TWMS J. App. and Eng. Math. V.11, N.3, 2021, pp. 804-813

COMMON FIXED POINT THEOREMS UNDER RATIONAL CONTRACTIONS USING TWO MAPPINGS AND SIX MAPPINGS AND COUPLED FIXED POINT THEOREM IN BICOMPLEX VALUED b-METRIC SPACE

I. H. JEBRIL[1], S. K. DATTA[2], R. SARKAR[3], N. BISWAS[4], §

Abstract. During the past decades, enormous works by different researchers have been carried out in fixed point theory on metric spaces. In this paper, we prove some common fixed point theorems in bicomplex valued metric space for two mappings and for six mappings. Also, we have introduced the concept of bicomplex valued b-metric space and coupled fixed point theorem in bicomplex valued b-metric space.

Keywords: Coupled fixed point, Contractive type mapping, Complete bicomplex valued metric space, Complex valued b-metric space, Compatible mappings.

AMS Subject Classification: 30D30, 30D35, 37C25, 47H10.

1. Introduction

Fixed point theory is one of the famous and traditional theories in Mathematics in which contraction is one of the main tools to prove the existence and uniqueness of a fixed point.

The Banach Contraction principle [10] is a very popular and effective tool in solving existence problems and it is an active area of research since 1922. Due to simplicity and usefulness of this classic and celebrated theorem, it has become a very popular source of existence and uniqueness theorem in different branches of mathematical analysis. This theorem provides an impressive illustration of the unifying power of functional analytic methods and their usefulness in various disciplines. The famous Banach theorem [10] was stated as "*Let (X, d) be a complete metric space and T be a mapping of X into itself*

[1] Department of Mathematics, Al-Zaytoonah University of Jordan, Amman Jordan.
e-mail: i.jebril@zuj.edu.jo; ORCID: https://orcid.org/0000-0003-4348-6197.
[2] Department of Mathematics, University of Kalyani, Kalyani, Nadia, West Bengal-741235, India.
e-mail: sanjibdatta05@gmail.com; ORCID: https://orcid.org/0000-0002-0905-8933.
[3] Gour Mahavidyalaya, P.O. Mangalbari, Dist., Malda, West Bengal, India.
e-mail: rakeshsarkar.malda@gmail.com; ORCID: https://orcid.org/0000-0001-7453-9023.
[4] Department of Mathematics, Chakdaha College, Chakdaha, Nadia, West Bengal-741222, India.
e-mail: nityamaths@gmail.com; ORCID: https://orcid.org/0000-0001-9230-1457.
§ Manuscript received: June 12, 2019; accepted: : October 11, 2019.

The second author acknowledges that his research work has been supported by the DST PURSE Programme of University of Kalyani, India.

satisfying $d(Tx,Ty) \leq kd(x,y)$, $\forall\ x,y \in X$, *where* k *is a constant in* $(0,1)$. *Then,* T *has a unique fixed point* $x^* \in X$."

Already there have been a number of generalizations of metric spaces such as, rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces.

During the last fifty years, fixed point theories with the treatment of complex valued metric spaces are emerging areas of works in the field of complex analysis as well as functional analysis. As an extension of complex valued metric spaces, one may think about bicomplex valued metric space. Naturally, the study of fixed point theory under the umbrella of bi complex analysis may be regarded as a virgin area of research and is still at an infancy stage. In 2011, Azam et. al [3] introduced the concept of complex valued metric space. Since then several authors studied the existence and uniqueness of fixed points in complex valued spaces (see [1, 2, 4, 9]). Some authors have proved the common fixed point of mappings satisfying rational inequality in complex valued metric space in [6] and a common fixed point result in complex valued metric spaces under contractive condition. The concept of the coupled fixed point was first introduced by Bhaskar and Laxikantham [5] in 2006. Some coupled fixed point theorems in complex valued metric space have been proved (see [8]).

The partial order relation $\preceq$ on the set of complex number $\mathbb{C}$ is defined as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re\,(z_2) \text{ and } Im\,(z_1) \leq Im\,(z_2)\,.$$

Thus $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$, (ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$, (iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$, (iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of $(ii), (iii)$ and (iv) is satisfied and we write $z_1 \prec z_2$ if only (iv) is satisfied.

Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}$ as follows:

(1) If $0 \precsim z_1 \precsim z_2$ then $|z_1| < |z_2|$;

(2) If $z_1 \precsim z_2,\ z_2 \precsim z_3$ then $z_1 \precsim z_3$ and

(3) If $z_1 \precsim z_2$ and $\lambda > 0$ is a real number then $\lambda z_1 \precsim z_2$.

The set of bicomplex numbers is defined as

$$\mathbb{C}_2 = \{w : w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3,\ p_k \in \mathbb{R},\ 0 \leq k \leq 3\},$$

or,

$$\mathbb{C}_2 = \{w = z_1 + i_2 z_2 \mid z_1, z_2 \in \mathbb{C}\},$$

where $z_1 = p_0 + i_1 p_1,\ z_2 = p_2 + i_1 p_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The inverse of $u = u_1 + i_2\, u_2$ exists if $u_1^2 +\ u_2^2 \neq 0$ i.e., if $\left|u_1^2 +\ u_2^2\right| \neq 0$ and it is defined as

$$u^{-1} = \frac{1}{u} = \frac{u_1 - i_2\, u_2}{u_1^2 +\ u_2^2},$$

and then u is called invertible.

Recently, J. Choi et. al. [7] have calculated and defined the conjugate of complex number, the bicomplex valued metric space and also defined the partial order relation $\precsim_{i_2}$ and a norm $\|\|$ on $\mathbb{C}_2$.

Example 1.1. *Consider* $X = \mathbb{C}$ *and a mapping* $d : X \times X \to \mathbb{C}_2$ *by* $d(z_1, z_2) = i_2\,|z_1 - z_2|\,,\ z_1, z_2 \in X$ *where* $|\ .\ |$ *is the complex modulus. One can easily check that* (X, d) *is a bicomplex valued metric space.*

Definition 1.1. *Two self mappings S, T on a bicomplex valued metric space X are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$ for all $x \in X$.*

Definition 1.2. *Let $\{z_n\}$ be a sequence in a bicomplex space X and $z \in X$. If for every $c \in \mathbb{C}_2$ with $0 < c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(z_n, z) < c$, then z is called the limit of $\{z_n\}$ and we write $\lim_{n\to\infty} z_n = z$ or, $z_n \to z$ as $n \to \infty$.*

Definition 1.3. *If every Cauchy sequence is convergent in a bicomplex valued metric space (X, d), then (X, d) is called a complete bicomplex valued metric space.*

Definition 1.4. *Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to X$ is called a bicomplex valued b-metric if for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $0 \precsim_{i_2} d(x, y)$, for all $x, y \in X$; (ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$ and (iv) $d(x, y) \precsim_{i_2} s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a bicomplex valued b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Definition 1.5. *An element $(z, z') \in X \times X$ is called a coupled fixed point of the mapping $S : X \times X \to X$ if*

$$S(z, z') = z \text{ and } S(z', z) = z'.$$

2. Lemmas

In this section, we present some lemmas, which will be needed in the sequel.

Lemma 2.1. [3] *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.2. *Let (X, d) be a bicomplex valued metric space and let $\{z_n\}$ be a sequence in X, then $\{z_n\}$ converges to z in $\mathbb{C}_2$ if and only if $\|d(z_n, z)\| \to 0$ as $n \to \infty$.*

Proof. Take $X = \mathbb{C}$. Here $z_n, z \in \mathbb{C}_2$; $n \in \mathbb{Z}_+$. Therefore $z_n = z_{n_1} + i_2 z_{n_2}$ and $z = z_1 + i_2 z_2$, where $z_{n_1}, z_{n_2}, z_1, z_2 \in \mathbb{C}$. It follows that the sequence

$$\|d(z_n, z)\| = \|z_n - z\| = \|(z_{n_1} - z_1) + i_2(z_{n_2} - z_2)\|$$

$$= \left(|z_{n_1} - z_1|^2 + |z_{n_2} - z_2|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

Now the sequence $\{z_n\}$ converges to the point z in $\mathbb{C}_2$ if and only if the sequences $\{z_{n_1}\}$ and $\{z_{n_2}\}$ converges to the points z_1 and z_2 respectively in $\mathbb{C}$. Also by Lemma 2.1, the sequences $\{z_{n_1}\}$ and $\{z_{n_2}\}$ converges to the points z_1 and z_2 respectively if and only if

$$|d(z_{n_1}, z_1)| \to 0 \text{ and } |d(z_{n_2}, z_2)| \to 0 \text{ as } n \to \infty,$$

$$\text{i.e., } |z_{n_1} - z_1| \to 0 \text{ and } |z_{n_2} - z_2| \to 0 \text{ as } n \to \infty. \tag{2.2}$$

Therefore by Equations (2.1) and (2.2), we can conclude that the sequence $\{z_n\}$ converges to the point z in $\mathbb{C}_2$ if and only if $\|d(z_n, z)\| \to 0$ as $n \to \infty$. This completes the proof of the lemma. $\square$

3. Main Results

Theorem 3.1. *Let (X, d) be a bicomplex valued b-metric space with coefficient $s \geq 1$. Let S, T, f and g be self mappings of X such that*

(i) The pairs $\{S, f\}$ and $\{T, g\}$ are weakly compatible,

(ii) $TX \subseteq fX$ and $SX \subseteq gX$,

(iii) fX or gX is a complete subspace of X and

(iv) $d(Sz, Tz') \precsim_{i_2} \lambda d(fz, gz') + \frac{\mu d(z,Sz)d(gz',Tz')}{1+d(fz,gz')}, \forall\, z, z' \in X$, where λ, μ are non-negative real numbers with $s\lambda + \mu < 1$.

Then S, T, f and g have a unique common fixed point.

Proof. Let $z_0 \in X$ be arbitrary. Using the condition (ii), we define a sequence $\{z'_n\}$ in X as $z'_{2k+1} = gz_{2k+1} = Sz_{2k}$ and $z'_{2k+2} = fz_{2k+2} = Tz_{2k+1},\ k = 0, 1, 2, ...,$ then we have

$$\begin{aligned} d(z'_{2k+1}, z'_{2k+2}) &= d(Sz_{2k}, Tz_{2k+1}) \\ &\precsim_{i_2} \lambda d(fz_{2k}, gz_{2k+1}) + \mu \frac{d(fz_{2k}, Sz_{2k})d(gz_{2k+1}, Tz_{2k+1})}{1 + d(fz_{2k}, gz_{2k+1})} \\ &= \lambda d(z'_{2k}, z'_{2k+1}) + \mu \frac{d(z'_{2k}, z'_{2k+1})d(z'_{2k+1}, z'_{2k+2})}{1 + d(z'_{2k}, z'_{2k+1})} \\ &\precsim_{i_2} \lambda d(z'_{2k}, z'_{2k+1}) + \mu d(z'_{2k+1}, z'_{2k+2}) \end{aligned}$$

$$i.e.,\ d(z'_{2k+1}, z'_{2k+2}) \precsim_{i_2} \frac{\lambda}{1-\mu} d(z'_{2k}, z'_{2k+1}). \tag{3.1}$$

Similarly, we can get

$$d(z'_{2k+2}, z'_{2k+3}) \precsim_{i_2} \frac{\lambda}{1-\mu} d(z'_{2k+2}, z'_{2k+1}). \tag{3.2}$$

Now put $h = \frac{\lambda}{1-\mu}$. From $0 \leq s\lambda + \mu < 1,\ s \geq 1, and\ \lambda + \mu < 1$ we get $0 \leq h < 1$. Thus using (3.1) and (3.2) for $n \in \mathbb{N}$, we get that

$$d(z'_n, z'_{n+1}) \precsim_{i_2} hd(z'_{n-1}, z'_n) \precsim_{i_2} h^2 d((z'_{n-2}, z'_{n-1}) \precsim_{i_2} ... \precsim_{i_2} h^{n-1} d(z'_1, z'_2).$$

So for $m, n \in \mathbb{N}$, we have

$$\begin{aligned} d(z'_n, z'_{n+m}) &\precsim_{i_2} s[d(z'_n, z'_{n+1}) + d(z'_{n+1}, z'_{n+m})] \\ &\precsim_{i_2} sd(z'_n, z'_{n+1}) + s^2[d(z'_{n+1}, z'_{n+2}) + d(z'_{n+2}, z'_{n+m})] \\ &\precsim_{i_2} sd(z'_n, z'_{n+1}) + s^2 d(z'_{n+1}, z'_{n+2}) + s^3[d(z'_{n+2}, z'_{n+3}) + d(z'_{n+3}, z'_{n+m})] \\ &\precsim_{i_2} sd(z'_n, z'_{n+1}) + s^2 d(z'_{n+1}, z'_{n+2}) + ... + s^{m-1} d(z'_{n+m-2}, z'_{n+m-1}) + s^{m-1} d(z'_{n+m-1}, z'_{n+m}) \\ &\precsim_{i_2} sd(z'_n, z'_{n+1}) + s^2 d(z'_{n+1}, z'_{n+2}) + ... + s^{m-1} d(z'_{n+m-2}, z'_{n+m-1}) + s^m d(z'_{n+m-1}, z'_{n+m}). \end{aligned}$$

Since $s \geq 1$, therefore we get

$$\begin{aligned} d(z'_n, z'_{n+m}) &\precsim_{i_2} sh^{n-1} d(z'_1, z'_2) + s^2 h^n d(z'_1, z'_2) + ... + s^{m-1} h^{n+m-3} d(z'_1, z'_2) + s^m h^{n+m-2} d(z'_1, z'_2) \\ &\precsim_{i_2} sh^{n-1}[1 + sh + s^2h^2 + ... + s^{m-1}h^{m-1}] d(z'_1, z'_2) \precsim_{i_2} \frac{sh^{n-1}}{1 - sh} d(z'_1, z'_2), \end{aligned}$$

$$i.e.,\ \left\| d(z'_n, z'_{n+m}) \right\| \leq \frac{sh^{n-1}}{1 - sh} \left\| d(z'_1, z'_2) \right\| \to 0 \text{ as } n \to 0, \text{ where } m \in \mathbb{N}.$$

Hence $\{z'_n\}$ is a Cauchy sequence in X. Since X is complete there exists a $z \in X$ such that $z'_n \to z$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} Sz_{2n} = \lim_{n\to\infty} gz_{2n+1} = \lim_{n\to\infty} Tz_{2n+1} = \lim_{n\to\infty} fz_{2n+2} = z. \tag{3.3}$$

Now if fX is a complete subspace of X. Therefore $\exists$ a $u \in X$ such that $fu = z$.

From the condition (iv), we have

$$d(Su, z) \precsim_{i_2} sd(Su, Tz_{2n+1}) + sd(Tz_{2n+1}, z)$$

$$\precsim_{i_2} s\left[\lambda d(fu, gz_{2n+1}) + \mu \frac{d(fu, Su)d(gz_{2n+1}, Tz_{2n+1})}{1 + d(fu, gz_{2n+1})}\right] + sd(Tz_{2n+1}, z)$$

$$= s\left[\lambda d(fu, z'_{2n+1}) + \mu \frac{d(fu, Su)d(z'_{2n+1}, z'_{2n+2})}{1 + d(fu, z'_{2n+1})}\right] + sd(z'_{2n+2}, z).$$

Therefore we have

$$\begin{aligned} \|d(Su, z)\| \quad &\leq \quad s\left[\lambda \left\|d(fu, z'_{2n+1})\right\| + \mu \frac{\sqrt{2}\,\|d(fu, Su)\| \left\|d(z'_{2n+1}, z'_{2n+2})\right\|}{\left\|1 + d(fu, z'_{2n+1})\right\|}\right] \\ &\quad +s\left\|d(z'_{2n+2}, z)\right\|. \end{aligned}$$

Letting $n \to \infty$ and using (3.3) and Lemma 2.2, we get that $\|d(Su, z\| \leq 0$. Thus $\|d(Su, z\| = 0$. i.e., $d(Su, z) = 0$ and hence $Su = z$. Since $SX \subseteq gX$, there exists $v \in X$ such that $gv = z$.

Again from condition (iv), we have

$$d(z, Tv) = d(Su, Tv) \precsim_{i_2} \lambda d(fu, gv) + \mu \frac{d(u, Su)d(gv, Tv)}{1 + d(fu, gv)} = 0.$$

Thus $d(z, Tv) = 0$ and hence $Tv = z$. Thus $fu = Su = z = gv = Tv$. Since f and S are weakly compatible so $fz = fSu = Sfu = Sz$. Now we will show that $Sz = z$.

From condition (iv), we get that

$$d(Sz, z) = d(Sz, Tv) \precsim_{i_2} \lambda d(fz, gv) + \mu \frac{d(fz, Sz)d(gv, Tv)}{1 + d(fz, gv)} = \lambda d(Sz, z).$$

Thus $(1 - \lambda)\,\|d(Sz, z)\| \leq 0$. Which shows that $d(Sz, z) = 0$ and hence $Sz = z$.

Similarly, since g and T are weakly compatible, therefore $gz = gTv = Tgv = Tz$. Also

$$d(z, Tz) = d(Sz, Tz) \precsim_{i_2} \lambda d(fz, gz) + \mu \frac{d(fz, Sz)d(gz, Tz)}{1 + d(fz, gz)} = \lambda d(z, Tz).$$

Thus $d(z, Tz) = 0$ and hence $Tz = z$. Therefore $Sz = fz = gz = Tz = z$ i.e., z is a common fixed point of four mappings S, T, f and g. Now we show that z is the unique common fixed point.

Let $z^* \in X$ be another common fixed point of four mappings S, T, f and g. Then we have $fz^* = Sz^* = gz^* = Tz^* = z^*$. Again by (iv), we have

$$d(z, z^*) = d(Sz, Tz^*) \precsim_{i_2} \lambda d(fz, gz^*) + \mu \frac{d(fz, Sz)d(gz^*, Tz^*)}{1 + d(fz, gz^*)} = \lambda d(z, z^*).$$

Thus $d(z, z^*) = 0$ and so $z = z^*$. Therefore z is the unique common fixed point of S, T, f and g.

If gX is complete, we can similarly prove the theorem. This completes the proof of the theorem. □

Theorem 3.2. *Let (X, d) be a bicomplex valued metric space and P, Q, R, S, T and U be six self mappings of X satisfying the conditions (i) $TU(X) \subseteq P(X)$ and $RS(X) \subseteq Q(X)$ and (ii) $d(RSz, TUz') \precsim_{i_2} ad(Pz, Qz') + b(d(Pz, RSz) + d(Qz', TUz')) + c(d(Pz, TUz') + d(Qz', RSz))$, for all $z, z' \in X$ where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. Assume that pairs (TU, Q) and (RS, P) are weakly compatible. Pairs (T, U), (T, Q), (U, Q), (R, S), (R, P)*

and (S,P) are commuting pairs of maps. Then T,U,R,S,Q and P have a unique common fixed point in X.

Proof. Let $z_0 \in X$. Then by (i) we can define inductively a sequence $\{z'_n\}$ in X such that $z'_{2n} = RSz_{2n} = Qz_{2n+1}$ and $z'_{2n+1} = TUz_{2n+1} = Pz_{2n+2}$ for all $n = 1,2,3,...$, then by (ii), we have

$$\begin{aligned}
d(z'_{2n}, z'_{2n+1}) &= d(RSz_{2n}, TUz_{2n+1}) \\
&\precsim_{i_2} ad(Pz_{2n}, Qz_{2n+1}) + b\left(d(Pz_{2n}, RSz_{2n}) + d(Qz_{2n+1}, TUz_{2n+1})\right) \\
&\quad +c(d(Pz_{2n}, TUz_{2n+1}) + d(Qz_{2n+1}, RSz_{2n})) \\
&= ad(z'_{2n-1}, z'_{2n}) + b(d(z'_{2n-1}, z'_{2n}) + d(z'_{2n}, z'_{2n+1})) + c(d(z'_{2n-1}, z'_{2n+1}) + d(z'_{2n}, z'_{2n})) \\
&\precsim_{i_2} (a+b+c)d(z'_{2n-1}, z'_{2n}) + (b+c)d(z'_{2n}, z'_{2n+1}),
\end{aligned}$$

$$i.e.,\ d(z'_{2n}, z'_{2n+1}) \precsim_{i_2} \frac{a+b+c}{1-b-c} d(z'_{2n-1}, z'_{2n}) = kd(z'_{2n-1}, z'_{2n}),\ k = \frac{a+b+c}{1-b-c}.$$

Similarly, we obtain that

$$d(z'_{2n+1}, z'_{2n+2}) \precsim_{i_2} kd(z'_{2n}, z'_{2n+1}).$$

Therefore, we have

$$\begin{aligned}
d(z'_n, z'_m) &\precsim_{i_2} d(z'_n, z'_{n+1}) + d(z'_{n+1}, z'_{n+2}) + ... + d(z'_{m-1}, z'_m) \\
&\precsim_{i_2} (k^n + k^{n+1} + ... + k^{m-1}) d(z'_1, z'_0) \precsim_{i_2} \frac{k^n}{1-k} d(z'_1, z'_0),
\end{aligned}$$

$$i.e.,\ \left\|d(z'_n, z'_m)\right\| \leq \frac{k^n}{1-k} \left\|d(z'_1, z'_0)\right\|.$$

Which implies that $\|d(z'_n, z'_m)\| \to 0$ as $n,m \to \infty$. Hence $\{z'_n\}$ is a Cauchy sequence.

Since X is complete, there exists a point $p \in X$ such that

$$\lim_{n\to\infty} RSz_{2n} = \lim_{n\to\infty} Qz_{2n+1} = \lim_{n\to\infty} TUz_{2n+1} = \lim_{n\to\infty} Pz_{2n+2} = p.$$

Since $TU(X) \subseteq P(X)$, there exists a point $u \in X$ such that $p = Pu$. Then by (ii), we have

$$\begin{aligned}
d(RSu, p) &\precsim_{i_2} d(RSu, TUz_{2n-1}) + d(TUz_{2n-1}, p) \\
&\precsim_{i_2} ad(Pu, Qz_{2n-1}) + b(d(Pu, RSu) + d(Qz_{2n-1}, TUz_{2n-1})) \\
&\quad +c(d(Pu, TUz_{2n-1}) + d(Qz_{2n-1}, RSu)) + d(TUz_{2n-1}, p).
\end{aligned}$$

Taking the limit as $n \to \infty$, we obtain that

$$d(RSu,p) \precsim_{i_2} ad(p,p) + b(d(p,RSu) + d(p,p)) + c(d(p,p) + d(p,RSu)) + d(p,p) = (b+c)d(p,RSu).$$

Therefore we have

$$\|d(RSu, p)\| \leq (b+c)\|d(p, RSu)\|.$$

Which is a contradiction, since $a,b,c \geq 0$ and $a+2b+2c < 1$. Therefore $\|d(RSu,p)\| = 0$, which implies $RSu = Pu = p$. Since $RS(X) \subseteq Q(X)$, there exists a point v in X such that $p = Qv$. Then by (ii), we have

$$d(p, TUv) = d(RSu, TUv)$$

$$\begin{aligned}
&\precsim_{i_2} ad(Pu, Qv) + b(d(Pu, RSu) + d(Qv, TUv)) + c(d(Pu, TUv) + d(Qv, RSu)) \\
&= ad(p,p) + b(d(p,p) + d(p, TUv) + c(d(p, TUv) + d(p,p)) = (b+c)d(p, TUv).
\end{aligned}$$

Therefore we have

$$\|d(p, TUv)\| \leq (b+c)\|d(p, TUv)\|.$$

Which is a contradiction, since $a,b,c \geq 0$ and $a+2b+2c < 1$. Therefore $TUv = Qv = p$ and so $RSu = Pu = TUv = Qv = p$.

Similarly, Q and TU are weakly compatible maps, so we have $TUp = Qp$. Now we claim that p is a fixed point of TU. If $TUp \neq p$, then by (ii), we have

$$d(p, TUp) = d(RSp, TUp)$$

$$\precsim_{i_2} ad(Pp, Qp) + b(d(Pp, RSp) + d(Qp, TUp)) + c(d(Pp, TUp) + d(Qp, TUp))$$

$$= ad(p, TUp) + b(d(p, p) + d(TUp, TUp)) + c(d(p, TUp) + d(TUp, p)) = (a + 2c)d(p, TUp).$$

So we get

$$\|d(p, TUp)\| \leq (a + 2c) \|d(p, TUp)\|.$$

Which is a contradiction. Therefore $TUp = p$. Hence $TUp = Qp = p$. We have therefore proved that $RSp = TUp = Pp = Qp = p$. So p is a common fixed point of P, Q, RS and TU.

By commuting conditions of pairs we have $Tp = T(TUp) = T(UTp) = TU(Tp)$, $Tp = T(Pp) = P(Tp)$, $Up = U(TUp) = (UT)(Up) = (TU)(Up)$ and $Up = U(Pp) = P(Up)$, which implies that Tp and Up are common fixed points of (TU, P). Therefore $Tp = p = Up = Pp = TUp$.

Similarly, $Rp = p = Sp = Qp = RSp$. Therefore p is a common fixed point of T, U, R, S, P and Q.

For uniqueness of p, let w be another common fixed point of T, U, R, S, P and Q. Then by (ii), we have

$$d(p, w) = d(RSp, TUw)$$

$$\precsim_{i_2} ad(Pp, Qw) + b(d(Pp, RSp) + d(Qw, TUw)) + c(d(Pp, TUw) + d(Qw, RSp))$$

$$= ad(p, w) + b(d(p, p) + d(w, w)) + c(d(p, w) + d(w, p)) = (a + 2c)d(p, w).$$

Hence we get

$$\|d(p, w)\| \leq (a + 2c) \|d(p, w)\|,$$

which is a contradiction. Therefore, we get $\|d(p, w)\| = 0$ i.e., $p = w$ is a unique common fixed point of T, U, R, S, P and Q. This proves the theorem. □

Theorem 3.3. *Let (X, d) be a complete bicomplex valued metric space. Let S and T:$X \times X \to X$, such that*

$$d(S(z, z'), T(u, v)) \precsim_{i_2} a\frac{d(z, u) + d(z', v)}{2} + b\frac{d(z, S(z, z')) + d(u, T(u, v))}{2},$$

$\forall\ z, z', u, v \in X$, where a and b are non-negative integers with $a + b < 1$. Then S and T have a unique common couple fixed point in $X \times X$.

Proof. Let z_0, $z'_0 \in X$ be arbitrary. We define two sequences $\{z_n\}$, $\{z'_n\}$ as$z_{2k+1} = S(z_{2k}, z'_{2k})$, $z_{2k+2} = T(z_{2k+1}, z'_{2k+1})$ and $z'_{2k+1} = S(z'_{2k}, z_{2k})$, $z'_{2k+2} = T(z'_{2k+1}, z_{2k+1})$. Now

$$d(z_{2k+1}, z_{2k+2}) = d(S(z_{2k}, z'_{2k}), T(z_{2k+1}, z'_{2k+1}))$$

$$\precsim_{i_2} a\frac{d(z_{2k}, z_{2k+1}) + d(z'_{2k}, z'_{2k+1})}{2} + b\frac{d(z_{2k}, S(z_{2k}, z'_{2k})) + d(z_{2k+1}, T(z_{2k+1}, z'_{2k+1}))}{2}$$

$$= a\frac{d(z_{2k}, z_{2k+1}) + d(z'_{2k}, z'_{2k+1})}{2} + b\frac{d(z_{2k}, z_{2k+1}) + d(z_{2k+1}, z_{2k+2})}{2}.$$

Therefore we get that

$$(2 - b)d(z_{2k+1}, z_{2k+2}) \precsim_{i_2} (a + b)d(z_{2k}, z_{2k+1}) + ad(z'_{2k}, z'_{2k+1}),$$

$$i.e.,\ d(z_{2k+1}, z_{2k+2}) \precsim_{i_2} \frac{(a + b)}{(2 - b)}d(z_{2k}, z_{2k+1}) + \frac{a}{(2 - b)}d(z'_{2k}, z'_{2k+1}). \tag{3.4}$$

Similarly, it can be shown that

$$d(z'_{2k+1}, z'_{2k+2}) \precsim_{i_2} \frac{(a+b)}{(2-b)} d(z'_{2k,} z'_{2k+1}) + \frac{a}{(2-b)} d(z_{2k}, z_{2k+1}). \tag{3.5}$$

By adding (3.4) and (3.5), we obtain

$$d(z_{2k+1}, z_{2k+2}) + d(z'_{2k+1}, z'_{2k+2}) \precsim_{i_2} \frac{(2a+b)}{(2-b)} \{d(z_{2k}, z_{2k+1}) + d(z'_{2k,} z'_{2k+1})\}.$$

Now, we take $h = \frac{2a+b}{2-b}$ then $0 \leq h < 1$, since $0 \leq a+b < 1$, therefore we have

$$d(z_{2k+1}, z_{2k+2}) + d(z'_{2k+1}, z'_{2k+2}) \precsim_{i_2} h\{d(z_{2k}, z_{2k+1}) + d(z'_{2k,} z'_{2k+1})\}.$$

Similarly, it can also be shown that

$$d(z_{2k+2}, z_{2k+3}) + d(z'_{2k+2}, z'_{2k+3}) \precsim_{i_2} h\{d(z_{2k+1}, z_{2k+2}) + d(z'_{2k+1,} z'_{2k+2})\}.$$

Thus for any $n \in \mathbb{N}$, we have

$$d(z_{n+2}, z_{n+1}) + d(z'_{n+2}, z'_{n+1}) \precsim_{i_2} h\{d(z_{n+1}, z_n) + d(z'_{n+1,} z'_n)\}$$

$$\precsim_{i_2} h^2\{d(z_n, z_{n-1}) + d(z'_n, z'_{n-1})\} \precsim_{i_2} ... \precsim_{i_2} h^{n+1}[d(z_1, z_0) + d(z'_1, z'_0)].$$

Now for $m > n$, we get that

$$d(z_{m,} z_n) + d(z'_m, z'_n) \precsim_{i_2} [d(z_n, z_{n+1}) + d(z'_n, z'_{n+1})] + [d(z_{n+1}, z_m) + d(z'_{n+1}, z'_m)]$$

$$d(z_{m,} z_n) + d(z'_m, z'_n) \precsim_{i_2} [d(z_n, z_{n+1}) + d(z'_n, z'_{n+1})] + [d(z_{n+1}, z_m) + d(z'_{n+1}, z'_m)]$$

$$\precsim_{i_2} [d(z_n, z_{n+1}) + d(z'_n, z'_{n+1})] + [d(z_{n+1}, z_{n+2}) + d(z'_{n+1}, z'_{n+2})] + [d(z_{n+1}, z_m) + d(z'_{n+1}, z'_m)]$$

$$\precsim_{i_2} ... \precsim_{i_2} [d(z_n, z_{n+1}) + d(z'_n, z'_{n+1})] + [d(z_{n+1}, z_{n+2}) + d(z'_{n+1}, z'_{n+2})] + [d(z_{m-1}, z_m) + d(z'_{m-1}, z'_m)]$$

$$\precsim_{i_2} [h^n + h^{n+1} + h^{n+2} + ... + h^{m-1}][d(z_1, z_0) + d(z'_1, z'_0)]$$

$$\precsim_{i_2} \frac{h^n}{1-h}[d(z_1, z_0) + d(z'_1, z'_0)] \to 0 \text{ as } n \to \infty.$$

Thus $d(z_{m,} z_n) \to 0$ and $d(z'_m, z'_n) \to 0$ as $m, n \to \infty$. Therefore $\{z_n\}$ and $\{z'_n\}$ are Cauchy sequences. Again since X is complete, there exist $z, z' \in X$ are such that $z_n \to z$ and $z'_n \to z'$, as $n \to \infty$. Therefore we have

$$d(S(z, z'), z) \precsim_{i_2} d(S(z, z'), z_{2k+2}) + d(z_{2k+2}, z)$$

$$= d(S(z, z'), T(z_{2k+1}, z'_{2k+1})) + d(z_{2k+2}, z)$$

$$\precsim_{i_2} a\frac{d(z, z_{2k+2}) + d(z', z'_{2k+2})}{2} + b\frac{d(z, S(z, z')) + d(z_{2k+1}, T(z_{2k+1}, z'_{2k+1}))}{2} + d(z_{2k+2}, z)$$

$$= a\frac{d(z, z_{2k+2}) + d(z', z'_{2k+2})}{2} + b\frac{d(z, S(z, z')) + d(z_{2k+1}, z_{2k+2})}{2} + d(z_{2k+2}, z).$$

Letting $k \to \infty$ and using Lemma 2.2, we get

$$d(S(z, z'), z) \precsim b\frac{d(z, S(z', z))}{2}.$$

Therefore $0 \leq a + b < 1$, and $d(S(z, z'), z) = 0$ shows that $S(z, z') = z$. Similarly, it can be shown that $S(z', z) = z'$. Again we have

$$d(z, T(z, z')) = d(S(z, z'), T(z, z'))$$

$$\precsim_{i_2} a\frac{d(z, z) + d(z', z')}{2} + b\frac{d(z, S(z, z')) + d(z, T(z, z'))}{2} = \frac{b}{2}d(z, T(z, z')).$$

Thus $(1 - \frac{b}{2}) \|d(z, T(z, z')\| \leq 0$ and hence $T(z, z') = z$. Similarly, we can show that $T(z', z) = z'$. Thus $S(z, z') = T(z, z') = z$ and $S(z', z) = T(z', z) = z'$. Therefore (z, z')

is a common coupled fixed point of S and T. For uniqueness, let $(p,q) \in X \times X$ such that$S(p,q) = T(p,q) = p$ and $S(q,p) = T(q,p) = q$. Now we have

$$\begin{aligned} d(z,p) = d(S(z,z'),T(p,q)) &\precsim_{i_2} a\frac{d(z,p)+d(z',q)}{2} + b\frac{d(z,S(z,z'))+d(p,T(p,q))}{2} \\ &= a\frac{d(z,p)+d(z',q)}{2} + b\frac{d(z,z)+d(p,p)}{2} = a\frac{d(z,p)+d(z',q)}{2}. \end{aligned} \tag{3.6}$$

Similarly, it can be shown that

$$d(z',q) \precsim_{i_2} a\frac{d(z,p)+d(z',q)}{2}. \tag{3.7}$$

By adding (3.6) and (3.7), we get

$$d(z,p) + d(z'q) \precsim_{i_2} a[d(z,p) + d(z',q)].$$

Thus $(1-a)[d(z,p) + d(z',q)] \precsim_{i_2} 0$. Therefore $d(z,p) + d(z',q) = 0$. Which shows that $d(z,p) = 0$ and $d(z',q) = 0$ hence $z = p$ and $z' = q$ i.e., $(z,z') = (p,q)$. Therefore (z,z') is the unique common fixed point of S and T. This completes the proof of the theorem. □

4. Acknowledgement

The authors are grateful to the referee for their valuable suggestions which has considerably improved the presentation of the paper.

References

[1] Abbas, M., Arsad, M. and Azam, A., (2013), Fixed points of asymptotically regular mappings in complex-valued metric spaces, Georgian Math. J., 20, pp. 213-221.

[2] Ahmad, J., Klin-Eam, C. and Azam, A., (2013), Common fixed points for multivalued mappings in complex valued metric spaces with applications, Abstr Appl. Anal., Article ID 854965, pp.1-12.

[3] Azam, A., Fisher, B. and Khan, M., (2011), Common Fixed Point Theorems in Complex Valued Metric Spaces, Num. Func. Anal. Opt.,32, pp. 243-253.

[4] Azam, A., Ahmad, J. and Kumam, p., (2013), Common fixed point theorems for multi-valued mappings in complexvalued metric spaces, J. Inequal. Appl., 2013:578, pp. 1-12.

[5] Bhaskar, T. G . and Laxikantam, V., (2006), Fixed point theorem in partially ordered metric spaces and applications, Nonlinear Analysis, 65,pp. 1379-1393.

[6] Bhatt, S., Chaukiyal, S. and Dimri, R. C., (2011), Common fixed point of mappings satisfying rational inequality in complex valued matrix space, International Journal of Pure and Applied Mathematics, 73 (2), pp. 159-164.

[7] Choi, J., Datta, S. K., Biswas, T. and Islam, N., (2017), Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces, Honam Mathematical J., 39 (1), pp. 115-126.

[8] Savitri and Hooda, N., (2015), A Common Coupled Fixed Point Theorem in Complex Valued Metric Space, International Journal of Computer Applications, 109 (4) pp. 10-12.

[9] Sintunavart, W. and Kumam, P., (2012), Generalized Common fixed point theorems in complex valued metric spaces and application, J. Inequal. Appl., 2012:84, pp. 1-12..

[10] Banach, S., (1922), Sur les operations dans les ensembles abstraita et leur application aux equations integrales, Fund. Math., 3 pp. 133-181.

Dr. Iqbal H. Jebril is a professor in the Department of Mathematics, Al-Zaytoonah University of Jordan, Amman, Jordan. His main teaching and research interests include Functional Analysis, Operator Theory and Fuzzy Logic. He has been the chief Organizer International Center for Scientific Research and Studies and chief editor of the International Journal of Open Problems in Computer Sciences and Mathematics since 2008, International Journal of open Problems in Complex Analysis since 2009, and General Mathematics Notes since 2013.

Prof. Sanjib Kumar Datta received his Ph.D. degree from University of Kalyani. He is currently working as a professor at the University of Kalyani, West Bengal, India. He has been working in several branches of Pure Mathematics namely Complex Analysis, Fuzzy Algebra, Operator Theory etc. He is a fellow of the Institute of Mathematics and its Applications, United Kingdom (UK) and a Member of The National Academy of Sciences, India.

Rakesh Sarkar received his Masters Degree in 2006 from University of North Bengal, Darjeeling, India. He is currently working as an Assistant Professor in Mathematics at Gour Mahavidyalaya, Malda, West Bengal. His research area of interest includes Complex Valued Metric Space, Bi-complex Valued Metric Space and Probabilistic Metric Space. He has been doing research in the mentioned area for two years.

Dr. Nityagopal Biswas received his masters degree in 2013 from Indian Institute of Technology and Ph.D. degree in 2019 from University of Kalyani, India. He is currently working as an assistant professor at the Department of Mathematics, Chakdaha College, India. He worked as a research associate at the Department of Mathematics, University of Kalyani, India. His area of interest includes Nevanlinna Theory, Complex Linear Differential Equations, Linear Difference, Differential-Difference Equations, Complex Valued Metric Space and Bi-complex Valued Metric Space.