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MINIMAL RESTRAINED MONOPHONIC SETS IN GRAPHS

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a *restrained monophonic set* S of a graph G is a *monophonic set* such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the *restrained monophonic number* of G and is denoted by $m_r(G)$. A restrained monophonic set S of G is called a *minimal restrained monophonic set* if no proper subset of S is a restrained monophonic set of G. The *upper restrained monophonic number* of G, denoted by $m_r^+(G)$, is defined as the maximum cardinality of a minimal restrained monophonic set of G. We determine bounds for it and find the upper restrained monophonic number of certain classes of graphs. It is shown that for any two positive integers a, b with $2 \leq a \leq b$, there is a connected graph G with $m_r(G) = a$ and $m_r^+(G) = b$. Also, for any three positive integers a, b and n with $2 \leq a \leq n \leq b$, there is a connected graph G with $m_r(G) = a$, $m_r^+(G) = b$ and a minimal restrained monophonic set of cardinality n. If p, d and k are positive integers such that $2 \leq d \leq p - 2$, $k \geq 3$, $k \neq p - 1$ and $p - d - k \geq 0$, then there exists a connected graph G of order p, monophonic diameter d and $m_r^+(G) = k$.

Keywords: restrained monophonic set, restrained monophonic number, minimal restrained monophonic set, upper restrained monophonic number.

AMS Subject Classification: 05C12.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [3]. A *block* of a graph is a maximal nonseparable subgraph. An *end-block* of G is a block containing exactly one cut-vertex

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of G. The *distance* $d(x, y)$ between two vertices x and y in a connected graph G is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic* [1]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$, the monophonic number of a graph and its related parameters was studied and discussed in [2, 5, 8]. A *restrained monophonic set* S of a graph G is a *monophonic set* such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the *restrained monophonic number* of G and is denoted by $m_r(G)$. The restrained monophonic number of a graph was introduced and studied in [9].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. A vertex u in G is a *monophonic eccentric vertex* of a vertex v in G if $e_m(u) = d_m(u, v)$. The monophonic distance was introduced and studied in [6, 7]. These concepts have interesting applications in Channel Assignment Problem in FM radio technologies. The monophonic matrix is used to discuss different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical Chemistry. For more applications of these parameters, one may refer to [4] and the references therein.

The following theorems will be used in the sequel.

Theorem 1.1. [9] Each extreme vertex of a connected graph G belongs to every restrained monophonic set of G.

Theorem 1.2. [9] If T is a tree of order p with k endvertices and $p - k \geq 2$, then $m_r(T) = k$.

Theorem 1.3. [9] For the complete graph $K_p (p \geq 2)$, $m_r(K_p) = p$.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Upper Restrained Monophonic Number

Definition 2.1. *A restrained monophonic set S of G is called a minimal restrained monophonic set if no proper subset of S is a restrained monophonic set of G. The upper restrained monophonic number of G, denoted by $m_r^+(G)$, is defined as the maximum cardinality of a minimal restrained monophonic set of G.*

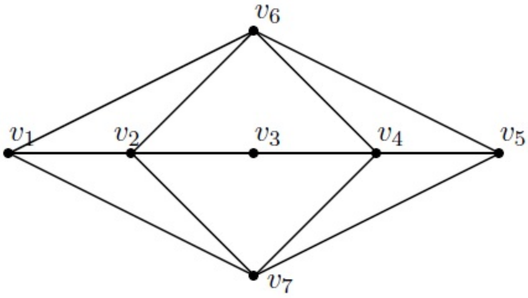

Figure 2.1: G

Example 2.1. *For the graph G given in Figure 2.1, the minimal restrained monophonic sets are $S_1 = \{v_1, v_5\}$ and $S_2 = \{v_3, v_6, v_7\}$. In this graph, the upper restrained monophonic number is 3 and the restrained monophonic number is 2.*

Note 2.1. *Every minimum restrained monophonic set is a minimal restrained monophonic set, and the converse need not be true. For the graph G given in Figure 2.1, S_2 is a minimal restrained monophonic set but it is not a minimum restrained monophonic set of G.*

Theorem 2.1. *Each extreme vertex of a connected graph G belongs to every minimal restrained monophonic set of G.*

Proof. Since every minimal restrained monophonic set of G is a restrained monophonic set of G, the theorem follows from Theorem 1.1. $\square$

Corollary 2.1. *For the complete graph K_p, $m_r^+(K_p) = p$.*

Remark 2.1. The converse of Corollary 2.1 need not be true. For the cycle C_4, no 2-element or 3-element subset $V(C_4)$ is a minimal restrained monophonic set of C_4. Thus, $V(C_4)$ is the unique minimal restrained monophonic set so that $m_r^+(C_4) = 4 = p$ and it is not a complete graph.

Theorem 2.2. *Let G be a connected graph with cutvertices and let S be a minimal restrained monophonic set of G. If v is a cutvertex of G, then every component of $G - v$ contains an element of S.*

Proof. Suppose that there is a component B of $G - v$ such that B contains no vertex of S. Let w be a vertex in B. Since S is a minimal restrained monophonic set of G, there exist vertices $x, y \in S$ such that w lies on some $x - y$ monophonic path $P : x = u_0, u_1, \ldots, w, \ldots, u_l = y$ in G. Let P_1 be the $x - w$ subpath of P and let P_2 be the $w - y$ subpath of P. Since v is a cutvertex of G, both P_1 and P_2 contains v so that P is not a path, which is a contradiction. Thus every component of $G - v$ contains an element of S. $\square$

Corollary 2.2. *Let G be a connected graph with cutvertices and let S be a minimal restrained monophonic set of G. Then every branch of G contains an element of S.*

Corollary 2.3. *For any tree T of order p with k-endvertices and $p - k \geq 2$, $m_r(T) = m_r^+(T) = k$.*

Proof. This follows from Theorems 1.2 and 2.1. $\square$

Since every end-block B is a branch of G at some cutvertex, it follows by Theorem 2.2 that every minimal restrained monophonic set of G contains at least one vertex from B that is not a cutvertex. Thus the following corollaries are consequences of Theorem 2.2 and Corollary 2.2.

Corollary 2.4. *If G is a connected graph with $k \geq 2$ end-blocks, then $m_r^+(G) \geq k$.*

Corollary 2.5. *If k is the maximum number of blocks to which a cutvertex in a graph G belongs, then $m_r^+(G) \geq k$.*

Theorem 2.3. *For any connected graph G, $2 \leq m_r(G) \leq m_r^+(G) \leq p$ except $m_r(G) = p-1$ and $m_r^+(G) = p-1$.*

Proof. It is clear from the definition of minimum restrained monophonic set that $m_r(G) \geq 2$. Since every minimal restrained monophonic set is a restrained monophonic set of G, $m_r(G) \leq m_r^+(G)$. It is clear that $V(G)$ induces a restrained monophonic set of G and $V(G) - \{z\}$ is not a restrained monophonic set of G for any vertex z in G. Hence $m_r^+(G) \leq p$, $m_r(G) \neq p-1$ and $m_r^+(G) \neq p-1$. □

Remark 2.2. *The bounds in Theorem 2.3 are sharp. For any non-trivial path P of order at least 4, $m_r(G) = 2$. It follows from Corollary 2.3 that for any tree T of order p with k-end vertices and $p-k \geq 2$, $m_r(T) = m_r^+(T) = k$. Also, by Corollary 2.1, $m_r^+(K_p) = p$.*

Theorem 2.4. *For any connected graph G, $m_r(G) = p$ if and only if $m_r^+(G) = p$.*

Proof. Let $m_r(G) = p$. Then by Theorem 2.3, $m_r^+(G) = p$. Conversely, let $m_r^+(G) = p$. Then $S = V(G)$ is the unique minimal restrained monophonic set of G. Since no proper subset of S is a restrained monophonic set, it is clear that S is the unique minimum restrained monophonic set of G and so $m_r(G) = p$. □

Theorem 2.5. *If G is a connected graph of order p with $m_r(G) = p-2$, then $m_r^+(G) = p-2$.*

Proof. Let $m_r(G) = p-2$. Then by Theorem 2.3, $m_r^+(G) = p-2$ or $m_r^+(G) = p$. If $m_r^+(G) = p$, then by Theorem 2.4, $m_r(G) = p$, which is a contradiction. Hence $m_r^+(G) = p-2$. □

Next, we determine the upper restrained monophonic number of some standard graphs.

Theorem 2.6. *For any cycle C_p $(p \geq 3)$,* $m_r^+(C_p) = \begin{cases} 3 & \text{if } p = 3 \text{ and } p \geq 5 \\ 4 & \text{if } p = 4. \end{cases}$

Proof. Let the cycle $C_p : v_1, v_2, \ldots, v_p, v_1$.
For $p = 3$, C_3 is complete, and by Corollary 2.1, $m_r^+(C_3) = 3$.
For $p = 4$, $m_r^+(C_4) = 4$ as seen in Remark 2.1.
For $p = 5$, it is clear that no 2-element subset of $V(C_5)$ is a restrained monophonic set of C_5. Any set of three consecutive vertices of C_5 is a minimal restrained monophonic set of C_5 and so $m_r^+(C_5) \geq 3$. It is clear that no subset S' of vertices with $|S'| \geq 4$ is a minimal restrained monophonic set and so $m_r^+(C_5) = 3$.
For $p \geq 6$, it is clear that the minimal restrained monophonic sets of C_p are either any sets $\{v_i, v_j\} (i \neq j)$ with $d(v_i, v_j) \geq 3$ or any set of three consecutive vertices of C_p. Hence it follows that $m_r^+(C_p) = 3$. □

Theorem 2.7. *For any wheel $W_p = K_1 + C_{p-1}$ $(p \geq 4)$,*

$$m_r^+(W_p) = \begin{cases} 4 & \text{if } p = 4 \\ 2 & \text{if } p \geq 5. \end{cases}$$

Proof. Let $W_p = K_1 + C_{p-1}$ be the wheel with $V(C_{p-1}) = \{v_1, v_2, \ldots, v_{p-1}\}$.
If $p = 4$, then W_4 is a complete graph, and so by Corollary 2.1, $m_r^+(W_p) = 4$.

If $p \geq 5$, then it is clear that any set of two non-adjacent vertices of C_{p-1} forms a minimal restrained monophonic set of W_p and so $m_r^+(W_p) \geq 2$. Now, let S be any restrained monophonic set of W_p such that $|S| \geq 3$. Then S contains at least two non-adjacent vertices $v_i, v_j (i \neq j)$ of C_{p-1} such that $S' = \{v_i, v_j\} \subset S$ so that S is not minimal. It follows that $m_r^+(W_p) = 2$. □

Theorem 2.8. *For the star $K_{1,p-1} (p \geq 2)$, $m_r^+(K_{1,p-1}) = p$.*

Proof. Since $V(K_{1,p-1})$ is the unique minimal restrained monophonic set of $K_{1,p-1}$, it follows that $m_r^+(K_{1,p-1}) = p$. □

Theorem 2.9. *For the complete bipartite graph $G = K_{m,n} (2 \leq m \leq n)$, $m_r^+(G) = \begin{cases} n+2 & \text{if } 2 = m \leq n \\ 4 & \text{if } 3 \leq m \leq n. \end{cases}$*

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the partite sets of G. If $m = n = 2$, then $G = K_{2,2}$ is a cycle of order 4 so that by Theorem 2.6, $m_r^+(G) = 4$. If $m = 2 < n$, then it is easily verified that V_1 and V_2 are the only two minimal monophonic sets. However, these are not restrained monophonic sets. Since no proper subset of $V_1 \cup V_2$ is a restrained monophonic set of G, it follows that $V_1 \cup V_2$ is the unique minimal restrained monophonic set of G and so $m_r^+(K_{2,n}) = n + 2$.

Now, if $m \geq 3$ and let $S = \{u_1, u_2, v_1, v_2\}$. Clearly, S is a minimal restrained monophonic set of G and so $m_r^+(G) \geq 4$. It is clear that any restrained monophonic set S of G must contain at least two vertices from each of V_1 and V_2. Now, any set formed by taking two vertices from V_1 and two vertices from V_2 is a restrained monophonic set of G. Hence it follows that any restrained monophonic set of cardinality at least 5 is not a minimal restrained monophonic set of G so that $m_r^+(G) = 4$. □

From the above results we observe that there are non-complete graphs G of order p with $m_r^+(G) = p$. This leads to the following open problem.

Problem 2.1. *Characterize the class of graphs G of order p for which $m_r^+(G) = p$.*

Theorem 2.10. *If x is an edge of K_p, then for the graph $G = K_p - x (p \geq 4)$, $m_r^+(G) = 2$.*

Proof. Let x be the edge $x = uv$. Since u and v are the only extreme vertices of G, by Theorem 2.1 every minimal restrained monophonic set of G contains S. It is clear that $S = \{u, v\}$ is a minimal restrained monophonic set so that $m_r^+(G) \geq 2$. Let S' be any restrained monophonic set of G such that $|S'| \geq 3$. Since u and v are extreme vertices, by Theorem 2.1, $u, v \in S'$ so that S' is not minimal. Hence $m_r^+(G) = 2$. □

3. Realization Results

In view of Theorem 2.3, we have the following realization result.

Theorem 3.1. *For any two positive integers a, b with $2 \leq a \leq b$, there is a connected graph G with $m_r(G) = a$ and $m_r^+(G) = b$.*

Proof. **Case 1.** $2 \leq a = b$. By Theorem 1.3 and Corollary 2.1, the complete graph of order a has the desired properties.

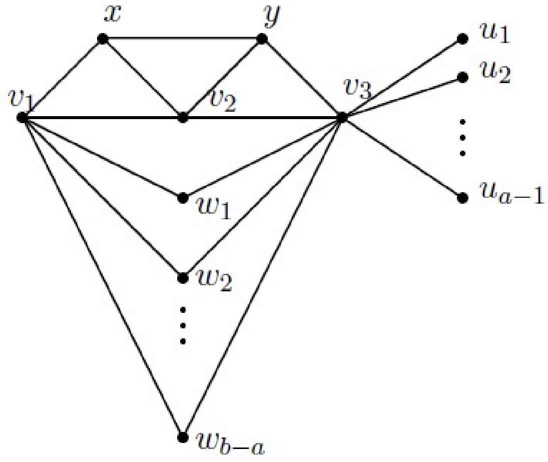

Figure 3.1: G

Case 2. $2 \leq a < b$. Let H be the graph obtained from the path $P_3 : v_1, v_2, v_3$ of order 3 by adding $b-1$ new vertices $w_1, w_2, ..., w_{b-a}, u_1, u_2, \cdots, u_{a-1}$ and joining $w_i (1 \leq i \leq b-a)$ to the vertices v_1 and v_3; and by joining $u_j (1 \leq j \leq a-1)$ to the vertex v_3. The graph G is obtained from H and the path $P_2 : x, y$ of order 2 by joining the vertex x with the vertices v_1 and v_2; and joining the vertex y with the vertices v_2 and v_3. The graph G is shown in Figure 3.1. Let $S = \{u_1, u_2, \cdots, u_{a-1}\}$ be the set of all endvertices of G. By Theorem 1.1, every restrained monophonic set of G contains S. Clearly, S is not a restrained monophonic set of G. Let $S_1 = S \cup \{v_1\}$. It is easily verified that S_1 is a restrained monophonic set of G and so $m_r(G) = a$.

Next we show that $m_r^+(G) = b$. Clearly $T = S \cup \{y, w_1, w_2, \ldots, w_{b-a}\}$ is a restrained monophonic set of G. We claim that T is a minimal restrained monophonic set of G. Let W be any proper subset of T. Then there exists a vertex, say v, such that $v \in T$ and $v \notin W$. By Theorem 2.1, $v \in \{y, w_1, w_2, \ldots, w_{b-a}\}$. It is easily verified that v is not an internal vertex of any $s-t$ monophonic path for some $s, t \in W$, it follows that W is not a restrained monophonic set of G. Hence T is a minimal restrained monophonic set of G and so $m_r^+(G) \geq b$.

Suppose that $m_r^+(G) > b$. Let M be a minimal restrained monophonic set of G with $|M| > b$. Then there exists at least one vertex, say, $v \in M$ such that $v \notin T$. Thus $v \in \{v_1, v_2, v_3, x\}$. If $v \in \{x, v_1\}$, then $M_1 = S \cup \{v\}$ is a restrained monophonic set of G and also it is a proper subset of M, which is a contradiction to M a minimal restrained monophonic set of G. Hence $v \in \{v_2, v_3\}$. If $v = v_2 \in M$ and $v_3 \notin M$, then $M - \{v\} = T$ is a restrained monophonic set of G and also it is a proper subset of M, which is a contradiction to M a minimal restrained monophonic set of G. Similarly, if $v = v_3 \in M$ and $v_2 \notin M$, we get a contradiction. If both $v_2, v_3 \in M$ and $T \subseteq M$, then M is either $(T - \{y\}) \cup \{v_2, v_3\}$ or $(T - \{w_i\}) \cup \{v_2, v_3\}$ $(1 \leq i \leq b-a)$. It is clear that M is not a monophonic set of G, which is a contradiction. If both $v_2, v_3 \in M$ and $T \subset M$, then T is a restrained monophonic set of G and also it is a proper subset of M, which is a contradiction. Thus there is no minimal restrained monophonic set M of G with $|M| > b$. Hence $m_r^+(G) = b$. □

Theorem 3.2. *For any three positive integers a, b and n with $2 \leq a \leq n \leq b$, there is a connected graph G with $m_r(G) = a$, $m_r^+(G) = b$ and a minimal restrained monophonic set of cardinality n.*

Proof. We consider four cases.

Case 1. $a = n = b$. Let G be the complete graph with a vertices. Then by Theorem 1.3 and Corollary 2.1, $m_r(G) = m_r^+(G) = a$ and $V(G)$ is the minimal restrained monophonic set of G.

Case 2. $a = n < b$. For the graph G given in Figure 3.1 of Theorem 3.1, it is proved that $m_r(G) = a, m_r^+(G) = b$ and $S = \{v_1, u_1, \ldots, u_{a-1}\}$ is a minimal restrained monophonic set of cardinality n.

Case 3. $a < n = b$. For the graph G given in Figure 3.1 of Theorem 3.1, it is proved that $m_r(G) = a$, $m_r^+(G) = b$ and $S = \{u_1, u_2, \ldots, u_{a-1}, w_1, w_2, \ldots, w_{b-a}, y\}$ is a minimal restrained monophonic set of cardinality n.

Case 4. $a < n < b$. Let $l = n - a + 1$ and $m = b - n + 1$. Let H_1 be the graph obtained from the path $P_{1,3} : v_{1,1}, v_{1,2}, v_{1,3}$ of order 3 by adding $l - 1$ new vertices $w_1, w_2, ..., w_{l-1}$ and joining $w_i (1 \leq i \leq l - 1)$ to the vertices $v_{1,1}$ and $v_{1,3}$. The graph G_1 is obtained from H_1 and the path $P_{1,2} : x_1, y_1$ by joining the vertex x_1 to both $v_{1,1}$ and $v_{1,2}$; and joining the vertex y_1 to both $v_{1,2}$ and $v_{1,3}$. Similarly, let H_2 be the graph obtained from the path $P_{2,3} : v_{2,1}, v_{2,2}, v_{2,3}$ of order 3 by adding $m - 1$ new vertices $v_1, v_2, ..., v_{m-1}$ and joining $v_j (1 \leq j \leq m - 1)$ to the vertices $v_{2,1}$ and $v_{2,3}$. The graph G_2 is obtained from H_2 and the path $P_{2,2} : x_2, y_2$ by joining the vertex x_2 to both $v_{2,1}$ and $v_{2,2}$; and joining the vertex y_2 to both $v_{2,2}$ and $v_{2,3}$. The graph G is obtained from G_1 and G_2 by identifying the vertices $v_{1,3}$ and $v_{2,1}$ (namely x); and also by adding $a - 2$ new vertices $u_1, u_2, \cdots, u_{a-2}$ and joining these vertices to the vertex x. The graph G is shown in Figure 3.2. Let $S = \{u_1, u_2, \cdots, u_{a-2}\}$ be the set of all endvertices of G. By Theorem 1.1, every restrained monophonic set of G contains S. Clearly, S is not a restrained monophonic set of G. Also, for any $v \in V(G) - S$, $S \cup \{v\}$ is not a restrained monophonic set of G. Let $S_1 = S \cup \{v_{1,1}, v_{2,3}\}$. It is easily verified that S_1 is a restrained monophonic set of G and so $m_r(G) = a$.

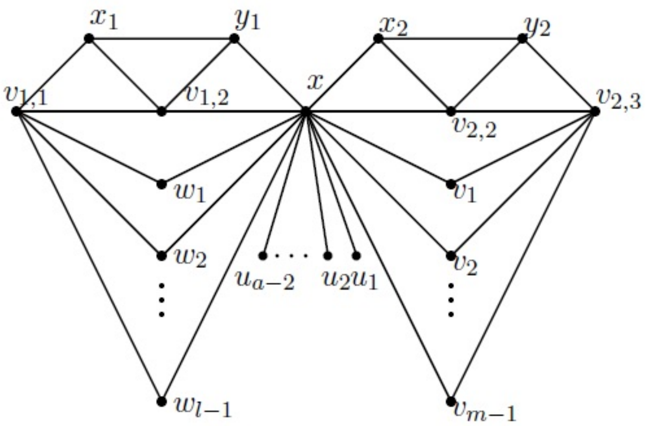

Figure 3.2: G

Next, we show that $m_r^+(G) = b$. Let $T = S \cup \{v_1, v_2, \cdots, v_{m-1}, w_1, w_2, \cdots, w_{l-1}, y_1, y_2\}$. It is clear that T is a restrained monophonic set of G. First, we claim that T is a minimal restrained monophonic set of G. Let W be any proper subset of T. Then there exists a vertex, say, $y \in T$ such that $y \notin W$. By Theorem 2.1, $y \in \{v_1, v_2, \cdots, v_{m-1}, w_1, w_2, \cdots, w_{l-1}, y_1, y_2\}$. It is clear that the vertex y is not an internal vertex of any monophonic path joining a pair of vertices in W. Hence W is not a monophonic set of G and so W is not a restrained monophonic set of G. Thus T is a minimal restrained monophonic set of G so that $m_r^+(G) \geq b$.

Now, we prove that $m_r^+(G) = b$. Suppose that $m_r^+(G) > b$. Let T' be a minimal restrained monophonic set of G with $|T'| > b$. Then there exists at least one vertex, say, $v \in T'$ such that $v \notin T$. Also, by Theorem 2.1, $v \in \{x, x_1, x_2, v_{1,1}, v_{1,2}, v_{2,2}, v_{2,3}\}$. If $v = x_1$ or $v = v_{1,1}$, then $T' - \{w_1, w_2, \cdots, w_{l-1}\}$ is a restrained monophonic set of G and it is a proper subset of T', which is a contradiction to T' a minimal restrained monophonic set of G. If $v = x_2$ or $v = v_{2,3}$, then $T' - \{v_1, v_2, \cdots, v_{m-1}\}$ is a restrained monophonic set of G and it is a proper subset T', which is a contradiction. Similarly, if $v \in \{x, v_{1,2}, v_{2,2}\}$, then $T' - \{v\}$ is a restrained monophonic set of G and it is a proper subset T', which is a contradiction. Hence $m_r^+(G) = b$.

Next we show that there is a minimal restrained monophonic set of cardinality n. Let $M = S \cup \{v_{2,3}, y_1, w_1, w_2, \cdots, w_{l-1}\}$. It is clear that M is a restrained monophonic set of G. We claim that M is a minimal restrained monophonic set of G. Assume, to the contrary, that M is not a minimal restrained monophonic set of G. Then there is a proper subset M' of M such that M' is a restrained monophonic set of G. Let $v \in M$ and $v \notin M'$. By Theorem 2.1, clearly $v \in M - S$. It is clear that the vertex v is not an internal vertex of any $s - t$ monophonic path for some $s, t \in M'$, which is a contradiction. Thus M is a minimal restrained monophonic set of G with cardinality n. Hence the theorem. □

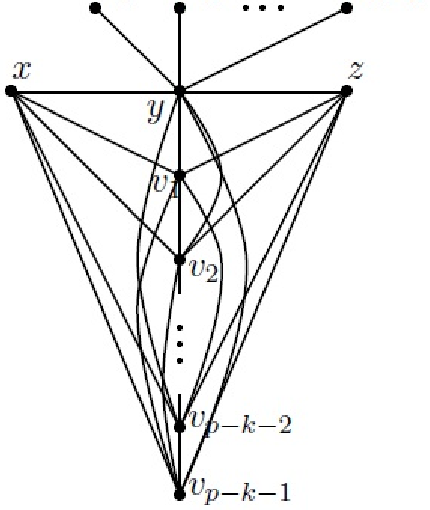

Figure 3.3: G

Theorem 3.3. *If* p, d *and* k *are positive integers such that* $2 \leq d \leq p-2$, $k \geq 3$, $k \neq p-1$ *and* $p-d-k \geq 0$, *then there exists a connected graph* G *of order* p, *monophonic diameter* d *and* $m_r^+(G) = k$.

Proof. We prove this theorem by considering two cases.
Case 1. $d = 2$ and $k \geq 3$. Let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained by adding $p - 3$ new vertices $v_1, v_2, \ldots, v_{p-k-1}, w_1, w_2, \ldots, w_{k-2}$ to P_3 and joining each $w_i (1 \leq i \leq k-2)$ to y; and joining each $v_i (1 \leq i \leq p-k-1)$ with x, y and z; and joining each $v_i (1 \leq i \leq p-k-2)$ with $v_j (i+1 \leq j \leq p-k-1)$. The graph G of order p is shown in Figure 3.3. It is clear that, for any vertex u in G, $1 \leq e_m(u) \leq 2$ and $e_m(x) = 2$ so that the monophonic diameter of G is 2. Let $S = \{w_1, w_2, w_3, \ldots, w_{k-2}, x, z\}$ be the set of all extreme vertices of G. By Theorem 2.1, every minimal restrained monophonic set of

G contains S. It is easily verified that S is the unique minimal restrained monophonic set of G so that $m_r^+(G) = k$.

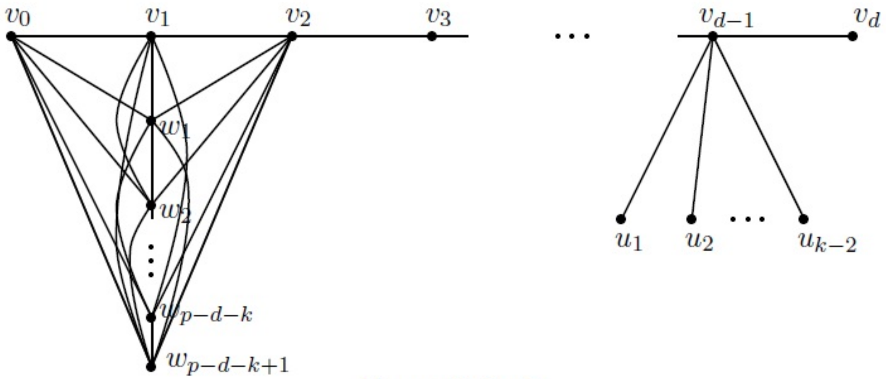

Figure 3.4: G

Case 2. $3 \leq d \leq p-2$ and $k \geq 3$. Let $P_{d+1} : v_0, v_1, \ldots, v_d$ be a path of length d. Let G be the graph obtained from P_{d+1} by adding $p-d-1$ new vertices $u_1, u_2, \ldots, u_{k-2}, w_1, w_2, \ldots, w_{p-d-k+1}$ to P_{d+1} and joining each $u_i (1 \leq i \leq k-2)$ with v_{d-1}; and joining each $w_j (1 \leq j \leq p-d-k+1)$ with v_0, v_1 and v_2; and joining each $w_i (1 \leq i \leq p-d-k)$ with $w_j (i+1 \leq j \leq p-d-k+1)$. The graph G of order p is shown in Figure 3.4. It is clear that, for any vertex x in G, $3 \leq e_m(x) \leq d$ and $e_m(v_0) = e_m(v_d) = e_m(u_i) = d (1 \leq i \leq k-2)$. Thus the monophonic diameter of G is d. Let $S = \{u_1, u_2, \ldots, u_{k-2}, v_0, v_d\}$ be the set of all extreme vertices of G. By Theorem 2.1, every minimal restrained monophonic set of G contains S. It is easily verified that S is the unique minimal restrained monophonic set of G so that $m_r^+(G) = k$. □

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