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ON OSCILLATORY SECOND ORDER NONLINEAR IMPULSIVE DELAY DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. In this study, we have found some sufficient conditions for the oscillation of a class of second order impulsive delay dynamic equations on time scale by using impulsive inequality and Riccati transformation technique. Some examples are given to illustrate our main results.

Keywords: Oscillation, nonoscillation, delay dynamic equation, impulse, time scales.

AMS Subject Classification: 34A37, 34A60, 39A12.

1. INTRODUCTION

Many evolution processes in nature are characterized by the fact that at certain moments of time called impulse, they experience an abrupt change of state. Impulsive differential/difference equations has many applications in real life situations. These equations arises in population dynamics, vibrating masses attached to an elastic bar, networks containing lossless transmission lines etc. ([12], [15]). In the last few decade, the oscillation theory for impulsive difference/differential equations has been extensively developed (see for e.g. [6], [14], [12]). In the literature, most of the results obtained for difference equations is the discrete analogues of differential equations and vice versa. Hence it was an immediate question to find a way for which one can unify the qualitative properties of both equations. In 1988 Stefen Hilger introduced the concept of time scales calculus, which unify the continuous and discrete calculus in his Ph.D. thesis [8].

In [11], Huang has considered the second order impulsive dynamic equation of the form

$$\begin{cases} [r(t)(u^{\Delta}(t))^{\gamma}]^{\Delta} + f(t, u^{\sigma}(t)) = 0, \ t \in \mathbb{J}_{\mathbb{T}} := [0, \infty) \cap \mathbb{T}, \ t \neq \tau_k, \ t \geq t_0, \\ u(\tau_k^+) = I_k(u(\tau_k)), \ u^{\Delta}(\tau_k^+) = J_k(u^{\Delta}(\tau_k)), \ k \in \mathbb{N}, \\ u(t_0^+) = u_0, \ u^{\Delta}(t_0^+) = u_0^{\Delta} \end{cases}$$

and improve the results of [9] and [10].

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In [5], Chhatria has studied the oscillation properties of the solution of second order impulsive delay dynamic equations of the form

$$\begin{cases} [r(t)(u^{\Delta}(t))^{\gamma}]^{\Delta} + p(t)x(\sigma(t)-\delta)) = 0, \ t \in \mathbb{J}_{\mathbb{T}} := [0,\infty) \cap \mathbb{T}, \ t \neq \tau_k, \ t \geq t_0, \\ u(\tau_k^+) = I_k(u(\tau_k)), \ u^{\Delta}(\tau_k^+) = J_k(u^{\Delta}(\tau_k)), \ k \in \mathbb{N}, \\ u(t_0^+) = u_0, \ u^{\Delta}(t_0^+) = u_0^{\Delta} \end{cases}$$

and improve the results of [11].

To the best of our knowledge, there is no work on the oscillation of impulsive nonlinear delay dynamic equations on time scales. Following this trends, we consider a class of second order impulsive nonlinear dynamic equations of the form:

$$(E)\begin{cases} [r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} + q(t)f(x(t-\delta)) = 0, \ t \in \mathbb{J}_{\mathbb{T}} := [0,\infty) \cap \mathbb{T}, \ t \neq \theta_k, \ t \geq t_0, & (1) \\ x(\theta_k^+) = g_k(x(\theta_k)), \ x^{\Delta}(\theta_k^+) = h_k(x^{\Delta}(\theta_k)), \ k \in \mathbb{N}, & (2) \\ x(t) = \phi(t), \ t_0 - \delta \leq t \leq t_0 & (3) \end{cases}$$

where $\gamma \geq 1$ is the quotient of odd positive integers, $\mathbb{T}$ is an unbouned above time scale with $0 \in \mathbb{T}$ and $\theta_k \in \mathbb{T}$ are the fixed moment of impulsive effect satisfying the properties $0 \leq t_0 < \theta_1 < \theta_2 < \cdots < \theta_k$, $\lim_{k\to\infty} \theta_k = \infty$.

$$x(\theta_k^+) = \lim_{h\to 0^+} x(\theta_k + h), \qquad x^{\Delta}(\theta_k^+) = \lim_{h\to 0^+} x^{\Delta}(\theta_k + h),$$

which represent the right limit of $x(t)$ at $t = \theta_k$ in the sense of time scale, if θ_k is right scattered, then $x(\theta_k^+) = x(\theta_k)$, $x^{\Delta}(\theta_k^+) = x^{\Delta}(\theta_k)$. Similarly, we can define $x(\theta_k^-)$, $x^{\Delta}(\theta_k^-)$;

Through out this paper, we suppose that the following conditions hold:

- (H_1) $r(t) > 0$, $\delta \in \mathbb{R}_+$, $t - \delta \in \mathbb{T}$;
- (H_2) $q(t) \in C_{rd}(\mathbb{T}, [t_0,\infty)_{\mathbb{T}})$ and $f(u) \in C(\mathbb{R}^+, \mathbb{R})$, $f(u)$ is nondeceasing, $uf(u) > 0$ for $u \neq 0$;
- (H_3) $g_k, h_k : \mathbb{R} \to \mathbb{R}$ are continuous function and there exist positive numbers a_k, a_k^*, b_k, b_k^* such that $a_k^* \leq \frac{g_k(u)}{u} \leq a_k$, $b_k^* \leq \frac{h_k(u)}{u} \leq b_k$, $u \neq 0$, $k \in \mathbb{N}$;

In this work, our objective is to extend the work of [11] and [5] to the second order nonlinear impulsive delay dynamic equations (1)-(3). About the time scale concept and fundamentals of time scale calculus we refer the monographs [3] and [4] and the references cited there in.

AC^i={$x : \mathbb{J}_{\mathbb{T}} \to \mathbb{R}$ is i-times Δ-differentiable, whose i th delta derivative $x^{\Delta^{(i)}}$ is absolutely continuous}.

PC={$x : \mathbb{J}_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous at the points $\theta_k, k \in \mathbb{N}$ for which $x(\theta_k^-)$, $x(\theta_k^+)$, $x^{\Delta}(\theta_k^-)$ and $x^{\Delta}(\theta_k^+)$ exist with $x(\theta_k^-)$=$x(\theta_k)$, $x^{\Delta}(\theta_k^-)$=$x^{\Delta}(\theta_k)$}.

Definition 1.1. *A solution of $x(t)$ of (E) is said to be regular if it is defined on some half line $[\theta_x,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$ and* $\sup\{|x(t)| : t \geq t_x\} > 0$. *A regular solution $x(t)$ of (E) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $x(t) > 0$ $(x(t) < 0)$, for $t \geq t_1$.*

Definition 1.2. *A function $x(t) \in PC \cap AC^2(\mathbb{J}_{\mathbb{T}} \setminus \{\theta_1, \theta_2, \cdots\}, \mathbb{R})$ is called a solution of (E) if: (i) it satisfies* (1) *a.e on $\mathbb{J}_{\mathbb{T}} \setminus \{\theta_k\}, k \in \mathbb{N}$ (ii) for $t = \theta_k, k \in \mathbb{N}$, $x(t)$ satisfies* (2) *(iii) and satisfies the initial condition* (3).

Definition 1.3. *A nontrivial solution $x(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0 \geq 0$ such that $x(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $x(t)$ is said to be oscillatory.*

2. Preliminary Results

We need the time scale version of the following well known results for our use in the sequel.

Lemma 2.1. [1] *Let $y, f \in C_{rd}$ and $p \in \mathcal{R}$. Then*

$$y^{\Delta}(t) \le p(t)y(t) + f(t),$$

implies that for all $t \in \mathbb{T}$

$$y(t) \le y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s.$$

Lemma 2.2. [10] *Assume that $m \in PC \cap AC^1(\mathbb{J}_{\mathbb{T}} \setminus \{\theta_k\}, \mathbb{R})$ satisfies*

$$\begin{aligned} &m^{\Delta}(t) \le p(t)m(t) + v(t), t \in \mathbb{J}_{\mathbb{T}} = [0, \infty) \cap \mathbb{T}, t \ne \theta_k \\ &m(\theta_k^+) \le d_k m(\theta_k) + e_k. \end{aligned}$$

for $k \in \mathbb{N}$ and $t \ge t_0$. Then the following inequality holds

$$\begin{aligned} m(t) \le m(t_0) \prod_{t_0<\theta_k<t} d_k e_p(t_0, t) &+ \int_{t_0}^{t} \prod_{s<\theta_k<t} d_k e_p(t, \sigma(s))v(s)\Delta s \\ &+ \sum_{t_0<\theta_k<t} \Big(\prod_{\theta_k<\theta_j<t} d_j e_p(t, \theta_k)\Big) e_k, t \ge t_0. \end{aligned}$$

Lemma 2.3. *Let $x(t)$ be a solution of (E). Assume that there exists $T \ge t_0$ such that $x(t) > 0 (< 0)$ for $t \ge T$ and*

(H_4) $\int_T^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(s)} \prod_{T<\theta_k<s} \frac{b_k^*}{a_k} \ \Delta s = \infty$

hold. Then $x^{\Delta}(\theta_k^+) \ge 0 (\le 0)$ and $x^{\Delta}(t) \ge 0 (\le 0)$ for $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}$ and $\theta_k \ge T$.

Proof. The proof of the lemma is same as that in [Lemma 2.3, [5]]. □

3. Main Results

Theorem 3.1. *Let all conditions of Lemma 2.3 hold. Furthermore, assume that*

(H_5) *there exists $\lambda > 0$ such that $|f(u)| \ge \lambda |u^{\gamma}|$;*

(H_6) *there exists a function $\beta(t) \in C_{rd}([0, \infty)_{\mathbb{T}}, [0, \infty)_{\mathbb{T}})$ such that*

$$\int_{t_0}^{t} \prod_{t_0<\theta_k<s} \frac{1}{b_k^{\gamma}} \left(\lambda q(s)\beta(s) - \frac{(\beta^{\Delta}(s))^2 r(s-\delta)}{4\gamma \left(\frac{s-\delta}{2}\right)^{\gamma-1} \beta(s)} \right) \Delta s = \infty.$$

Then every solution of (E) oscillates..

Proof. Let $x(t)$ be a nonoscillatory solution of (E). Without loss of generality, assume that $x(t) > 0$, $x(t-\delta) > 0$ for $t \ge t_1$. Due to Lemma 2.3, there exists $t_2 > t_1$ such that $x^{\Delta}(t) > 0$ for $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}, k \in \mathbb{N}$ and $\theta_k \ge t_2$. Using (H_5) in (E), we get

$$\begin{cases} [r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} + \lambda q(t)x^{\gamma}(t-\delta) \le 0, \ t \ne \theta_k, \ t \ge t_2, \\ x(\theta_k^+) = g_k(x(\theta_k)), \ x^{\Delta}(\theta_k^+) = h_k(x^{\Delta}(\theta_k)), \ k \in \mathbb{N}. \end{cases}$$

Let

$$w(t) = \beta(t)\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t-\delta)}. \tag{4}$$

Then $w(\theta_k^+) \geq 0$ and $w(t) \geq 0$ for $\theta_k \geq t_3$. From (4), for $t \neq \theta_k$ we have

$$\begin{aligned}
w^\Delta(t) &= [r(t)(x^\Delta(t))^\gamma]^\Delta \frac{\beta(t)}{x^\gamma(t-\delta)} + r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma \Big[\frac{\beta^\Delta(t)x^\gamma(t-\delta) - \beta(t)(x^\gamma(t-\delta))^\Delta}{x^\gamma(t-\delta)x^\gamma(\sigma(t)-\delta)}\Big] \\
&\leq -\lambda q(t)\beta(t) + \frac{\beta^\Delta(t) r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma}{x^\gamma(\sigma(t)-\delta)} - \frac{\beta(t)r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma (x^\gamma(t-\delta))^\Delta}{x^\gamma(t-\delta)x^\gamma(\sigma(t)-\delta)} \\
&= -\lambda q(t)\beta(t) + \frac{\beta^\Delta(t)}{\beta(\sigma(t))} w(\sigma(t)) - \frac{\beta(t)r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma (x^\gamma(t-\delta))^\Delta}{x^\gamma(t-\delta)x^\gamma(\sigma(t)-\delta)}.
\end{aligned}$$

Therefore,

$$w^\Delta(t) \leq -\lambda q(t)\beta(t) + \frac{\beta^\Delta(t)}{\beta(\sigma(t))} w(\sigma(t)) - \frac{\beta(t)r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma (x^\gamma(t-\delta))^\Delta}{x^\gamma(t-\delta)x^\gamma(\sigma(t)-\delta)}. \tag{5}$$

Since

$$(x^\gamma(t))^\Delta = \frac{x^\gamma(\sigma(t)) - x^\gamma(t)}{\mu(t)}$$

and using the inequality [7]

$$x^l - y^l \geq l y^{l-1}(x-y)^l \text{ for all } x, y \geq 0,\ l \geq 1,$$

we have

$$\begin{aligned}
(x^\gamma(t-\delta))^\Delta &= \frac{x^\gamma(\sigma(t)-\delta) - x^\gamma(t-\delta)}{\mu(t-\delta)} \\
&\geq \frac{\gamma x^{\gamma-1}(t-\delta)}{\mu(t-\delta)}(x(\sigma(t)-\delta) - x(t-\delta))
\end{aligned}$$

and hence

$$(x^\gamma(t-\delta))^\Delta \geq \gamma x^{\gamma-1}(t-\delta) x^\Delta(t-\delta). \tag{6}$$

Using (6) in (5), we get

$$w^\Delta(t) \leq -\lambda q(t)\beta(t) + \frac{\beta^\Delta(t)}{\beta(\sigma(t))} w(\sigma(t)) - \frac{\beta(t)r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma \gamma x^{\gamma-1}(t-\delta)x^\Delta(t-\delta)}{x^\gamma(t-\delta)x^\gamma(\sigma(t)-\delta)}. \tag{7}$$

Clearly, $(r(t)(x^\Delta(t))^\gamma)^\Delta \leq 0$, $r(t) \geq 0$ and $x^\Delta(t) \geq 0$ for all $t \geq t_2$. Therefore, for $t_3 \geq 2t_2$

$$\begin{aligned}
x(t) =& x(t_2) + \int_{t_2}^t x^\Delta(s)\Delta s \geq \int_{t_2}^t x^\Delta(s)\Delta s \\
&\geq x^\Delta(t)\int_{t_2}^t \Delta s = x^\Delta(t)(t-t_2),
\end{aligned}$$

that is,

$$x(t-\delta) \geq \Big(\frac{t-\delta}{2}\Big) x^\Delta(t-\delta) \text{ for all } t \geq t_4 > t_3,$$

that is,

$$\gamma x^{\gamma-1}(t-\delta) \geq \gamma\Big(\frac{t-\delta}{2}\Big)^{\gamma-1}(x^\Delta(t-\delta))^{\gamma-1},$$

that is,

$$\gamma x^{\gamma-1}(t-\delta)x^\Delta(t-\delta) \geq \gamma\Big(\frac{t-\delta}{2}\Big)^{\gamma-1}(x^\Delta(t-\delta))^\gamma, \tag{8}$$

Also, $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0$ implies that $r(t)(x^{\Delta}(t))^{\gamma} \geq r(\sigma(t))(x^{\Delta}(\sigma(t)))^{\gamma}$ and hence

$$(x^{\Delta}(t-\delta))^{\gamma} \geq \frac{r(\sigma(t)-\delta)}{r(t-\delta)}(x^{\Delta}(\sigma(t)-\delta))^{\gamma} \tag{9}$$

Using (8) and (9) in (7), we get

$$w^{\Delta}(t) \leq -\lambda q(t)\beta(t) + \frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}w(\sigma(t)) - \frac{\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)(w(\sigma(t))^2}{\beta^2(\sigma(t))r(t-\delta)}.$$

Using the fact that $y - my^2 \leq \frac{1}{4m}$ the preceding inequality reduces to

$$\begin{aligned} w^{\Delta}(t) &\leq -\lambda q(t)\beta(t) + \frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}w(\sigma(t)) + \frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\Big[w(\sigma(t)) - \frac{\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)(w(\sigma(t))^2}{\beta^{\Delta}(t)\beta(\sigma(t))r(t-\delta)}\Big] \\ &\leq -\lambda q(t)\beta(t) + \frac{(\beta^{\Delta}(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)}, \end{aligned}$$

that is,

$$w^{\Delta}(t) \leq -\left[\lambda q(t)\beta(t) - \frac{(\beta^{\Delta}(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)}\right],\ t \neq \theta_k. \tag{10}$$

For $t = \theta_k$,

$$w(\theta_k^+) = \frac{r(\theta_k^+)(x^{\Delta}(\theta_k^+))^{\gamma}}{x^{\gamma}(\theta_k^+ - \delta)} \leq \frac{b_k^{\gamma} r(\theta_k)(x^{\Delta}(\theta_k))^{\gamma}}{x^{\gamma}(\theta_k - \delta)} = b_k^{\gamma} w(\theta_k).$$

Therefore, we have

$$\begin{aligned} & w^{\Delta}(t) \leq -\lambda q(t)\beta(t) + \frac{(\beta^{\Delta}(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)}, t \neq \theta_k \\ & w(\theta_k^+) \leq b_k^{\gamma} w(\theta_k), k \in \mathbb{N} \end{aligned}$$

and by Lemma 2.2, we get

$$\begin{aligned} w(t) &\leq w(t_4)\prod_{t_4<\theta_k<t} b_k^{\gamma} - \int_{t_4}^{t}\prod_{s<\theta_k<t} b_k^{\gamma}\left[\lambda q(s)\beta(s) - \frac{(\beta^{\Delta}(s))^2 r(s-\delta)}{4\gamma\left(\frac{s-\delta}{2}\right)^{\gamma-1}\beta(s)}\right]\Delta s \\ &\leq \prod_{t_4<\theta_k<t} b_k^{\gamma}\left[w(t_4) - \lambda\int_{t_4}^{t}\prod_{t_4<\theta_k<s}\frac{1}{b_k^{\gamma}}\left(\lambda q(s)\beta(s) - \frac{(\beta^{\Delta}(s))^2 r(s-\delta)}{4\gamma\left(\frac{s-\delta}{2}\right)^{\gamma-1}\beta(s)}\right)\right]\Delta s \\ &\to -\infty \text{ as } t \to \infty \end{aligned}$$

due to (H_7), a contradiction to $w(t) > 0$ for $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}, k \in \mathbb{N}$. This completes the proof of the theorem. □

Theorem 3.2. *Let all conditions of Lemma 2.3 hold. Assume that there exists a positive integer* k_0 *such that* $a_k^* \geq 1$, $b_k \leq 1$ *for* $k \geq k_0$, $\theta_{k+1} - \theta_k > \delta$. *Furthermore, assume that*
(H_7) $\int_{\pm c}^{\pm\infty} \frac{du}{f(u)} < \infty,\ c > 0$
and

(H_8) $\quad \sum_{k=0}^{\infty}\int_{\theta_k}^{\theta_{k+1}} \frac{1}{r^{\frac{1}{\gamma}}(s-\delta)}\Big(\int_T^{\infty}\prod_{T<\theta_k<v}\frac{1}{b_k^{\gamma}}q(v)\Delta v\Big)\Delta s=\infty$

hold. Then every solution of (E) *oscillates.*

Proof. Let $x(t)$ be a nonoscillatory solution of (E). By Lemma 2.3, we have $x^{\Delta}(t)>0$ and $x^{\Delta}(\theta_k^+)>0$ for $t\in(\theta_k,\theta_{k+1}]_{\mathbb{T}}, k\in\mathbb{N}$, $t\geq t_2$. Due to $a_k^*\geq 1,\ k\in\mathbb{N}$, we get

$$x(\theta_k)\leq x(\theta_k^+)$$

and thus $x(t)$ is nondecreasing for $t\in(\theta_k,\theta_{k+1}]_{\mathbb{T}}, k\in\mathbb{N}$. Especially,

$$x(t_2^+)\leq x(\theta_1)\leq x(\theta_1^+)\leq x(\theta_2)<\cdots \tag{11}$$

and hence $x(t)$ is nondecreasing for $t\in[t_2,\infty)_{\mathbb{T}}$. From (E), we get

$$\begin{cases}[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta}=-q(t)f(x(t-\delta)), t\neq\theta_k, t\geq t_2\\ x^{\Delta}(\theta_k^+)\leq b_k x^{\Delta}(\theta_k), k\in\mathbb{N}.\end{cases}$$

let $m(t)=r(t)(x^{\Delta}(t))^{\gamma}$, then

$$\begin{cases}m^{\Delta}(t)=-q(t)f(x(t-\delta)), t\neq\theta_k, t\geq t_2\\ m^{\Delta}(\theta_k^+)\leq b_k^{\gamma}m(\theta_k), k\in\mathbb{N}.\end{cases}$$

Due to Lemma 2.2, we get

$$m^{\Delta}(t)\leq m^{\Delta}(s)\prod_{s<\theta_k<t}b_k^{\gamma}-\int_s^t\prod_{u<\theta_k<t}b_k^{\gamma}q(u)f(x(u-\delta))\Delta u,\ s\geq t_2,$$

that is,

$$r(t)(x^{\Delta}(t))^{\gamma}\leq r(s)(x^{\Delta}(s))^{\gamma}\prod_{s<\theta_k<t}b_k^{\gamma}-\int_s^t\prod_{u<\theta_k<t}b_k^{\gamma}q(u)f(x(u-\delta))\Delta u,\ s\geq t_2 \tag{12}$$

implies that

$$x^{\Delta}(s)\geq\frac{1}{r^{\frac{1}{\gamma}}(s)}\left(\int_s^t\prod_{s<\theta_k<u}\frac{1}{b_k^{\gamma}}q(u)f(x(u-\delta))\Delta u\right)^{\frac{1}{\gamma}}.$$

Consequently,

$$\frac{x^{\Delta}(s)}{f^{\frac{1}{\gamma}}(x(s-\delta))}\geq\frac{1}{r^{\frac{1}{\gamma}}(s)}\left(\int_s^t\prod_{s<\theta_k<u}\frac{1}{b_k^{\gamma}}q(u)\Delta u\right)^{\frac{1}{\gamma}}.$$

We may note that, (12) implies that $r(t)(x^{\Delta}(t))^{\gamma}\leq r(s)(x^{\Delta}(s))^{\gamma}\prod_{s<\theta_k<t}b_k,\ s\geq t_2$. Ultimately,

$$r(s)(x^{\Delta}(s))^{\gamma}\leq r(s-\delta)(x^{\Delta}(s-\delta))^{\gamma}\prod_{s-\sigma<\theta_k<t}b_k,\ s\geq t_2+\delta. \tag{13}$$

Let $s\in(\theta_k,\theta_{k+1}]_{\mathbb{T}}$. Using the fact that $\theta_{k+1}-\theta_k>\delta$, $b_k\leq 1$ and due to (13), we get

$$\begin{aligned}\int_{\theta_k}^{\theta_{k+1}}\frac{r^{\frac{1}{\gamma}}(s)x^{\Delta}(s)}{r^{\frac{1}{\gamma}}(s-\delta)f^{\frac{1}{\gamma}}(x(s-\delta))}\Delta s&\leq\int_{\theta_k}^{\theta_{k+1}}\prod_{s-\sigma<\theta_k<s}b_k\frac{x^{\Delta}(s-\delta)}{f^{\frac{1}{\gamma}}(x(s-\delta))}\Delta s\\ &\leq\int_{\theta_k}^{\theta_{k+1}}\frac{x^{\Delta}(s-\delta)}{f^{\frac{1}{\gamma}}(x(s-\delta))}\Delta s=\int_{x(\theta_k-\delta)}^{x(\theta_{k+1}-\delta)}\frac{\Delta v}{f^{\frac{1}{\gamma}}(v)},\end{aligned}$$

that is,

$$\int_{\theta_k}^{\theta_{k+1}} \frac{1}{r^{\frac{1}{\gamma}}(s-\delta)}\Big(\int_s^t \prod_{s<\theta_k<u} \frac{1}{b_k^\gamma} q(u)\Delta u\Big)\Delta s \le \int_{x(\theta_k-\delta)}^{x(\theta_{k+1}-\delta)} \frac{\Delta v}{f^{\frac{1}{\gamma}}(v)}.$$

Since (11) holds, the above inequality becomes

$$\sum_{k=1}^{\infty}\int_{\theta_k}^{\theta_{k+1}} \frac{1}{r^{\frac{1}{\gamma}}(s-\delta)}\Big(\int_s^t \prod_{s<\theta_k<u} \frac{1}{b_k^\gamma} q(u)\Delta u\Big)\Delta s \le \int_{x(\theta_1-\delta)}^{\infty} \frac{\Delta v}{f^{\frac{1}{\gamma}}(v)} < \infty$$

due to (H_7), a contradiction to (H_8). This completes the proof of the theorem. □

Theorem 3.3. *Let all conditions of Lemma 2.3 hold. Assume that there exists a positive integer k_0 such that $b_k \le 1$ for $k \ge k_0$. Furthermore, assume that (H_7), $f(ab) \ge f(a)f(b)$ for any $ab > 0$, $\theta_{k+1} - \theta_k = \delta$ and*

(H_9) $\quad \sum_{k=0}^{\infty}\int_{\theta_k}^{\theta_{k+1}} \frac{1}{r^{\frac{1}{\gamma}}(s-\delta)}\Big(\int_T^\infty \prod_{T<\theta_k<v} \frac{1}{d_k} q(v)\Delta v\Big)\Delta s = \infty,$

where

$$d_k = \begin{cases} b_1^\gamma, & \text{if } k = 1, \\ \frac{b_k^\gamma}{f(a_{k-1}^*)}, & \text{if } k = 2, 3, \cdots \end{cases}$$

hold. Then every solution of (E) oscillates.

Proof. Let $x(t)$ be a nonoscillatory solution of (E). By Lemma2.3, we get $x^\Delta(t) > 0$ and $x^\Delta(\theta_k^+) > 0$ for $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}, k \in \mathbb{N}$, $t \ge t_2$. Indeed, $x^\Delta(t-\delta) > 0$ for $t \ge t_3 \ge t_2 + \delta$. Let

$$w(t) = \frac{r(t)(x^\Delta(t))^\gamma}{f(x(t-\delta))}.$$

Then $w(\theta_k^+) \ge 0$ and $w(t) \ge 0$ for $\theta_k \ge t_3$. From (4), for $t \ne \theta_k$ we have

$$\begin{aligned} w^\Delta(t) &= \frac{[r(t)(x^\Delta(t))^\gamma]^\Delta f(x(t-\delta)) - r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma f^\Delta(x(t-\delta))}{f(x(t-\delta))f(x(\sigma(t)-\delta))} \\ &\le \frac{[r(t)(x^\Delta(t))^\gamma]^\Delta}{f(x(t-\delta))} - \frac{r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma f^\Delta(x(t-\delta))}{f(x(t-\delta))f(x(t-\delta))} \\ &\le -q(t), \end{aligned}$$

where we have used the fact that $x^\Delta(t) > 0$, $\sigma(t) \ge t$ and $f(t)$ is nonincreasing. Now for $t = \theta_k$, if $k = 1$

$$w(\theta_1^+) = \frac{r(\theta_1^+)(x^\Delta(\theta_1^+))^\gamma}{f(x(\theta_1^+ - \delta))} \le \frac{b_1^\gamma r(\theta_1)(x^\Delta(\theta_1))^\gamma}{f(x(\theta_1-\delta))} = d_1 w(\theta_1).$$

If $k = 2, 3, \cdots$.

$$\begin{aligned} w(\theta_k^+) &= \frac{r(\theta_k^+)(x^\Delta(\theta_k^+))^\gamma}{f(x(\theta_k^+ - \delta))} \le \frac{b_k^\gamma r(\theta_k)(x^\Delta(\theta_k))^\gamma}{f(x(\theta_{k-1}^+ - \delta))} \le \frac{b_k^\gamma r(\theta_k)(x^\Delta(\theta_k))^\gamma}{f(a_{k-1}^* x(\theta_{k-1}-\delta))} \\ &\le \frac{b_k^\gamma r(\theta_k)(x^\Delta(\theta_k))^\gamma}{f(a_{k-1}^*)f(x(\theta_{k-1}-\delta))} \le \frac{b_k^\gamma r(\theta_k)(x^\Delta(\theta_k))^\gamma}{f(a_{k-1}^*)f(x(\theta_k-\delta))} = d_k w(\theta_k). \end{aligned}$$

Consider the impulsive dynamic inequality

$$\begin{aligned} &w^\Delta(t) \le -q(t), t \ne \theta_k,\ t \ge t_3 \\ &w^\Delta(\theta_k^+) \le d_k w(\theta_k),\ k \in \mathbb{N}. \end{aligned}$$

By Lemma 2.2, we get

$$w(t) \leq w(s) \prod_{s<\theta_k<t} d_k - \int_s^t \prod_{u<\theta_k<t} d_k q(u)\Delta u, \ s \geq t_3.$$

The rest of the proof follows from the proof of the Theorem 3.2 and hence the details are omitted. □

Theorem 3.4. *Let all the conditions of Lemma 2.3 and $b_k \geq 1$ hold. Assume that*

$$(H_{10}) \quad \limsup_{k\to\infty} \frac{1}{t^m} \int_{t_0}^{\theta_{k+1}} (t-s)^m \left[\lambda q(t)\beta(t) - \frac{(\beta^{\Delta}(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)} \right] \Delta s = \infty \text{ for some } m > 1,$$

then every solution of (E) oscillates.

Proof. Proceeding as in the proof of Theorem 3.1, we get

$$w^{\Delta}(t) \leq -\left[\lambda q(t)\beta(t) - \frac{(\beta^{\Delta}(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)} \right], \text{ for } t \neq \theta_k.$$

Multiplying $(t-s)^m$ $(t>s)$ to the preceding inequality and integrating from θ_k to θ_{k+1}, we get

$$\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^{\Delta}(s)ds \leq -\int_{\theta_k}^{\theta_{k+1}} (t-s)^m \left[\lambda q(t)\beta(t) - \frac{(\beta^{\Delta}(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)} \right] \Delta s.$$

Indeed,

$$\begin{aligned}
&\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^{\Delta}(s)\Delta s \\
&= (t-s)^m u(s)\big|_{\theta_k}^{\theta_{k+1}} - \int_{\theta_k}^{\theta_{k+1}} ((t-s)^m)^{\Delta_s} w(s)\Delta s \\
&= \int_{\theta_k}^{\theta_{k+1}} m(t-s)^{m-1} w(s)\Delta s + (t-\theta_{k+1})^m w(\theta_{k+1}) - (t-\theta_k)^m w(\theta_k^+),
\end{aligned}$$

because $((t-s)^m)^{\Delta_s} = -m(t-s)^{m-1}$. As a result,

$$\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^{\Delta}(s)\Delta s \geq -(t-\theta_k)^m w(\theta_k^+).$$

Therefore,

$$\begin{aligned}
\int_{\theta_k}^{\theta_{k+1}} (t-s)^m \left[Lq(t)\beta(t) - \frac{(\beta^{\Delta}(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)} \right] \Delta s &\leq -\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^{\Delta}(s)\Delta s \\
&\leq (t-\theta_k)^m w(\theta_k^+) \\
&\leq b_k (t-\theta_k)^m w(\theta_k)
\end{aligned}$$

that is,

$$\frac{1}{t^m}\int_{\theta_k}^{\theta_{k+1}}(t-s)^m\left[\lambda q(t)\beta(t)-\frac{(\beta^\Delta(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)}\right]\Delta s\le b_k\left(\frac{t-\theta_k}{t}\right)^m w(\theta_k).$$

and hence

$$\limsup_{k\to\infty}\frac{1}{t^m}\int_{\theta_k}^{\theta_{k+1}}(t-s)^m\left[\lambda q(t)\beta(t)-\frac{(\beta^\Delta(t))^2 r(t-\delta)}{4\gamma\left(\frac{t-\delta}{2}\right)^{\gamma-1}\beta(t)}\right]\Delta s<\infty,$$

a contradiction to (H_{10}). This completes the proof of the theorem. □

4. Discussion and Examples

Example 4.1. *Consider*

$$\begin{cases} x^{\Delta\Delta}(t)+\frac{\lambda}{(t+2)^2}x(t-2)=0, & t>2, t\neq\theta_k,\\ x(\theta_k^+)=\frac{k+1}{k}x(\theta_k),\ \ x^\Delta(\theta_k^+)=x^\Delta(\theta_k), & k\in\mathbb{N}, \end{cases}\tag{14}$$

where $\gamma=1$, $r(t)=1$, $\delta=2$, $q(t)=\frac{1}{(2+t)^2}\ge 0$, $a_k^*=a_k=\frac{k+1}{k}$, $b_k^*=b_k=1$, $\theta_k=3k$, $\theta_{k+1}-\theta_k=3>2, k\in\mathbb{N}$, $f(u)=\lambda u,\lambda>0$. *Let's choose* $\beta(t)=t+2$. *Then, from* (H_4)

$$\begin{aligned}
&\int_T^\infty\prod_{T<\theta_k<s}\frac{b_k^*}{a_k}\,\Delta s\\
&=\int_2^\infty\prod_{2<\theta_k<s}\frac{k}{k+1}ds\\
&=\int_2^{\theta_1}\prod_{2<\theta_k<s}\frac{k}{k+1}\Delta s+\int_{\theta_1^+}^{\theta_2}\prod_{2<\theta_k<s}\frac{k}{k+1}\Delta s+\int_{\theta_2^+}^{\theta_3}\prod_{2<\theta_k<s}\frac{k}{k+1}\Delta s+\cdots\\
&=\frac12(\theta_1-2)+\frac12\times\frac23(\theta_2-\theta_1)+\frac12\times\frac23\times\frac34(\theta_3-\theta_2)+\cdots\\
&=\frac12\times 2+\frac13\times 3+\frac14\times 3+\frac15\times 3+\cdots\\
&\ge\frac12+\frac13+\frac14+\frac15+\cdots=\sum_{i=2}^\infty\frac1i=\infty
\end{aligned}$$

and from (H_6)

$$\begin{aligned}
&\int_2^\infty\prod_{2<\theta_k<s}\frac{1}{b_k^\gamma}\Big[\frac{\lambda}{(s+2)}-\frac{1}{4(s+2)}\Big]\Delta s\\
&=\Big[\int_2^{\theta_1}\prod_{2<\theta_k<s}+\int_{\theta_1^+}^{\theta_2}\prod_{2<\theta_k<s}+\cdots+\int_{\theta_{k-1}^+}^{\infty}\prod_{2<\theta_k<s}\Big]\Big(\frac{\lambda}{(s+2)}-\frac{1}{4(s+2)}\Big)\Delta s\\
&\to\infty \text{ if } \lambda>\frac14.
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 are satisfied and hence (14) *has a oscillatory solution.*

Example 4.2. *Consider*

$$\begin{cases} [\frac{1}{t+1}x^{\Delta}(t)]^{\Delta}+(1+\frac{1}{t})x^3(t-1)=0, & t>1, t\neq\theta_k,\\ x(\theta_k^+)=\frac{k-1}{k}x(\theta_k), & k\in\mathbb{N}, k>k_0,\\ x^{\Delta}(\theta_k^+)=\frac{1}{k}x^{\Delta}(\theta_k), & k\in\mathbb{N}, k>k_0, \end{cases} \tag{15}$$

where $\gamma=1$, $\delta=1$, $r(t)=\frac{1}{t+1}$, $q(t)=1+\frac{1}{t}\geq 0$, $a_k^*=a_k=\frac{k-1}{k}$, $b_k^*=b_k=\frac{1}{k}$, $\theta_k=2^k$, $\theta_{k+1}-\theta_k=2^k>1, k\in\mathbb{N}, k>k_0=1$, $f(u)=u^3$. *Clearly, from* (H_4) *we have*

$$\begin{aligned}
&\int_T^{\infty}\prod_{T<\theta_k<s}\frac{b_k^*}{a_k}\,\Delta s\\
&=\int_1^{\infty}\prod_{1<\theta_k<s}\frac{1}{k-1}\Delta s\\
&=\int_1^{\theta_2}\prod_{1<\theta_k<s}\frac{1}{k-1}\Delta s+\int_{\theta_2^+}^{\theta_3}\prod_{1<\theta_k<s}\frac{1}{k-1}\Delta s+\int_{\theta_3^+}^{\theta_4}\prod_{1<\theta_k<s}\frac{1}{k-1}\Delta s+\cdots\\
&=(\theta_2-1)+\frac{1}{2}\times(\theta_3-\theta_2)+\frac{1}{2}\times\frac{1}{3}\times(\theta_4-\theta_3)+\cdots\\
&=2+\frac{1}{2}\times 2^2+\frac{1}{2}\times\frac{1}{3}\times 2^3+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times 2^4+\cdots\\
&\geq 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=1+\sum_{i=2}^{\infty}\frac{1}{i}=\infty
\end{aligned}$$

and from (H_8),

$$\sum_{k=2}^{\infty}\int_{\theta_k}^{\theta_{k+1}}\frac{1}{t}\Big(\int_1^{\infty}\Big[\prod_{1<\theta_k<v}k\Big(1+\frac{1}{v}\Big)\Big]\Delta v\Big)\Delta s=\infty.$$

All conditions of Theorem 3.2 are satisfied for (15) *and hence* (15) *has a oscillatory solution.*

4.1. **Future Directions.** To the best of the author's r knowledge, this is the first investigation of nonlinear impulsive delay dynamic equations with deviating arguments on times scales. That means there are many directions in which future investigations can proceed. We mention only a few here.

(1) Are there other kinds of non-linearities which are of interest for (E) and for which results can be found?
(2) What can be said about the forced oscillation, that is, the oscillation of

$$[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta}+q(t)f(x(t-\delta))=g(t),\ t\in\mathbb{J}_{\mathbb{T}}:=[0,\infty)\cap\mathbb{T},\ t\neq\theta_k,\ t\geq t_0,$$
$$x(\theta_k^+)=g_k(x(\theta_k)),\ x^{\Delta}(\theta_k^+)=h_k(x^{\Delta}(\theta_k)),\ k\in\mathbb{N},$$
$$x(t)=\phi(t),\ t_0-\delta\leq t\leq t_0,$$

where $g(t)$ is oscillatory?
(3) What can be said about the positive solution of (E) and its various generalization?

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