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FIXED POINTS IN ϕ-ORDERED PARTIAL QUASI-METRIC SPACE

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Abstract. We obtain fixed points for an order preserving mapping in a preordered left p-complete partial quasi-metric space using a preorder induced by an appropriate function. In particular, we do not make use of any contraction condition.

Keywords: Partial quasi-metric; ϕ-order; fxed point; left p-complete; weakly left-related.

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1. Introduction

Partial metrics were introduced by Matthews[7] in 1992. They generalize the concept of a metric space in the sense that the self-distance from a point to itself need not be equal to zero. In [5], by dropping the symmetry condition in the definition of a partial metric, Künzi et al. studied another variant of partial metrics, namely partial quasi-metrics. Motivated in part by the fact that partial quasi-metric spaces provide suitable frameworks in several areas of asymmetric functional analysis, domain theory, complexity analysis and in modelling partially defined information, which often appears in computer science, the development of the fixed point theory for theses spaces appears to be an interesting focus for current research. In this setting, the problem of fixed point theorems arises in a natural way. This problem is indeed more interesting due to the fact that asymmetric structures present a natural inclination for different type of completeness.

On the other hand, Bhashkar[2] and Lakshmikantham[6] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ (where X a non-empty set) and established some coupled fixed point theorems in partially ordered complete metric spaces.

The aim of this paper is to analyze the existence of coupled and common coupled fixed points for mapping defined on a left p-complete partial quasi-metric space (X, p). The

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approach we use in the present manuscript brings a new feature to the fixed point theory, as we need not use any contractive condition satisfied by the mapping. Moreover the technique of proof is simpler and different from the ones used in fixed point theory.

We conclude this introductory part by recalling some pertinent notions and properties on partial quasi-metric spaces and also from order theory which will be useful later on.

Definition 1.1. *Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two preordered sets. A map $T : X \to Y$ is said to be **preorder-preserving** or **isotone** if for any $x, y \in X$,*

$$x \preceq_X y \Longrightarrow Tx \preceq_Y Ty.$$

*Similarly, for any family $(X_i, \preceq_{X_i})$, $i = 1, 2, \cdots, n$; $(Y, \preceq_Y)$ of posets, a mapping $F : X_1 \times X_2 \times \cdots \times X_n \to Y$ is said to be **preorder-preserving** or **isotone** if for any for any $(x_1, x_2, \cdots, x_n), (z_1, z_2, \cdots, z_n) \in X_1 \times X_2 \times \cdots \times X_n$,*

$$x_i \preceq_{X_i} z_i \text{ for all } i = 1, 2, \cdots, n \Longrightarrow F(x_1, x_2, \cdots, x_n) \preceq_Y F(z_1, z_2, \cdots, z_n).$$

Definition 1.2. *(Compare [7]) A partial metric type on a set X is a function $p : X \times X \to [0, \infty)$ such that:*

(pm1) *$x = y$ iff $(p(x, x) = p(x, y) = p(y, y)$ whenever $x, y \in X$,*
(pm2) *$p(x, x) \leq p(x, y)$ whenever $x, y \in X$,*
(pm3) *$p(x, y) = p(y, x)$; whenever $x, y \in X$,*
(pm4)

$$p(x, y) + p(z, z) \leq p(x, z) + p(z, y)$$

for any points $x, y, z \in X$.

The pair (X, p) is called a partial metric space.

It is clear that, if $p(x, y) = 0$, then, from (pm1) and (pm2), $x = y$.

Definition 1.3. *([5, Definition 1.]) A partial quasi-metric on a set X is a function $p : X \times X \to [0, \infty)$ such that:*

(1a) *$p(x, x) \leq p(x, y)$ whenever $x, y \in X$,,*
(1b) *$p(x, x) \leq p(y, x)$ whenever $x, y \in X$,*
(2) *$p(x, z) + p(y, y) \leq K(p(x, y) + p(y, z))$ whenever $x, y, z \in X$, for some $K \geq 1$,*
(3) *$x = y$ iff $(p(x, x) = p(x, y)$ and $p(y, y) = p(y, x))$ whenever $x, y \in X$.*

The pair (X, p) will be called partial quasi-metric space.

If p satisfies all these conditions except possibly (1b), we shall speak of a lopsided partial quasi-metric type or a lopsided partial quasi-metric.

Remark 1.1. *If p is a partial quasi-metric on X satisfying (4) $p(x, y) = p(y, x)$ whenever $x, y \in X$, then p is called a partial metric on X in the sense of Shukla [8].*

Lemma 1.1. *([5, Lemma 2.])*

(a) *Each quasi-metric p on X is a partial quasi-metric on X with $p(x, x) = 0$ whenever $x \in X$.*
(b) *If p is a partial quasi-metric on X, then so is its conjugate $p^{-1}(x, y) = p(y, x)$ whenever $x, y \in X$.*
(c) *If p is a partial quasi-metric on X, then p^+ defined by $p^+(x, y) = p(x, y) + p^{-1}(x, y)$ is a partial metric on X.*

The notions such as convergence, completeness, Cauchy sequence in the setting of partial metric spaces, can be found in [1, 7] and references therein.

For every partial quasi-metric space (X,p), the collection of balls

$$p(x,\epsilon)=\{y\in X: p(x,y)<\epsilon+p(x,x)\}$$

yields a base for a T_0-Topology $\tau(p)$ on X.

Now, we define Cauchy sequence and convergent sequence in a partial quasi-metric space.

Definition 1.4. *Let (X,p) be a partial quasi-metric space. Let $(x_n)_{n\geq 1}$ be any sequence in X and $x\in X$. Then:*

(1) *The sequence $(x_n)_{n\geq 1}$ is said to be convergent with respect to $\tau(p)$ (or $\tau(p)$-convergent) and converges to x, if $\lim_{n\to\infty} p(x,x_n)=p(x,x)$. We write*

$$x_n \xrightarrow{p} x.$$

(2) *The sequence $(x_n)_{n\geq 1}$ is said to be convergent with respect to $\tau(p^+)$ (or $\tau(p^+)$-convergent) and converges to x, if $\lim_{n\to\infty} p^+(x_n,x)=p^+(x,x)$.*

(3) *The sequence $(x_n)_{n\geq 1}$ is said to be a left p-Cauchy sequence if*

$$\lim_{n\leq m, n,m\to\infty} p(x_n,x_m)$$

exists and is finite.

(4) *(X,p) is said to be left p-complete if for every left p-Cauchy sequence $(x_n)_{n\geq 1}\subseteq X$, there exists $x\in X$ such that:*

$$\lim_{n<m, n,m\to\infty} p(x_n,x_m)=\lim_{n\to\infty} p(x,x_n)=p(x,x).$$

(5) *(X,p) is said to be p-sequentially complete if every $\tau(p^+)$-Cauchy sequence is $\tau(p)$-convergent, i.e. there exists $x\in X$ such that:*

$$\lim_{n,m\to\infty} p^+(x_n,x_m)=\lim_{n\to\infty} p(x,x_n)=p(x,x).$$

(6) *(X,p) is said to be $\tau(p)$-Smyth complete if every left p-Cauchy sequence is $\tau(p^+)$-convergent.*

Dually, we define right p-Cauchy sequence and right p-complete spaces.

Definition 1.5. *Let (X,p) be a partial quasi-metric space. A function $T:X\to X$ is called p-**sequentially continuous** or **left-sequentially continuous** if for any $\tau(p)$-convergent sequence (x_n) with $\lim_{n\to\infty} p(x,x_n)=p(x,x)$, the sequence (Tx_n) $\tau(p)$-converges to Tx, i.e.*

$$\lim_{n\to\infty} p(Tx,Tx_n)=p(Tx,Tx).$$

*Similarly, a function $T: X\times X\to X$ is said to be d-**sequentially continuous** or **left-sequentially continuous** if for any $\tau(p)$-convergent sequences (x_n), (y_n) with*

$$\lim_{n\to\infty} p(x,x_n)=p(x,x),\quad \lim_{n\to\infty} p(y,y_n)=p(y,y)\,,$$

we have

$$\lim_{n\to\infty} p(T(x,y),T(x_n,y_n))=p(T(x,y),T(x,y)).$$

Definition 1.6. *Let X be a non-empty set. An element $(x,y)\in X\times X$ is called:*

(1) *a coupled fixed point of the mappings* $F : X \times X \to X$ *if* $F(x, y) = x$ *and* $F(y, x) = y$*;*
(2) *a coupled coincidence point of the mappings* $F : X \times X \to X$ *and* $T : X \to X$ *if* $F(x, y) = Tx$ *and* $F(y, x) = Ty$ *, and in this case* (Tx, Ty) *is called the coupled point of coincidence;*
(3) *a common coupled fixed point of the mappings* $F : X \times X \to X$ *and* $T : X \to X$ *if* $F(x, y) = Tx = x$ *and* $F(y, x) = Ty = y$*.*

Definition 1.7. *Let* X *be a non-empty set. An element* $(x, y) \in X \times X$ *is called:*
(1) *a common coupled coincidence point of the mappings* $F : X \times X \to X$ *and* $T, R : X \to X$ *if* $F(x, y) = Tx = Rx$ *and* $F(y, x) = Ty = Ry$ *, and in this case* (Tx, Ty) *is called the coupled point of coincidence;*
(2) *a common coupled fixed point of the mappings* $F : X \times X \to X$ *and* $T : X \to X$ *if* $F(x, y) = Tx = Rx = x$ *and* $F(y, x) = Ty = Ry = y$*.*

Our proofs are merely copies by the recent work of Gaba[3] and Agyingi[4].

2. First results

The following lemma is useful in this paper.

Lemma 2.1. *A sequence* (x_n) *in a partial quasi-metric space* (X, p) *is a left* p*-Cauchy, if and only if for any* ε*, there exists* $N \in \mathbb{N}$ *such that for all* $N \le n \le m$*, we have*

$$p(x_n, x_m) - p(x_m, x_m) < \varepsilon.$$

Proof. Since (x_n) is left p-Cauchy, there exists $\lambda \ge 0$ such that for any ε, there exists $N \in \mathbb{N}$ such that for all $N \le n \le m$,

$$|p(x_n, x_m) - \lambda| < \frac{\varepsilon}{2}.$$

For $n = m \ge N$, then $|p(x_n, x_n) - \lambda| < \frac{\varepsilon}{2}$. Therefore

$$|p(x_n, x_m) - p(x_n, x_n)| \le |p(x_n, x_m) - \lambda| + |p(x_n, x_n) - \lambda| < \varepsilon.$$

By (1a) in the definition of a partial quasi-metric, $|p(x_n, x_m) - p(x_n, x_n)| = p(x_n, x_m) - p(x_n, x_n) < \varepsilon$. Conversely it is obvious. □

Lemma 2.2. *Let* (X, p) *be a partial quasi-metric space and* $\phi : X \to \mathbb{R}$ *a map. Define the binary relation "*$\preceq$*" on* X *as follows:*

$$x \preceq y \Longleftrightarrow p(x, y) - p(x, x) \le \phi(y) - \phi(x).$$

Then "$\preceq$*" is a partial order on* X*. It will be called the partial order induced by* ϕ*.*

Proof.

- Reflexivity: For all $x \in X$;

$$p(x, x) - p(x, x) = 0 \le \phi(x) - \phi(x)\,,$$

  hence $x \preceq x$, i.e., "$\preceq$" is reflexive.
- Antisymmetry: If $x \preceq y$ and $y \preceq x$, then

$$p(x, y) - p(x, x) \le \phi(y) - \phi(x) \text{ and } p(y, x) - p(y, y) \le \phi(x) - \phi(y).$$

  This implies $p(x, y) - p(x, x) + p(y, x) - p(y, y) = 0$.

Moreover, from the axioms (1a) and (1b) in the definition of a partial quasi-metric, we have that

$$0 \leq p(x,y) - p(x,x) \qquad \text{and } 0 \leq p(y,x) - p(y,y).$$

Hence

$$p(x,y) - p(x,x) = 0 = p(y,x) - p(y,y).$$

This entails that $p(x,x) = p(x,y)$ and $p(y,y) = p(y,x)$ and from the axiom (3) in the definition of a partial quasi-metric, we have that $x = y$.

- Transitivity: For $x, y, z \in X$ s.t. $x \preceq y$ and $y \preceq z$, we have

$$p(x,y) - p(x,x) \leq \phi(y) - \phi(x) \text{ and } p(y,z) - p(y,y) \leq \phi(z) - \phi(y).$$

$$\begin{aligned} p(x,z) - p(x,x) &\leq p(x,y) + p(y,z) - p(y,y) - p(x,x) \\ &= p(x,y) - p(x,x) + p(y,z) - p(y,y) \\ &\leq \phi(y) - \phi(x) + \phi(z) - \phi(y) \\ &= \phi(z) - \phi(x) \end{aligned}$$

i.e. $x \preceq z$.

Thus, "$\preceq$" is transitive, and so the relation "$\preceq$" is an order on X.

□

Example 2.1. *Let $X = [0, \infty)$ and $p(x,y) = \max\{0, x-y\} + x$, then (X,p) is a partial quasi-metric space. Let $\phi : X \to \mathbb{R}$, $\phi(x) = 2x$. Then for $x, y \in X$*

$$\begin{aligned} x \preceq_\phi y := x \preceq y &\Longleftrightarrow p(x,y) - p(x,x) \leq \phi(y) - \phi(x) \\ &\Longleftrightarrow \max\{0, x-y\} \leq 2y - 2x. \end{aligned}$$

It follows that

$$2 \preceq 4, \qquad \frac{1}{4} \preceq \frac{1}{2}, \qquad 2 \preceq 2.$$

3. Coupled fixed points

This section presents the first main results of this paper.

Theorem 3.1. *Let (X,p) be a Hausdorff left p-complete partial quasi-metric space, $\phi : X \to \mathbb{R}$ be a bounded from above function and $\preceq$ the the preorder induced by ϕ. Let $F : X \times X \to X$ be a preorder-preserving and p-sequentially continuous mapping on X such that there exist two elements $x_0; y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \textit{and} \quad y_0 \preceq F(y_0, x_0).$$

Then F has a couple fixed point in X^2.

Proof. Let $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \preceq F(y_0, x_0).$$

We construct the sequences $(x_n), (y_n)$ as follows:

$$x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 0.$$

We show that

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_n \preceq y_{n+1} \quad \text{for all } n \geq 0. \tag{1}$$

To achieve this, we shall use the mathematical induction.
Since $x_0 \preceq F(x_0, y_0)$ and $y_0 \preceq F(y_0, x_0)$ and as $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \preceq x_1$ and $y_0 \preceq y_1$. Thus (1) holds for $n = 0$.
Suppose now that (1) holds for some $k > 0$. Then, since $x_k \preceq x_{k+1}$ and $y_k \preceq y_{k+1}$ and F is preorder preserving, we have

$$x_{k+1} = F(x_k, y_k) \preceq F(x_{k+1}, y_{k+1}) = x_{k+2}$$

and

$$y_{k+1} = F(y_k, x_k) \preceq F(y_{k+1}, x_{k+1}) = y_{k+2}.$$

Thus, by mathematical induction, we conclude that (1) holds for all $n \geq 0$. Therefore,

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots ,$$

and

$$y_0 \preceq y_1 \preceq y_2 \preceq \cdots \preceq y_n \preceq .$$

By definition of the preorder, we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \cdots \leq \phi(x_n) \leq \cdots ,$$

and

$$\phi(y_0) \leq \phi(y_1) \leq \phi(y_2) \leq \cdots \leq \phi(y_n) \leq \cdots .$$

Hence, the sequences $(\phi(x_n))$ and $(\phi(y_n))$ are non-decreasing sequences of real numbers. Since ϕ is bounded from above, the sequences $(\phi(x_n))$ and $(\phi(y_n))$ are convergent and, therefore, Cauchy. This entails that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $m \geq n \geq n_0$, we have $\phi(x_m) - \phi(x_n) < \varepsilon$ and $\phi(y_m) - \phi(y_n) < \varepsilon$. It follows that

$$p(x_n, x_m) - p(x_n, x_n) \leq \phi(x_m) - \phi(x_n) < \varepsilon, \quad p(y_n, y_m) - p(y_n, y_n) \leq \phi(y_m) - \phi(y_n) < \varepsilon.$$

We conclude that (x_n) and (y_n) are left p-Cauchy in X and since X is left p-complete, there exist $x^*, y^* \in X$ such that

$$\lim_{n<m, n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x^*, x_n) = p(x^*, x^*) ,$$

$$\lim_{n<m, n,m\to\infty} p(y_n, y_m) = \lim_{n\to\infty} p(y^*, y_n) = p(y^*, y^*).$$

Since F is p-sequentially continuous and X Hausdorff, we have

$$x_n \xrightarrow{p} x^* \Longleftrightarrow x_n = F(x_{n-1}, y_{n-1}) \xrightarrow{p} F(x^*, y^*) = x^*,$$

and

$$y_n \xrightarrow{p} y^* \Longleftrightarrow y_n = F(y_{n-1}, x_{n-1}) \xrightarrow{p} F(y^*, x^*) = y^*.$$

Thus, we have proved that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, i.e. (x^*, y^*) is a coupled fixed point of F.

□

Corollary 3.1. *Let (X,p) be a Hausdorff right p-complete partial quasi-metric space, $\phi : X \to \mathbb{R}$ be a bounded from above function and $\preceq$ the the preorder induced by ϕ. Let $F : X \times X \to X$ be a preorder-preserving and p^{-1}-sequentially continuous mapping on X such that there exist two elements $x_0; y_0 \in X$ with*

$$F(x_0, y_0) \preceq x_0 \quad \text{and} \quad F(y_0, x_0) \preceq y_0.$$

Then F has a couple fixed point in X^2.

Example 3.1. *Let $X = [0,\infty)$ and $p(x,y) = \max\{0, x-y\} + x$, then (X,p) is a left p-complete partial quasi-metric space and $\leq$ is the usual ordering. We define $\phi : X \to \mathbb{R}$ as $\phi(x) = 2x$ and $F : X \times X \to X$ as*

$$F(x,y) = x(1+y)$$

which is obviously a non-decreasing function on X^2.
We choose $x_0 = 1$ and $y_0 = 0$. Then

$$F(x_0, y_0) = 1.(1+0) = 1 \quad \text{and} \quad F(y_0, x_0) = 0.(1+1) = 0.$$

We see that $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$. Also

$$F(0,y) = 0 \quad \text{and} \quad F(0,x) = 0.$$

Hence $(0,0)$ is a coupled fixed point of F.

4. Common coupled fixed point

Now, we define the concept of weakly related mappings on preordered spaces as follows:

Definition 4.1. *(Compare* [3, Definition 4.2]*)*
*Let $(X,\preceq)$ be a preordered space, and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then the pair $\{F, g\}$ is said to be **weakly left-related** if the following two conditions are satisfied:*

(C1)

$$F(x,y) \preceq gF(x,y) \quad \text{and} \quad gx \preceq F(gx, gy)$$

for all $(x,y) \in X^2$,

(C2)

$$F(y,x) \preceq gF(y,x) \quad \text{and} \quad gy \preceq F(gy, gx)$$

for all $(x,y) \in X^2$.

One can easily deduce the concept of weakly right-related.

We are now ready to state our first common coupled fixed point existence theorem.

Theorem 4.1. *Let (X,p) be a Hausdorff left p-complete partial quasi-metric space, $\phi : X \to \mathbb{R}$ be a bounded from above function and $\preceq$ the the preorder induced by ϕ. Let $F : X \times X \to X$ and $G : X \to X$ be two p-sequentially continuous mappings such that the pair $\{F, G\}$ is weakly left-related. If there exist two elements $x_0; y_0 \in X$ with*

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \preceq F(y_0, x_0).$$

Then, F and G have a common couple fixed point.

Proof. Let $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \preceq F(y_0, x_0).$$

We construct the sequences $(x_n), (y_n)$ as follows:

$$x_{2n+1} = F(x_{2n}, y_{2n}) \quad \text{and} \quad x_{2n+2} = Gx_{2n+1} \quad \text{for all } n \geq 0$$

and

$$y_{2n+1} = F(y_{2n}, x_{2n}) \quad \text{and} \quad y_{2n+2} = yx_{2n+1} \quad \text{for all } n \geq 0.$$

We show that

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_n \preceq y_{n+1} \quad \text{for all } n \geq 0. \tag{2}$$

Since $x_0 \preceq F(x_0, y_0)$ and using $x_{2n+1} = F(x_{2n}, y_{2n})$, we have $x_0 \preceq x_1$. Again, since the $\{F, G\}$ is weakly left-related, we have $x_1 = F(x_0, y_0) \preceq GF(x_0, y_0) = Gx_1 = x_2$, i.e. $x_1 \preceq x_2$. Also, since $x_2 = Gx_1 \preceq F(Gx_1, Gy_1) = F(x_2, y_2) = x_3$, we have that $x_2 \preceq x_3$. Similarly, using the fact that the pair $\{F, G\}$ is weakly left-related and the relations $x_{2n+1} = F(x_{2n}, y_{2n}),\ x_{2n+2} = Gx_{2n+1}$, we get

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots .$$

A similar reasoning, using fact that the pair $\{F, G\}$ is weakly left-related and the relations $y_{2n+1} = F(y_{2n}, x_{2n}),\ y_{2n+2} = yx_{2n+1}$, leads to

$$y_0 \preceq y_1 \preceq y_2 \preceq \cdots \preceq y_n \preceq \cdot$$

By definition of the preorder, we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \cdots \leq \phi(x_n) \leq \cdots ,$$

and

$$\phi(y_0) \leq \phi(y_1) \leq \phi(y_2) \leq \cdots \leq \phi(y_n) \leq \cdots .$$

Hence, the sequences $(\phi(x_n))$ and $(\phi(y_n))$ are non-decreasing sequences of real numbers. Since ϕ is bounded from above, the sequences $(\phi(x_n))$ and $(\phi(y_n))$ are convergent and, therefore, Cauchy. This entails that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $m \geq n \geq n_0$, we have $\phi(x_m) - \phi(x_n) < \varepsilon$ and $\phi(y_m) - \phi(y_n) < \varepsilon$. It follows that

$$p(x_n, x_m) - p(x_n, x_n) \leq \phi(x_m) - \phi(x_n) < \varepsilon, \quad p(y_n, y_m) - p(y_n, y_n) \leq \phi(y_m) - \phi(y_n) < \varepsilon.$$

We conclude that (x_n) and (y_n) are left p-Cauchy in X and since X is left p-complete, there exist $x^*, y^* \in X$ such that

$$\lim_{n<m, n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x^*, x_n) = p(x^*, x^*)\,,$$

$$\lim_{n<m, n,m\to\infty} p(y_n, y_m) = \lim_{n\to\infty} p(y^*, y_n) = p(y^*, y^*).$$

Since F and G are p-sequentially continuous, it is easy to see that

$$x_{2n+1} \xrightarrow{p} x^* \Longleftrightarrow x_{2n+1} = F(x_{2n}, y_{2n}) \xrightarrow{p} F(x^*, y^*) = x^*,$$

and

$$x_{2n+2} \xrightarrow{p} x^* \Longleftrightarrow x_{2n+2} = G(x_{2n+1}) \xrightarrow{p} G(x^*) = x^*,$$

and hence

$$Gx^* = x^* = F(x^*, y^*).$$

Similarly, since F and G are p-sequentially continuous, we easily derive that

$$Gy^* = y^* = F(y^*, x^*).$$

Thus, we have proved that (x^*, y^*) is a common coupled fixed point of F and G.

□

Example 4.1. *Let $X = [0, \infty)$ and $p(x, y) = \max\{0, x - y\} + x$, then (X, p) is a left p-complete partial quasi-metric space and $\leq$ is the usual ordering. We define, for $a \geq 0$, $\phi_a : X \to \mathbb{R}$ as $\phi_a(x) = ax$ and $F : X \times X \to X$ and $G : X \to X$ as*

$$F(x, y) = x + |\sin(xy)| \quad \text{and } Gx = 5x.$$

We set $x_0 = 1$ and $y_0 = 0$ then $F(x_0, y_0) = 1 + 0 = 1$ and $F(y_0, x_0) = 0 + 0 = 0$. So $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$. We have

$$F(x, y) = x + |\sin(xy)| \leq 5(x + |\sin(xy)|) = GF(x, y)$$

and

$$Gx = 5x \leq 5x + |\sin(25xy)| = F(Gx, Gy) = F(5x, 5y).$$

Again $F(y, x) = y + |\sin(xy)|$ and $Gy = 5y$ so we get $F(y, x) \leq GF(y, x)$ and $Gy \leq F(Gx, Gy)$ and so the pair $\{F, G\}$ is weakly left-related.
Hence we see that all the conditions of our theorem are satisfied. Also we have $F(0, y) = 0, G0 = 0$ and $F(0, x) = 0$ implying $F(0, y) = 0 = G0$ and $F(0, x) = 0 = G0$. Thus $(0, 0)$ is a common coupled fixed point of the pair $\{F, G\}$.

Corollary 4.1. *Let (X, p) be a Hausdorff right p-complete partial quasi-metric space, $\phi : X \to \mathbb{R}$ be a bounded from above function and $\preceq$ the the preorder induced by ϕ. Let $F : X \times X \to X$ and $G : X \to X$ be two p^{-1}-sequentially continuous mappings such that the pair $\{F, G\}$ is weakly left-related. If there exist two elements $x_0; y_0 \in X$ with*

$$F(x_0, y_0) \preceq x_0 \quad \text{and} \quad F(y_0, x_0) \preceq y_0.$$

Then, F and G have a common couple fixed point.

Next we give a common coupled fixed point theorem for three maps. This result is more interesting since we drop the hypothesis $x_0 \preceq F(x_0, y_0)$ and $y_0 \preceq F(y_0, x_0)$.

Theorem 4.2. *Let (X, p) be a Hausdorff left p-complete partial quasi-metric space, $\phi : X \to \mathbb{R}$ be a bounded from above function and $\preceq$ the the preorder induced by ϕ. Let $F : X \times X \to X$ and $G, H : X \to X$ be three p-sequentially continuous mappings such that the pairs $\{F, G\}$ and $\{F, H\}$ are weakly left-related.*
Then, F, G and H have a common couple fixed point.

Proof. The reasoning follows the same steps as in the previous proofs.

Let $x_0, y_0 \in X$. We construct the sequences (x_n) and (y_n) as follows:

$$Hx_{3n-3} = x_{3n-2},\ x_{3n-1} = F(x_{3n-2}, y_{3n-2}),\ x_{3n} = Gx_{3n-1} \quad \text{for all } n \geq 1,$$

and

$$Hy_{3n-3} = y_{3n-2},\ y_{3n-1} = F(y_{3n-2}, x_{3n-2}),\ y_{3n} = Gy_{3n-1} \quad \text{for all } n \geq 1.$$

We have $Hx_0 = x_1$, $Hy_0 = y_1$ and since the pairs $\{F, H\}$ and $\{F, G\}$ are weakly left-related, it follows that

$$x_1 \preceq Hx_0 \preceq F(Hx_0, Hy_0) = F(x_1, y_1) = x_2,$$

and

$$x_2 = F(x_1, y_1) \preceq GF(x_1, y_1) = Gx_2 = x_3,$$

i.e. $x_1 \preceq x_2 \preceq x_3$.
Using repeatedly that the pairs $\{F, G\}$ and $\{F, H\}$ are weakly left-related, we get

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots .$$

A similar reasoning, using fact that the pairs $\{F, H\}$ and $\{F, G\}$ are weakly left-related, leads to

$$y_0 \preceq y_1 \preceq y_2 \preceq \cdots \preceq y_n \preceq \cdot$$

By definition of the preorder, we have

$$\phi(x_0) \leq \phi(x_1) \leq \phi(x_2) \leq \cdots \leq \phi(x_n) \leq \cdots ,$$

and

$$\phi(y_0) \leq \phi(y_1) \leq \phi(y_2) \leq \cdots \leq \phi(y_n) \leq \cdots .$$

Hence, the sequences $(\phi(x_n))$ and $(\phi(y_n))$ are non-decreasing sequences of real numbers. Since ϕ is bounded from above, the sequences $(\phi(x_n))$ and $(\phi(y_n))$ are convergent and, therefore, Cauchy. This entails that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $m \geq n \geq n_0$, we have $\phi(x_m) - \phi(x_n) < \varepsilon$ and $\phi(y_m) - \phi(y_n) < \varepsilon$. It follows that

$$p(x_n, x_m) - p(x_n, x_n) \leq \phi(x_m) - \phi(x_n) < \varepsilon, \quad p(y_n, y_m) - p(y_n, y_n) \leq \phi(y_m) - \phi(y_n) < \varepsilon.$$

We conclude that (x_n) and (y_n) are left p-Cauchy in X and since X is left p-complete, there exist $x^*, y^* \in X$ such that

$$\lim_{n<m, n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x^*, x_n) = p(x^*, x^*),$$

$$\lim_{n<m, n,m\to\infty} p(y_n, y_m) = \lim_{n\to\infty} p(y^*, y_n) = p(y^*, y^*).$$

Since F, G and H are p-sequentially continuous, it is easy to see that

$$x_{3n-1} \xrightarrow{p} x^* \Longleftrightarrow x_{3n-1} = F(x_{3n-2}, y_{3n-2}) \xrightarrow{p} F(x^*, y^*) = x^*,$$

$$x_{3n} \xrightarrow{p} x^* \Longleftrightarrow x_{3n} = Gx_{3n-1} \xrightarrow{p} x^* \Longleftrightarrow Gx^* = x^*,$$

and

$$x_{3n-2} \xrightarrow{p} x^* \Longleftrightarrow x_{3n-2} = Hx_{3n-3} \xrightarrow{p} x^* \Longleftrightarrow Hx^* = x^*,$$

and hence

$$Hx^* = Gx^* = x^* = F(x^*, y^*).$$

Similarly, since F, G and H are p-sequentially continuous, we easily derive that

$$Hx^* = Gy^* = y^* = F(y^*, x^*).$$

Thus, we have proved that (x^*, y^*) is a common coupled fixed point of F, G and H.

□

Corollary 4.2. *Let (X, p) be a Hausdorff left p-complete partial quasi-metric space, $\phi : X \to \mathbb{R}$ be a bounded from above function and $\preceq$ the the preorder induced by ϕ. Let $F : X \times X \to X$ and $G, H : X \to X$ be three p^{-1}-sequentially continuous mappings such that the pairs $\{F, G\}$ and $\{F, H\}$ are weakly right-related.*
Then, F, G and H have a common couple fixed point.

Conflict of interest.

The author declares that there is no conflict of interests regarding the publication of this article.

Data availability

No data was used in the present manuscript.

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