha} + d_{k-1,\alpha}) + \sum_{m=0}^{k-2}(\xi_{m+1} - 2\xi_m + \xi_{m-1})d_{m,\alpha}.\end{aligned} \tag{22}$$

Then

$$\xi_k = \frac{(-\xi_{k-2} + 2\xi_{k-1})(w_{k,\alpha} + d_{k-1,\alpha}) + \sum_{m=0}^{k-2}(-\xi_{m+1} + 2\xi_m - \xi_{m-1})d_{m,\alpha}}{\frac{4}{h^2\varsigma_K}\sin^2\left(\frac{wh}{2}\right) + (w_{k,\alpha} + d_{k-1,\alpha})}.$$

For $k = 1$, then

$$\left(w_{1,\alpha} + d_{0,\alpha}\right)\xi_1 < \xi_1\left(w_{1,\alpha} + d_{0,\alpha} + \frac{4}{h^2\varsigma_K}\sin^2\left(\frac{wh}{2}\right)\right) = \xi_0\left(w_{1,\alpha} + d_{0,\alpha}\right).$$

For $k = 2$, we get

$$(w_{2,\alpha} + d_{1,\alpha})\xi_2 < \xi_0(w_{2,\alpha} + d_{1,\alpha}).$$

Repeating the process until N we obtain

$$\xi_N < \xi_0. \tag{23}$$

Note that

$$|\beta_k^i| = \xi_k < \xi_0 = |f_i|.$$

Consequently, $\|\beta\|_{L^2} \leq \|f\|_{L^2}$.

□

6. Conclusion

In this work, we consider a novel finite difference discretization scheme to solve numerically the diffusion-wave equation involving a Caputo- Fabrizio fractional derivative supplemented with initial and boundary conditions. Also, we prove this new scheme is unconditionally stable in L^2.

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