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IMPACT OF INITIAL TIME DIFFERENCE ON STABILITY CRITERIA OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, an impulsive differential system is investigated for the first time for several stability criteria relative to initial time difference. The investigations are carried out by perturbing Lyapunov functions and by using comparison results. A generalized Lyapunov function has been used for the investigation.The results that are obtained to investigate the stability significantly depend on the moment of impulses. An example is given to illustrate the derived result.

Keywords: Impulsive differential system, stability, perturbed Lyapunov function, initial time difference.

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1. Introduction

Stability is one of the most important feature in the qualitative theory of differential equations [14, 13]. However, while considering the real world problems, it is sometimes not possible to investigate the stability of the solutions by keeping the initial time same. Stability with initial time difference (ITD) is a generalization of the basic concept of stability of solutions. The concept of stability with respect to ITD was firstly investigated by Lakshmikantham and Vatsala [9] and Lakshmikantham et al. [10]. In the past, different types of stability were studied for variety of differential equations with reference to ITD like ordinary differential equations [11, 19], fractional differential equations [4], delay differential equations [1, 6], caputo fractional differential equations [2, 3] etc. without impulse effects. In dealing with real world problems, impulsive differential equations are more suitable and have been investigated to study various types of stabilities [7, 16], but the investigation of impulsive differential equations with respect to ITD is at its initial stage and has not been investigated much in the past. However, S.G. Hristova [5] investigated the stability behavior of impulsive differential equations with respect to ITD by employing Lyapunov function and some comparison results under few rigid conditions.

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In the present paper, for the first time, we study the various stability criteria for a system of impulsive differential equation with ITD by perturbing the Lyapunov function. Lyapunov function is widely recognized as a tool for investigating the stability properties of nonlinear differential equations. When a Lyapunov function does not seem to satisfy all the necessary criteria to deduce the required properties, then it becomes worth perturbing the Lyapunov function rather than discarding it. The concept of perturbing Lyapunov function to study the nonuniform properties of solutions of differential equations was firstly given by Lakshikantham [8] and further extended to investigate various stability criteria for impulsive differential equations [12, 20, 17]. The notion of perturbing Lyapunov function was used by McRae and Song et al. [15, 18] to study the stability properties of differential equations without impulse effect relative to ITD, but in this paper, we are using the method of perturbing Lyapunov function for impulsive differential equations by introducing initial time difference. Here, a generalized Lyapunov function is utilised to investigate the desired stabilities. We carry out our results with the help of some comparison results [9, 7].

In the present manuscript, the main motive is to investigate the impact of initial time difference on stability criteria of impulsive differential equations. The paper is arranged into five sections; namely preliminaries, comparison results, main results, examples and conclusion. In preliminaries, we have introduced some basic definitions and notations followed by a section, where we have carried out two comparison results for our investigations. In main results, we have given some criteria to bring the equi-stability and equi-asymptotic stability of impulsive differential equation with respect to initial time difference. An example is also given to support the deduced result. Finally, on the basis of these results, conclusion is drawn.

2. Preliminaries

Let R^n denotes the n dimensional Euclidean space and let $R_+ = [0, \infty)$,
Consider the impulsive differential system :

$$\begin{cases} x' = f(t,x), & t \neq t_i \\ \Delta x = I_i(x), & t = t_i \end{cases} \tag{1}$$

where $f \in PC[R_+ \times R^n, R^n]$ is piece-wise continuous function and $I_i \in C(R^n, R^n)$ is continuous function for every i where $i = 1, 2, 3...$
Let $\Delta x = I_i(x(t)) = x(t_i^+) - x(t_i)$, where $0 < t_1 < t_2 < ...t_i < t_{i+1}..., t_i \to \infty$ as $i \to \infty$
Let $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$ be two solutions of the system (1) through (t_0, x_0) and (τ_0, y_0) respectively, where $t_0, \tau_0 \in R_+$. Here both $x(t)$ and $y(t)$ are piecewise continuous having discontinuity of first kind.
Here, we will study the stability criteria with respect to the solution $x(t) = x(t; t_0, x_0)$.
Let $\gamma = \tau_0 - t_0 > 0$.
Denote $S(x, \rho) = \{x \in R^n : \|x - y\| < \rho\}$ for every $y \in S(\rho) = \{y \in R^n : \|y\| < \rho\}$.
Consider the complimentary sets of $S(x, \rho)$ and $S(\rho)$, respectively, as $S^c(x, \rho)$ and $S^c(\rho)$.
Also, consider the following function:

$$K = \{\phi \in C(R_+, R_+) : \phi \text{ is strictly increasing and } \phi(0) = 0\}$$

In order to study the stability of impulsive differential systems with respect to ITD, firstly we will discuss some of the definitions as given below:

Definition 2.1 [15]. Let $z(t) = x(t; t_0, x_0) - y(t + \gamma; \tau_0, y_0)$ such that $z_0 = x_0 - y_0$. Then, the solution $x(t) = x(t; t_0, x_0)$ of impulsive differential system (1) is:

(A1) equistable with respect to solution $y(t) = y(t; \tau_0, y_0)$ of the system (1) through (τ_0, y_0), if for a given $\epsilon > 0$ there exist $\delta = \delta(t_0, \epsilon) > 0$ and $\sigma = \sigma(t_0, \epsilon) > 0$ such that $\|z_0\| < \delta$ and $|\gamma| < \sigma$ implies $\|z(t)\| < \epsilon,\ t \geq t_0$;

(A2) uniformly stable if the condition (A1) holds true, where both δ and σ are not dependent on t_0;

(A3) equi-asymptotically stable with respect to solution $y(t) = y(t; \tau_0, y_0)$ of the system (1) through (τ_0, y_0), if the condition (A1) holds true and for a given $\epsilon > 0$ there exist $\delta_0 = \delta_0(t_0) > 0, \sigma_0 = \sigma_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that $\|z_0\| < \delta_0$ and $|\gamma| < \sigma_0$ implies $\|z(t)\| < \epsilon,\ t \geq t_0 + T$;

(A4) uniformly asymptotically stable if both (A2) and (A3) holds, with δ_0, σ_0 and T in (A3) are independent of t_0.

Definition 2.2 [7]. Let Lyapunov function $V : R_+ \times R^n \to R_+$ belongs to class V_0 such that

(i) V is continuous function on each of the sets $(t_{i-1}, t_i] \times R^n; \forall x \in R^n$ and $i = 1, 2, 3...$, and there exists a limit $\lim_{(t,y)\to(t_i^+,x)} V(t, y) = V(t_i^+, x)$;

(ii) V is Lipschitz in the local neighborhood of x
i.e. $\|V(t, x) - V(t, y)\| \leq L\|x - y\|$ holds $\forall\ \|x - y\| < \rho$ where $\rho > 0$ and $L > 0$.
Now, define

$$D_-V(t, x) = \lim_{s\to 0^-} inf \frac{1}{s} \{V(t + s, x + sf(t, x)) - V(t, x)\}$$

where $(t, x) \in (t_{i-1}, t_i] \times R^n$,

Consider the nonlinear impulsive differential system (1), for $V \in V_0$, we define the generalized derivative [18] depending upon the difference of solutions $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$, with respect to the system (1) as follows:

$$D_-V(t, x - y) = \lim_{s\to 0^-} inf \frac{1}{s} [V(t + s, x - y + s(f(t, x) - f(t, y)) - V(t, x - y))] \qquad (2)$$

In the following, we will consider the two comparison principles which will eventually support our further investigations.

Lemma 2.1 [7] Let $m \in PC[R_+, R]$ piece-wise continuous and $m(t)$ is left continuous at $t_i, i = 1, 2, 3...$, with $t_i \to \infty$ as $i \to \infty$ where $D_-m(t) = \lim_{s\to 0} inf \frac{1}{s} [m(t + s) - m(t)]$. Suppose that

(a) $g \in C[R_+ \times R^n \to R], \psi_i : R \to R, \psi_i(u)$ is non-decreasing in u and for each $i = 1, 2, 3...$,

$$\begin{cases} D_-m(t) \leq g(t, m(t)), & t \neq t_i, \\ m(t_i^+) \leq \psi_i(m(t_i)), & t = t_i, \\ m(t_0) \leq u_0; \end{cases}$$

(b) r(t) is the maximal solution of

$$\begin{cases} u' = g(t, u), & t \neq t_i, \\ u(t_i^+) = \psi_i(u(t_i)), & t_i > t_0 \geq 0 \\ u(t_0) = u_0 \end{cases}$$

Then, for $t \geq t_0$ we have $m(t) \leq r(t)$

Lemma 2.2 [9] Let us consider that

(a) $m \in C[R_+, R_+], h : R_+^2 \to R$ is continuous and non- decreasing in t for each w and $\tau_0 \geq t_0$. Also, $D_-m(t) \leq h(t, m(t))$, $m(t_0) \leq w_0$ and $t_0 \geq 0$,

(b) the maximal solution $r(t) = r(t; \tau_0, w_0)$ of $w' = h(t, w), w(\tau_0) = w_0 \geq 0, \tau_0 \geq 0$, exists for $t \geq \tau_0$;

Then, for $t \geq t_0$, we have $m(t) \leq r(t + \gamma)$ and for $t \geq \tau_0$ we have $m(t - \gamma) \leq r(t)$.

3. Comparison Results

For our further investigation, we will use the following comparison system:

$$\begin{cases} u' = g_1(t, u), & t \neq t_i, \\ u(t_i^+) = J_i(u(t_i)), & t = t_i, i = 1, 2, 3... \\ u(\tau_0) = u_0, & \tau_0 > t_0 \end{cases} \tag{3}$$

existing for $t \geq \tau_0$, where $u(t; \tau_0, u_0)$ is any solution of (3) and

$$\begin{cases} v' = g_2(t, v), & t \neq t_i, \\ v(t_i^+) = F_i(v(t_i)), & t = t_i, i = 1, 2, 3... \\ v(\tau_0) = v_0, & \tau_0 > t_0 \end{cases} \tag{4}$$

existing for $t \geq \tau_0$, where $v(t; \tau_0, v_0)$ is any solution of (4).
In order to prove the desired results, we will first prove two comparison principles for the system (1) under consideration.

Theorem 3.1. [9] *Let us suppose that:*

(a) $m \in PC[R_+, R_+]$ *is piece-wise continuous,* $g_1 \in C[R_+^2, R]$ *is continuous and non decreasing and* $J_i : R_+ \to R_+$ *is non-decreasing such that*

$$\begin{cases} D_-m(t) \leq g_1(t, m(t)), & t \neq t_i, \\ m(t_i^+) \leq J_i(m(t_i)), & t = t_i, i = 1, 2, 3... \end{cases} \tag{5}$$

(b) *Let* $r(t) = r(t; \tau_0, u_0)$ *is the maximal solution of (3).*

Then $m(t_0) \leq u_0$ *implies* $m(t) \leq r(t + \gamma)$ *for* $t \geq t_0$ *and* $m(t - \gamma) \leq r(t)$ *for* $t \geq \tau_0$.

Proof. For $t \in [t_0, t_1]$, we have by Lemma 2.2, $m(t) \leq r(t + \gamma)$.
Hence, by knowing the facts that J_1 is non-decreasing and $m(t_1) \leq r(t_1 + \gamma)$, we obtain

$$m(t_1^+) \leq J_1(m(t_1)) \leq J_1(r(t_1 + \gamma)) = u_1^+.$$

Now, for $t_1 < t \leq t_2$, again by Lemma 2.2, it follows that $m(t) \leq r(t + \gamma)$ where $r(t) = r(t; \tau_1, u_1^+)$ is the maximal solution of (3) for $t \in [t_1, t_2]$.
Continuing as above, we get

$$m(t_2^+) \leq J_1(m(t_2)) \leq J_1(r(t_2 + \gamma)) = u_2^+.$$

Repeating in the same way, we finally get the desired results and hence, the proof. □

Theorem 3.2. [18] *Let* $V \in PC[R_+ \times R^n \to R_+]$ *and* $V \in V_0$.
Let us consider that

$$\begin{cases} D_-V(t, z) \leq g_1(t, V(t, z)) \\ V(t_i^+, z(t_i^+)) \leq J_i(V(t_i, z(t_i))) \end{cases} \tag{6}$$

where $g_1 \in C[R_+^2, R_+]$ *is continuous and non - decreasing,* $J_i : R_+ \to R_+$ *is non-decreasing. Let* $r(t) = r(t; \tau_0, u_0)$ *be the maximal solution of the comparison system (3).*

Then, $V(t, z(t)) \leq r(t+\gamma, \tau_0, u_0),\ t \geq t_0$ *holds provided* $V(t_0, z_0) \leq u_0$.

Proof. Let $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$ be the solutions of the system (1) such that $V(t_0, z_0) \leq u_0$.
Define $m(t) = V(t, z(t))$ such that for adequately small but positive s, we have

$$\begin{aligned} m(t+s) - m(t) &= V(t+s, z(t+s)) - V(t, z(t)) \\ &= V(t+s, x(t+s) - y(t+\gamma+s)) - V(t, x(t) - y(t+\gamma)) \end{aligned} \tag{7}$$

After adding and subtracting a common term (i.e. $V(t+s, x(t) - y(t+\gamma) + s(f(t, x(t)) - f(t, y(t+\gamma))$) on right hand side of above equation (7) and then multiplying by s^{-1} (where $s \to 0^-$) on both sides of the same, we get

$$\begin{aligned} D_- m(t) &\leq \lim_{s\to 0^-} inf L \| \frac{[x(t+s) - x(t)]}{s} - f(t, x(t) \| \\ &\quad + \lim_{s\to 0^-} inf L \| \frac{[y(t+\gamma+s) - y(t+\gamma)]}{s} - f(t, y(t+\gamma) \| + D_- V(t, x(t) - y(t+\gamma)) \\ &= D_- V(t, x(t) - y(t+\gamma)) \\ &= D_- V(t, z(t)) \\ &\leq g_1(t, m(t)) \end{aligned}$$

Now consider

$$\begin{aligned} m(t_i^+) &= V(t_i^+, z(t_i^+)) \\ &\leq J_i(m(t_i)) \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are fulfilled and we get the desired result. □

4. Main Results

Here, we will deduce the sufficient conditions for the stability of impulsive differential equations with ITD. For the same, we will apply the method of perturbed Lyapunov function along with comparison results.

Theorem 4.1. *Suppose that the following conditions are fulfilled:*

(i) *Let* $V_1 \in PC[R_+ \times S(x, \rho), R_+]$ *and* $V_1(t, x) \in V_0$ *such that*

$$\begin{cases} D_- V_1(t, z) \leq g_1(t, V_1(t, z)), & t \neq t_i \\ V_1(t_i^+, z(t_i^+)) \leq J_i(V_1(t_i, z(t_i))), & i = 1, 2, 3.. \end{cases} \tag{8}$$

where g_1 *and* J_i *are same as defined in Theorem 3.2.*

(ii) *For* $0 < \eta < \rho$ *, there exist a* $V_2 \in PC[R_+ \times S(\rho) \bigcap S^c(\eta), R_+]$*, and* $V_2(t, x) \in V_0$ *such that*

$$b(\|x\|) \leq V_2(t, x) \leq a(\|x\|), \quad a, b \in K$$

and

$$\begin{cases} D_-[V_1(t, z) + V_2(t, z)] \leq g_2(t, V_1(t, z) + V_2(t, z)), & t \neq t_i \\ [V_1(t_i^+, z(t_i^+)) + V_2(t_i^+, z(t_i^+))] \leq F_i(V_1(t_i, z(t_i)) + V_2(t_i, z(t_i))), & t = t_i, i = 1, 2, 3.. \end{cases}$$

where $g_2 : R_+^2 \to R_+$ *is continuous and* $F_k : R_+ \to R_+$ *is non-decreasing.*

(iii) *The zero solution of comparison system (3) is equi-stable and the zero solution of comparison system (4) is uniformly stable.*
Then, the solution $x(t) = x(t; t_0, x_0)$ of impulsive differential system (1) is equi-stable.

Proof. Let $0 < \epsilon < \rho$ and considering condition (iii), as the zero solution of (4) is uniformly stable, we have for a given $b(\epsilon) > 0$ and $\tau_0 \in R_+$, there exist a $\delta_0 = \delta_0(\epsilon) > 0$, such that $v_0 < \delta_0$ implies

$$v(t; \tau_0, v_0) < b(\epsilon), t \geq \tau_0 \tag{9}$$

where $v(t; \tau_0, v_0)$ is any solution of (4).
Choose $\delta_1 = \delta_1(\epsilon) > 0$, such that

$$a(\delta_1) < \frac{\delta_0}{2} \tag{10}$$

Also, from the stability of zero solution of (3), for given $\frac{\delta_0}{2} > 0$ and $\tau_0 \in R_+$, there exist a $\delta_2 = \delta_2(\tau_0, \epsilon) > 0$ such that

$$u(t; \tau_0, u_0) < \frac{\delta_0}{2}, \ \ t \geq \tau_0 \tag{11}$$

provided $u_0 < \delta_2$ where $u(t; \tau_0, u_0)$ is any solution of (3).
Choose $u_0 = V_1(t_0, z_0)$, such that for some $\delta_3 > 0$,

$$\|z_0\| < \delta_3 \text{ and } V_1(t_0, z_0) < \delta_2 \tag{12}$$

hold concurrently.
Therefore, $\lim_{(t_0,x_0)\to(\tau_0,y_0)} \|z(t)\| = 0$, which follows that for a given $\epsilon > 0$, there exist $\delta_4 = \delta_4(\epsilon) > 0$ and $\sigma_1 = \sigma_1(\epsilon)$, such that $\|z_0\| < \delta_4$ and $\gamma < \sigma_1$ implies

$$\|z(t)\| < \delta_1, \ \ t_0 \leq t \leq \tau_0 \tag{13}$$

Choose $\delta = min(\delta_1, \delta_3, \delta_4)$ and $\sigma = \sigma_1$
Now, we claim that $\|z_0\| < \delta$ and $\|z(t)\| < \epsilon$, for $t \geq t_0$. If it does not hold true, then there exist a solution $y(t) = y(t; \tau_0, y_0)$ of impulsive differential system (1) with $\|z_0\| < \delta$, $\gamma < \delta$ and $t_2 > t_1 > \tau_0$ satisfying

$$\|z(t_1)\| = \delta_1; \|z(t_2)\| = \epsilon \text{ and } \delta_1 < \|z(t)\| < \epsilon \text{ where } t_1 < t < t_2. \tag{14}$$

By using Theorem (3.2) and condition (i) of Theorem (4.1), we get

$$V_1\left(t_1, z(t_1)\right) \leq r_1(t_1 + \gamma; \tau_0, u_0), \ \ t \geq t_0 \tag{15}$$

where $r_1(t_1; \tau_0, u_0)$ is the maximal solution of (3).
Thus, by using (11), we have

$$V_1\left(t_1, z(t_1)\right) \leq \frac{\delta_0}{2} \tag{16}$$

In addition, from condition (ii) of Theorem 4.1, (10) and (14), we have

$$V_2\left(t_1, z(t_1)\right) \leq a\left(\|z(t_1)\|\right) \leq a(\delta_1) < \frac{\delta_0}{2} \tag{17}$$

Thus, we get

$$V_1\left(t_1, z(t_1)\right) + V_2\left(t_1, z(t_1)\right) \leq \frac{\delta_0}{2} \tag{18}$$

Now, let $m(t) = V_1\left(t, z(t)\right) + V_2\left(t, z(t)\right), \ \ t \in [t_1, t_2]$.
From condition (ii), we get

$$D_- m(t) \leq g_2(t, m(t)), \ \ t \in [t_1, t_2]$$

and

$$\begin{cases} m(t_i^+) = V_1\left(t_i^+, z(t_i^+)\right) + V_2\left(t_i^+, z(t_i^+)\right) \\ \qquad\quad \leq F_i\left(V_1(t_i, z(t_i)) + V_2(t_i, z(t_i))\right) \\ \qquad\quad = F_i(m(t_i)) \end{cases}$$

Thus, by using equation (18) and Lemma 2.1, we get

$$m(t) \leq r_2(t; t_1, m(t_1)), t \in [t_1, t_2] \tag{19}$$

where $r_2(t; t_1, m(t_1))$ is the maximal solution of (4).
Thus, by using (9) and (19), we have

$$V_1\left(t_2, z(t_2)\right) + V_2\left(t_2, z(t_2)\right) < b(\epsilon) \tag{20}$$

Also, by using equation (14), condition (ii) and as $V_1(t, x) \geq 0$, we obtain

$$b(\epsilon) = b(\|z(t_2)\|) \leq V_2(t_2, z(t_2)) \leq V_1(t_2, z(t_2)) + V_2(t_2, z(t_2)) < b(\epsilon)$$

which leads to a contradiction, proving the equistability of (1). □

Remark: If the condition (iii) of Theorem 4.1 is strengthen by assuming the zero solution of both comparison systems (3) and (4) as uniformly stable, then the solution $x(t) = x(t; t_0, x_0)$ of the system (1) is uniformly stable.

Theorem 4.2. *Assume that all the conditions of Theorem (4.1) are fulfilled except condition (i) which is replaced as follows:*
Let $V_1 \in PC[R_+ \times S(x, \rho), R_+]$ *and* $V_1(t, x) \in V_0$ *such that*

$$\begin{cases} D_- V_1(t, z) + p(t, z) \leq g_1(t, V_1(t, z)), \qquad t \neq t_i \\ V_1(t_i^+, z(t_i^+)) + \int_{t_0}^{t_i} p(s, z(s))ds \leq J_i(V_1(t_i, z(t_i))), \quad t = t_i, i = 1, 2, 3.. \end{cases} \tag{21}$$

where $g_1 \in C(R_+^2, R_+)$ *is non - decreasing,* $p(t, x) : R_+ \times S(\rho) \to R_+$ *is piecewise continuous and integrable such that* $p(t, x) \geq b_0(\|x\|)$ *where* $b_0 \in K$, $D_- p(t, x)$ *is bounded above or below and* $J_i : R_+ \to R_+$ *is non- decreasing.*
Then, the solution $x(t) = x(t; t_0, x_0)$ *of impulsive differential system (1) will be equi-asymptotically stable.*

Proof. According to Theorem 4.1, we have already proved that the system (1) is equistable. Therefore, for $\epsilon = \alpha$, let $\delta^* = \delta^*(t_0, \alpha) > 0$ and $\sigma^* = \sigma^*(t_0, \alpha) > 0$, such that $\|z_0\| < \delta^*$ and $|\gamma| < \sigma^*$ implies

$$\|z(t)\| < \alpha, \;\; t \geq t_0$$

Now, we will prove that,

$$\|z(t)\| \to 0 \tag{22}$$

as $t \to \infty$ when $\|z_0\| < \delta^*$ and $|\gamma| < \sigma^*$.
As $p(t, x) \geq b_0(\|x\|)$, in order to prove (22), it is sufficient to prove that

$$lim_{t \to \infty} p\left(t, z(t)\right) = 0$$

when $\|z_0\| < \delta^*$ and $|\gamma| < \sigma^*$.
If it is not true, then there will be two divergent sequences $\{t_n\}$ and $\{t_n^*\}$, such that for $\beta > 0$, we have

$$p(t_i, z(t_i)) = \frac{\beta}{2}; \quad p(t_i^*, z(t_i^*)) = \beta \text{ and } \frac{\beta}{2} \leq p(t, z(t)) \leq \beta \quad t \in (t_i, t_i^*) \tag{23}$$

As $D_{-}p(t,x)$ is bounded above, there exist a constant M such that $D_{-}p(t,x) \leq M$. Therefore, we have

$$\int_{t_i}^{t_i^*} D_{-}p(s,z(s))ds \leq M(t_i^* - t_i)$$

By using (22), we get

$$t_i^* - t_i \geq \frac{\beta}{2M} \quad \text{for each } i \tag{24}$$

$$\text{Let } G(t,z) = V_1(t,z) + \int_{t_0}^{t} p(s,z(s))ds \tag{25}$$

Then, by using inequality (21), we have

$$\begin{aligned} D_{-}G(t,z) &\leq D_{-}V_1(t,z) + p(t,z) \\ &\leq g_1(t, V_1(t,z)) \\ &\leq g_1(t, G(t,z)) \end{aligned}$$

Also,

$$\begin{aligned} G(t_i^+, z(t_i^+)) &= V_1(t_i^+, z(t_i^+) + \int_{t_0}^{t_i} p(s,z(s))ds \\ &\leq J_i(V_1(t_i, z(t_i))) \\ &\leq J_i(G(t_i, z(t_i))) \end{aligned}$$

Hence, by Theorem (3.2), we have

$$G\,(t, z(t)) \leq r_1(t+\gamma, \tau_0, u_0), \;\; t \geq t_0 \tag{26}$$

where $r_1(t, \tau_0, u_0)$ is the maximal solution of (3).
Thus, from (23)- (26), we have,

$$\begin{aligned} 0 &\leq V_1\,(t, z(t)) \\ &\leq r_1(t+\gamma, \tau_0, u_0) - \int_{t_0}^{t} p(s,z(s))ds \\ &\leq \frac{\delta_0}{2} - \frac{\beta}{2} \sum_{1 \leq i \leq n} (t_i^* - t_i) \\ &< \frac{\delta_0}{2} - \frac{\beta}{2}\frac{n\beta}{2M} < 0 \end{aligned}$$

which gives a contradiction for a sufficiently large value of n.
Therefore, we get

$$lim_{t\to\infty} p\,(t, z(t)) = 0 \text{ for } \|z_0\| < \delta^* \text{ and } |\gamma| < \sigma^*.$$

Also, $p(t,x) \geq b_0(\|x\|)$, therefore, we have

$$lim_{t\to\infty} \|z(t)\| = 0 \text{ for } \|z_0\| < \delta^* and |\gamma| < \sigma^*.$$

Hence, the proof is completed. □

Remark: If the assumptions of Theorem 4.2 are strengthen by assuming the zero solution of both comparison systems (3) and (4) as uniformly stable, then the solution $x(t) = x(t; t_0, x_0)$ of the system (1) is uniformly asymptotically stable.

In this section, we will present an example to support the proved results.

5. Example

Consider the following impulsive differential system:

$$\begin{cases} x_1' = (\sin\ln(t+1) + \cos\ln(t+1) - 2)\, x_1 + l x_2, & t \neq t_i; \\ x_2' = l x_1 + (\sin\ln(t+1) + \cos\ln(t+1) - 2)\, x_2, & t \neq t_i; \\ \triangle x_1 = \lambda_i x_1; \quad \triangle x_2 = 0, & t = t_i \end{cases} \tag{27}$$

where $-1 \leq \lambda_i \leq 0$.
Let $x(t) = x(t; t_0, x_0)$ and $y(t) = y(t; \tau_0, y_0)$ be two solutions of (27), such that $\gamma = \tau_0 - t_0 > 0$
Define $\mu(t) = \exp\left[-2(t+1)(2 - \sin\ln(t+1))\right]$.
Set $V_1(x) = (x_1 + x_2)^2, V_2(x) = (x_1)^2 + (x_2)^2, a(x) = 2x^2$ and $b(x) = \frac{1}{2}x^2$.
Then, $V_2(x)$ clearly satisfies $b(\|x\|) \leq V_2(t,x) \leq a(\|x\|)$. As, $\mu'(t) \leq 0$, it is clear that $\mu(t)$ is non-increasing. Also, $\mu'(t)$ is non-decreasing for $t \geq t_0$ as $\mu''(t) \geq 0$.
Now,

$$\begin{aligned} D_- V_1(z) &= D_- V_1(x-y) \\ &= 2(x_1 + x_2 - y_1 - y_2)^2 \left(\sin\ln(t+1) + \cos\ln(t+1) - 2 + l\right) \\ &= \left(\frac{\mu'(t)}{\mu(t)} + 2l\right) V_1(x-y) \\ &\leq \left(\frac{\mu'(t)}{2\mu(t)} + 2l\right) V_1(x-y) \\ &= \left(\frac{\mu'(t)}{2\mu(t)} + 2l\right) V_1(z) \end{aligned}$$

and

$$\begin{aligned} V_1(z(t_i^+)) &= V_1(x(t_i^+) - y(t_i^+ + \gamma)) \\ &= \left((x_1 - y_1)(1 + \lambda_i) + (x_2 - y_2)\right)^2 \\ &\leq (x_1 - y_1)^2 + (x_2 - y_2)^2 + 2(x_1 - y_1)(x_2 - y_2) \\ &= V_1(x-y) \\ &= V_1(z(t_i)) \end{aligned}$$

Set $u' = g_1(t,u) = \left(\frac{\mu'(t)}{2\mu(t)} + 2l\right) u, \;\; u(t_i^+) = J_i(u) = u, \;\; u(\tau_0) = u_0 \geq 0, \;\; \tau_0 > t_0$ where $g_1(t,u)$ is non-decreasing in t for $t \geq \tau_0$.
The general solution is

$$\begin{aligned} u(t, \tau_0, u_0) &= \frac{u_0}{\exp(2l\tau_0)\mu(\tau_0)^{\frac{1}{2}}} \exp(2lt)\mu^{\frac{1}{2}}(t) \\ &= \frac{u_0}{\exp(2l\tau_0)\mu(\tau_0)^{\frac{1}{2}}} \exp\left((2l + \sin\ln(t+1) - 2)t + \sin\ln(t+1) - 2\right). \end{aligned}$$

Now, $u(t, \tau_0, u_0) \to 0$, as $t \to \infty$ if $l < \frac{1}{2}$; hence, u is stable if $l < \frac{1}{2}$.
Likewise, we have

$$\begin{aligned} D_- V_1(z) + D_- V_2(z) &\leq \left(\frac{\mu'(t)}{\mu(t)} + 2l\right) V_1(z) + \left(\frac{\mu'(t)}{\mu(t)} + 2l\right) V_2(z) \\ &\leq \left(\frac{\mu'(t)}{\mu(t)} + 2l\right) (V_1(z) + V_2(z)). \end{aligned}$$

Also,

$$\begin{aligned}[V_1(z(t_i^+)) + V_2(z(t_i^+))] &= [V_1(x(t_i^+) - y(t_i^+ + \gamma)) + V_2(x(t_i^+) - y(t_i^+ + \gamma))] \\ &= [(x_1 - y_1)(1 + \lambda_i) + (x_2 - y_2)]^2 \\ &\quad + \left[(x_1 - y_1)^2(1 + \lambda_i)^2 + (x_2 - y_2)^2\right] \\ &\leq V_1(x - y) + V_2(x - y) \\ &= V_1(z(t_i)) + V_2(z(t_i))\end{aligned}$$

Set $v' = g_2(t,u) = \left(\frac{\mu'(t)}{\mu(t)} + 2l\right) v, \;\; v(t_i^+) = F_i(v) = v, \;\; v(\tau_0) = v_0 \geq 0, \;\; \tau_0 > t_0.$
The general solution of above system is

$$v(t, \tau_0, u_0) = \frac{v_0}{\exp(2l\tau_0)\mu(\tau_0)} \exp(2lt)\mu(t)$$

Clearly, $v \equiv 0$ is uniformly stable if $l < 1$.
Thus, as per Theorem 4.1, the solution $x(t, t_0, x_0)$ is equistable with reference to initial time difference if $l < 1/2$.

6. Conclusions

In this paper, for the first time, we develop two comparison principles to investigate the stability criteria for impulsive differential equations with initial time difference by employing perturbed Lyapunov function. McRae and Song et al. [15, 18] discussed same stability criteria for differential equations without impulse effect. We extended the criteria for impulsive differential equations with initial time difference. We have applied the technique of perturbing Lyapunov function to obtain the sufficient conditions under much weaker assumptions in comparison to the stability investigated by S.G. Hristova [5] by using Lyapunov function and some comparison results under some rigid conditions. An example is also given to support the proved results.

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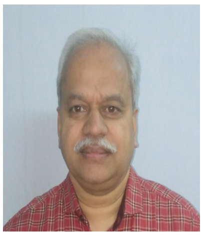

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