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EXTENDING THE NOTION OF 3-FOLD-3-POINT-SPLITTING FROM GRAPHS TO BINARY MATROIDS

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ABSTRACT. Slater defined *r-fold-n-point-splitting* operation on graphs and proved that, if G is an n-connected graph and H is a graph obtained from G by an r-fold-n-point-splitting, then H is n-connected. In this article we extend this notions from graphs to binary matroids and give some similar results to matroids. Moreover, we examine the Eulerianity of the resulting matroid obtained by this operation when the original matriod is Eulerian.

Keywords: Binary matroid, n-connected matroid, splitting operation, r-fold-n-point-splitting, cocircuit.

AMS Subject Classification: 05B35.

1. Introduction

Fleischner [3] defined the splitting operation on graphs as follows: Let G be a connected graph and v be a vertex of degree at least three in G. If $x = vv_1$ and $y = vv_2$ are two edges incident at v, then the splitting away the pair x, y from v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ adjacent to v_1 and v_2. The transition from G to $G_{x,y}$ is called the splitting operation on G.

Raghunathan et al. [8] extended the notion of the splitting operation from graphs to binary matroid for every pair x, y of it's elements. Shikare et al. [9] generalized this operation for any subset $T \subseteq E(M)$ for binary matroids as follows:
Let M be a binary matroid on a set E and A be a matrix over $GF(2)$ representing M. Let T be a subset of E and A_T be the matrix that is obtained by adding an extra row to A in which the row being zero everywhere except for the columns corresponding to T, where it takes the value 1. Let M_T be the matroid represented by the matrix A_T, we say that M_T has been obtained from M by the splitting the set T.

Slater [10] defined the *n-point-splitting* operation on graphs as follows: Let G be a graph and u be a vertex of G such that $deg(u) \geq 2n - 2$. Let H be the graph obtained from

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G by replacing u by two adjacent vertices u_1 and u_2, if a vertex x is adjacent to u in G, written x adj u, then make x adj u_1 or adj u_2 (but not both) such that $deg(u_1) \geq n$ and $deg(u_2) \geq n$. We call the transition from G to H an n-point-splitting operation.

Azadi [1] extended the notion of n-point-splitting operation from graphs to binary matroids as follows: Let M be a binary matroid on a set E and A be a matrix over $GF(2)$ that representing M. Let T be a subset of E, and A'_T be the matrix obtained by adjoining an extra row to A in which the row being zero everywhere except for the columns corresponding to the elements of T where it takes the value 1, and adjoining an extra column (corresponding to a, $a \notin E(M)$) with this column being zero everywhere except for the last row where it takes the value 1. Let M'_T be the matroid represented by the matrix A'_T, we say that M'_T has been obtained from M by the element splitting of the set T.

The following proposition presents the set of circuits of the element splitting matroids.

Proposition 1.1. [1] *Let $M = (E, \mathcal{C})$ be a binary matroid, T be a subset of E, and let $a \notin E$. Then the set of circuits of element splitting are as follows:*
i): $\mathcal{C}_0$=$\{C \in \mathcal{C} : C$ contains an even number of elements of $T\}$,
ii): $\mathcal{C}_1$= set of minimal members of $\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \emptyset$ and each C_1 and C_2 contains an odd number of elements of T such that $C_1 \cup C_2$ contains no member of $\mathcal{C}_0\}$; and
iii): $\mathcal{C}_2 = \{C \cup \{a\} : C \in \mathcal{C}$ and C contains odd number of elements of $T\}$.

The following lemma is an easy consequence of the definition of M_T and M'_T.

Lemma 1.1. *Let M be a binary matroid on E and $T \subseteq E(M)$. Then $M_T = M'_T \backslash a$ and $M = M'_T/a$*

Various properties concerning the element splitting matroid have been studied in [2]. In that paper, authors consider the problem of determining precisely which graphic matroids M have the property that the element splitting operation, by every pair of elements on M yields a graphic matroid. The problem is solved by proving that there is exactly one minor-minimal matroid that does not have this property.

Let M be a matroid with ground set E. For $X \subseteq E$, let

$$\lambda_M(X) = r(X) + r(E - X) - r(M).$$

We call λ_M the connectivity function of M. Let k be a positive integer. A k-separation of M is a partition $\{X, Y\}$ of E(M) such that $min\{|X|, |Y|\} \geq k$, and $\lambda_M(X) \leq k - 1$. For all $n \geq 2$, M is n-connected if, for all k in $\{1, 2, ..., n-1\}$, M has no k-separation.

Slater [11] defined the *r-fold-n-point-splitting* operation on graphs as follows: Suppose $u \in V(G)$ and u is adjacent to $v_1, ..., v_t$ where $t \geq n$. Let H be the graph obtained from G by replacing u by the complete graph K_r, say on points $u_1, u_2, ..., u_r$, where $2 \leq r \leq n$ and for each v_i, make v_i adjacent to exactly one u_i. If $deg(u_i) \geq n$, $i = 1, 2, ..., r$, in H, then H will be said to be obtained from G by r-fold-n-point splitting.

We illustrate this construction with the help of Figure 1.

Theorem 1.1. [11] *If G is an n-connected graph and H is obtained from G by r-fold-n-point-splitting, then H is n-connected.*

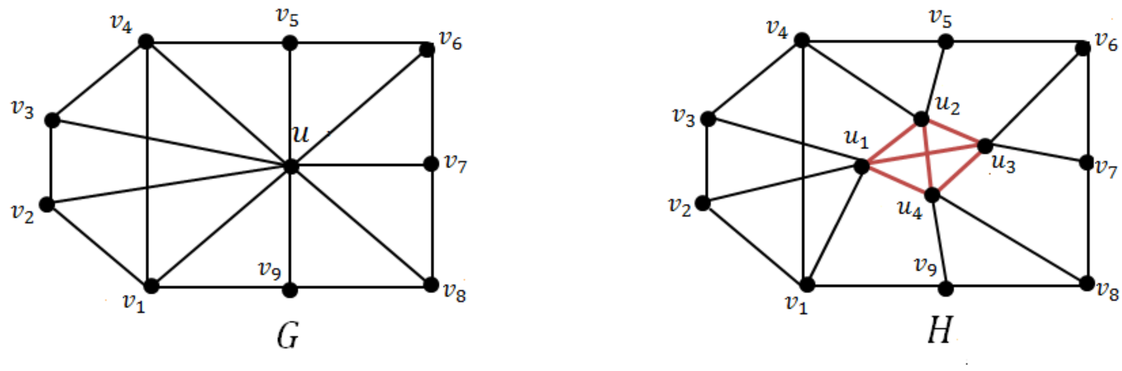

FIGURE 1. Graph H is an 4-fold-n-point splitting $(n = 4, 5)$ of graph G.

Corollary 1.1. *Suppose G is an n-connected graph with $n \geq 3$. Then any graph H obtained by an 3-fold-n-point-splitting of G is 3-connected.*

Note that for $r = 2$, r-fold-n-point splitting is n-point splitting. Slater defined both these operations (n-point splitting and r-fold-n-point splitting) to obtain another n-connected graph of a given graph G ([10] and [11]). However, the extension of n-point splitting from n-connected graphs to binary matroids does not preserve n-connectedness, in general. In this paper, we extend r-fold-n-point splitting operation to binary matroids and prove that this operation for $r = 3$, preserves 3-connectivity.

Various properties of the splitting matroid are explored in [8] and [9]. For the standard terminology in matroid we refer to [7].

In section 2 of this article, we extend the notion of r-fold-n-point-splitting from graphs to binary matroids when $r = 3$ and extend Theorem 1.1 and Corollary 1.1 to binary matroids. In section 3, we study the Eulerianity of a binary matroid and a 3-fold of it's.

2. 3-FOLD IN BINARY MATROID

In this section, we extend the notion of 3-fold-n-point-splitting from graphs to binary matroids and prove that, this operation preserves connectivity.

Remark 2.1. *Let G be the graph shown in Figure 2. A 3-fold-3-point-splitting of G is the graph G'' (see Figure 2). We see that $M(G'')$ is obtained by adding an element c to the matroid $((M(G))'_{\{1,2\}})'_{\{1\}}$ such that $\{a, b, c\}$ form a circuit in which $a, b \notin E(M(G))$.*

Definition 2.1. *Let A be a matrix that representating a rank-r binary matroid M. Suppose $\emptyset \neq T'_1 \subsetneqq T_1$ in which, T_1 is a proper subset of a cocircuit of M. Set $A_1 = (A'_{T_1})'_{T'_1}$. A 3-fold of matroid $M[A]$ w.r.t. T_1 and T'_1 is a matroid obtained by adding an element c to matroid $M[A_1]$ such that $\{a, b, c\}$ is a circuit in which $a, b \notin E(M)$ and are those elements added to $E(M)$ in two element splitting operation that occured. We denote it by M'' (see Figure 3).*

It is clear that $E(M'') = E(M) \cup \{a, b, c\}$.

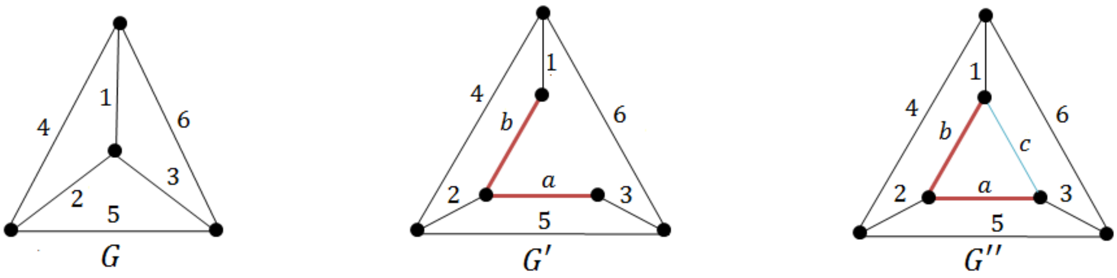

FIGURE 2.

We illustrate this construction with the help of Figure 2 in which $M(G^{''})$ is a 3-fold of $M(G)$. Let $M_1 = M(G)$, $M_2 = M(G^{'})$ and $M_3 = M(G^{''})$. We see that M_2 is obtained from M_1 by two element splitting operations with respect to $T_1 = \{1, 2\}$ and $T_1^{'} = \{2\}$, respectively; and M_3 is obtained from M_2 by adding element c to matroid M_2 such that $\{a, b, c\}$ forms a circuit in M_3.

By Definition 2.1, the 3-fold operation for binary matroids is related to the element splitting operation. Also, Lemma 1.1 provides a relation between the element splitting operation and the splitting operation. These two splitting operations do not preserve n-connectedness of the given, in general. The sufficient conditions to preserve n-connectedness under the splitting operation and under the element splitting operation are obtained in [6] and [5], respectively. However, we prove that the 3-fold operation preserves k-connectedness of the given binary matroid for $k \in \{2, 3\}$.

We have the following Lemma from the definition 2.1.

Lemma 2.1. *Let M be a binary matroid and let M'' be a 3-fold of M. Then $M = M''/\{a, b, c\}$.*

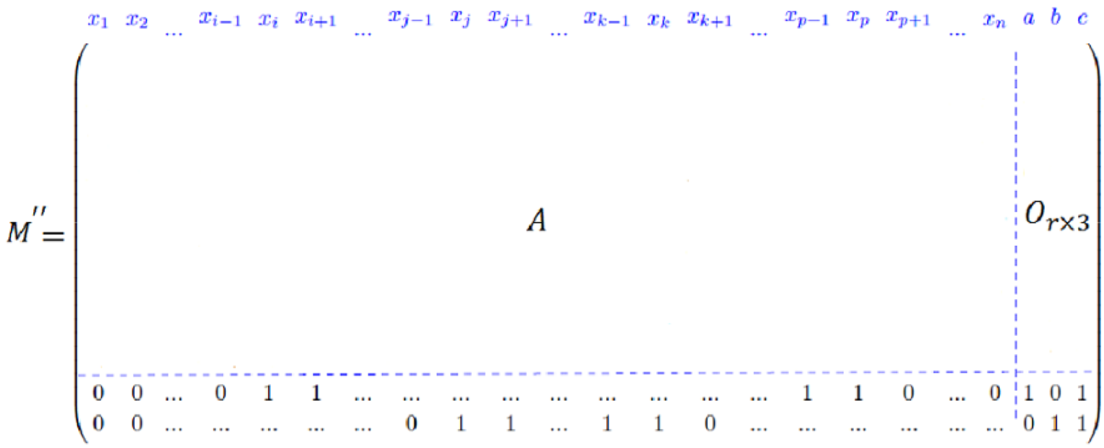

FIGURE 3. A 3-fold of $M[A]$ w.r.t. $T_1 = \{x_i, x_{i+1}, ..., x_p\}$ and $T_1^{'} = \{x_j, x_{j+1}, ..., x_k\}$

Proposition 2.1. *([7], Proposition 8.2.1). If M is an n-connected matroid and $|E(M)| \geq 2(n-1)$, then all circuits and all cocircuits of M have at least n elements.*

Proposition 2.2. *([7], Proposition 9.2.2). Let A be a binary representation of a rank-r binary matroid M. Then the cocircuit space of M equals the row space of A. Moreover, this space has dimension r and is the orthogonal subspace of the circuit space of M.*

Corollary 2.1. *Let M be a binary matroid on E and $T \subseteq E$. Then $M = M_T$ if and only if T is a union of the disjoint cocircuits of M.*

Corollary 2.2. *Let M be a binary matroid on E and $T \subseteq E$. If T is not a union of the disjoint cocircuits of M, then $M \neq M_T$ and $r(M_T) = r(M) + 1$.*

We will need the following proposition of Oxley [7] (see also [4]).

Proposition 2.3. *([7], Proposition 9.2.2). Let M be a binary matroid and $S \subseteq E(M)$. Then S is a member of cocircuit space (circuit space) of M if and only if $|S \cap T|$ is even for every member T of circuit space (cocircuit space) of M.*

Lemma 2.2. *Let M be a connected binary matroid and $T \subseteq E(M)$. If T is a proper subset of a cocircuit of M, then M'_T is connected.*

Proof. Let D be a component of M'_T containing a. Since T is properly contained in a cocircuit of M, so by Proposition 2.3, there is a circuit C in M containing an odd number of elements of T. Hence, by proposition 1.1, $C \cup \{a\}$ is a circuit of M'_T and thus it is contained in D. Now, suppose $x \in E(M)$. Let y be any element of C. As M is connected, there is a circuit Z containing x and y. Now, Z or $Z \cup \{a\}$ is a circuit of M'_T and thus Z is contained in D. This demonstrates that $E(M) \subseteq D$. Hence M'_T has only one component, and thus it is connected.

□

Corollary 2.3. *Let M be a connected binary matroid. Then M'' is a connected matroid.*

Proof. Let M be a binary matroid and let M'' be a 3-fold of M, w.r.t. T'_1 and T_1. Because T_1 and T'_1 are proper subsets of the cocircuits of M and M'_{T_1}, respectively, by Proposition 2.3 and Lemma 2.2, $(M'_{T_1})'_{T'_1}$ is connected and has only one component. Now the addition of element c such that $\{a, b, c\}$ forms a circuit, does not increase the number of the components of M''. Hence, M'' is connected.

□

The following proposition is neccesary in our discussion.

Proposition 2.4. *([7], Corollary 8.2.5). Let X, C, and D be subsets of the ground set E of a matroid M where C and D are disjoint. Then*

$$\lambda_{M\backslash D/C}(X - (D \cup C)) \leq \lambda_M(X).$$

Equivalently, for $Y = E - X$, if (X, Y) is k-separating in M, and N is a minor of M, then $(X \cap E(N), Y \cap E(N))$ is k-separating in N.

3. 3-connectivity of 3-fold matroids

In this section we prove that, if M is a binary 3-connected matriod, then the martroid M'' obtained by a 3-fold of M, is also 3-connected.

Lemma 3.1. *([7], Lemma 8.1.4). Let M be a matroid with ground set E. If $X \subseteq E$, then*

$$\lambda_M(X) = r(X) + r^*(X) - |X|.$$

Let M be a matroid on E, and T be a subset of E. Let $M/T = (M^*\backslash T)^*$, in which M^* is dual of M. We shall call M/T, the contraction of M onto $E - T$ or the contraction of T from M.

Lemma 3.2. *Let M be a 3-cnnected binary matroid with $|E(M)| \geq 4$. Then every cocircuit of M'' has at least three elements.*

Proof. Since M'' is a connected matroid, every cocircuit has at least two elements. So we prove that, M'' has no 2-cocircuit. Let X be a two-elements cocircuit of M''. Then $E(M'') - X$ is a hyperplane of M''; so has rank $r(M'') - 1$.

Case 1: If $X \subseteq \{a, b, c\}$, then by corollary 2.2, $r(M'') = r((M_{T_1})_{T_1'})$ and the fact that $r((M_{T_1})_{T_1'}) \leq r(E(M'') - X)$, we have $r(E(M'') - X) = r(M'')$, is a contradiction.

Case 2: Let X contain no element of the triangle $\{a, b, c\}$. Since M is a 3-connected matroid, so it has a basis, say B, which contains no element of X. Now the union of B with any two-elements subset of $\{a, b, c\}$ hase rank $r(M'')$, a contradiction.

Case 3: If X contains exactly one element of the triangle $\{a, b, c\}$, then by similar argument in case 2, we have $r(E(M'') - X) = r(M'')$, a contradiction.
So the lemma holds. □

The following corollary is an immediate consequence of the matrix representation of M''

Corollary 3.1. *Let M be a 3-connected binary matroid. Then every circuit of M'' has at least three elements.*

Theorem 3.1. *Let M be a 3-connected binary matroid with $|E(M)| \geq 4$. Then the matroid M'' on the set $E(M) \cup \{a, b, c\}$ is 3-connected.*

Proof. By Corollary 2.3, M'' is connected. It is sufficient to show that M'' has no 2-separation.

Suppose that (X, Y) is a 2-separation of M''. Then $min\{|X|, |Y|\} \geq 2$ and

$$\begin{aligned} 1 &= r''(X) + r''(Y) - r''(M'') \\ &= r''(X) + r''^*(X) - |X|. \end{aligned} \tag{1}$$

In which r'' and r''^* are the rank functions of M'' and it's dual respectively. Without loss of generality suppose that $min\{|X|, |Y|\} = |X|$.

Case 1: Let $|X| = 2$. By Lemma 3.2 and Corollary 3.1, every circuit and cocircuit of M'' has at least three elements, so $r''(X) \geq 2$ and $r''^*(X) \geq 2$. This is in contradiction with (1).

Case 2: Let $|X| = 3$. We consider two subcases.
i) If $r''(X) = 2$, then $X = \{a, b, c\}$ or $X \subseteq E(M)$. If $X = \{a, b, c\}$, then by Corollary 2.2, $r''(Y) = r'((M_{T_1})_{T_1'}) = r''(M'')$ where r' is the rank function of $(M_{T_1})_{T_1'}$, a contradiction. Let $X \subseteq E(M)$. Since M is 3-connected matriod, X is not a cocircuit of M. So there is a basis, say B, of M such that $B \cap X = \emptyset$. Now $B \cup \{a, b, c\}$ is contained in Y with rank $r''(M'')$, a contradiction.
ii) If $r''(X) = 3$, then $r''^*(X) = 2$ or $r''^*(X) = 3$; Both of them are contradictions.

Case 3: Let $|X| = 4$. Since M'' is binary and every circuit and cocircuit of M'' has at least three elements, so $r''^*(X) \geq 3$ and $r''(X) \geq 3$. Now

$$2 \leq r''(X) + r''^*(X) - |X| = 1$$

a contradiction.

Case 4: If $|X| \geq 5$; then $min\{|X-\{a,b,c\}|, |Y-\{a,b,c\}|\} \geq 2$. But by proposition 2.4 and the fact that $M''/\{a,b,c\} = M$, we have

$$\lambda_M(X-\{a,b,c\}) = \lambda_{M''/\{a,b,c\}}(X-\{a,b,c\}) \leq \lambda_{M''}(X) = 1.$$

This contradicts 3-connectivity of M. So M'' has no 2-separation. This completes the proof. □

4. Eulerianity of M and M''

Let M be an Eulerian binary matroid. In this section we give a sufficient condition for which M'' to be Eulerian.

Lemma 4.1. *([7], Corollary 9.4.7). Let C be a circuit of a binary matroid M and e be an element of $E(M)-C$. Then, in M/e, either C is a circuit or C is a disjoint union of two circuits. In both cases, M/e has no other circuits contained in C.*

Definition 4.1. [12] *A matroid M is said to be Eulerian if it's ground set is a union of disjoint circuits.*

The following lemma is an immediate consequence of the definition of Eulerian matroid.

Lemma 4.2. *Let M a binary matroid on E. Then M is Eulerian if and only if there is a matrix representing M, say A, in which has even non-zero entries in each rows.*

Proposition 4.1. *If a 3-fold matroid M'' of a binary matroid M is Eulerian, then M is also Eulerian matroid.*

Proof. Let $C_1 \cup C_2 ... \cup C_n$ be an Eulerian partition of $E(M'')$. Without loss of generality, let $a \in C_1$. Set $M_1 = M''/a$. Now $C_1 - \{a\}$ is a circuit of M_1. By Lemma 4.1, for any other circuit C_i, $2 \leq i \leq n$, C_i is a circuit of M_1 or is a union of two disjoint circuits of M_1. By replacing C_i, by the union of two disjoint circuits of M_1 in the last case, and C_1 by $C_1 - \{a\}$ in the above partition, we obtain an Eulerian partition for M_1.

By repeating this procedure, we have an Eulerian partition $C'_1 \cup C'_2 ... \cup C'_m$, for M, because $M = M''/\{a,b,c\}$. So M is Eulerian. □

Corollary 4.1. *Let M be a binary matroid and let M'' be a 3-fold of M. If M is not Eulerian then M'' can not be Eulerian.*

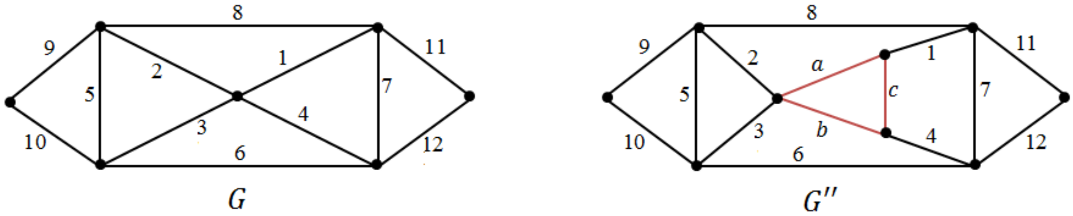

Figure 4.

The converse of Proposition 4.1 is not true.

Example 4.1. *Let $M = M(G)$ and $M^{''} = M(G^{''})$ be a 3-fold of M, w.r.t. $T_1 = \{2,3,4\}$ and $T_1^{'} = \{4\}$ (see Figure 4). It is clear that M is Eulerian but $M^{''}$ is not.*

In the following theorem we give a sufficient condition for the matroid $M^{''}$ to be Eulerian.

Theorem 4.1. *Let M be a Eulerian binary matroid and let M'' be a 3-fold of M, w.r.t. T_1' and T_1. If T_1 and T_1' has even cardinality, Then M'' is Eulerian matroid.*

Proof. Let A and A'' are the matrices that representing M and M'', respectively. Since M is Eulerian, so the rows of M'' corresponding to rows of A, have even non-zero entries. We consider the last two rows A''. The last row's have two 1's in columns corresponding to a, b, c and even non-zero entries in other columns. Now by lemma 4.2, the result holds. □

Corollary 4.2. *Let M be a binary matroid and let M'' be a 3-fold of M, w.r.t. T_1' and T_1 such that T_1' and T_1 has even cardinality. Then M is Eulerian if and only if M'' is Eulerian.*

5. Conclusion

The purpose of the current study was to determine another 3-connected matroid of a given 3-connected matroid by a 3-fold operation. Moreover, we examined the Eulerianity of the resulting matroid by this operation when the original matriod is Eulerian, and provided sufficient conditions through with the resulting matriod becomes Eulerian.

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