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ON THE CONNECTED DETOUR MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT. For a connected graph $G = (V, E)$ of order at least two, a connected detour monophonic set S of G is called a minimal connected detour monophonic set if no proper subset of S is a connected detour monophonic set of G. The upper connected detour monophonic number of G, denoted by $dm_c^+(G)$, is defined as the maximum cardinality of a minimal connected detour monophonic set of G. We determine bounds for it and find the same for some special classes of graphs. For any three positive integers a, b and n with $6 \leq a \leq n \leq b$, there is a connected graph G with $dm_c(G) = a$, $dm_c^+(G) = b$ and a minimal connected detour monophonic set of cardinality n.

Keywords: detour monophonic set, connected detour monophonic set, connected detour monophonic number, minimal connected detour monophonic set, upper connected detour monophonic number.

AMS Subject Classification: 05C12

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [3]. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an extreme vertex if the subgraph induced by its neighbors is complete.

The closed interval $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices of a graph G is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [1, 4] and further studied in [2].

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A chord of a path P is an edge joining two non-adjacent vertices of P. A path P is called a monophonic path if it is a chordless path. A longest $x - y$ monophonic path is called an $x - y$ detour monophonic path. A set S of vertices of G is a detour monophonic set if each vertex v of G lies on an $x - y$ detour monophonic path for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the detour monophonic number of G and is denoted by $dm(G)$. The detour monophonic number of a graph was introduced in [8] and further studied in [7]. A connected detour monophonic set of G is a detour monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected detour monophonic set of G is the connected detour monophonic number of G and is denoted by $dm_c(G)$. The connected detour monophonic number of a graph G was introduced and studied in [9].

For any two vertices u and v in a connected graph G, the monophonic distance $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The monophonic eccentricity $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The monophonic radius, $rad_m G$ of G is $rad_m G = \min \{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $diam_m G$ of G is $diam_m G = \max \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [5, 6]. The following theorems will be used in the sequel.

Theorem 1.1. [9] Each extreme vertex of a connected graph G belongs to every connected detour monophonic set of G.

Theorem 1.2. [9] Let G be a connected graph with cutvertices and let S be a connected detour monophonic set of G. If v is a cutvertex of G, then every component of $G - v$ contains an element of S.

Theorem 1.3. [9] Every cutvertex of a connected graph G belongs to every connected detour monophonic set of G.

Theorem 1.4. [9] Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $dm_c(G) = 2$.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Upper Connected Detour Monophonic Number

Definition 2.1. *Let G be a connected graph. A connected detour monophonic set S of G is called a minimal connected detour monophonic set if no proper subset of S is a connected detour monophonic set of G. The upper connected detour monophonic number of G, denoted by $dm_c^+(G)$, is defined as the maximum cardinality of a minimal connected detour monophonic set of G.*

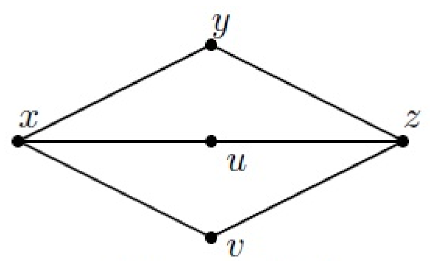

Figure 2.1: G

Example 2.1. For the graph G given in Figure 2.1, the minimal connected detour monophonic sets are $S_1 = \{x, y, z\}$, $S_2 = \{x, u, z\}$, $S_3 = \{x, v, z\}$, $S_4 = \{x, y, u, v\}$ and $S_5 = \{z, y, u, v\}$. Hence the upper connected detour monophonic number of G is 4.

Note 2.1. *Every minimum connected detour monophonic set is a minimal connected detour monophonic set, and the converse need not true. For the graph G given in Figure 2.1, S_4 is a minimal connected detour monophonic set and it is not a minimum connected detour monophonic set of G.*

Since every minimal connected detour monophonic set of G is a connected detour monophonic set of G, we have the following theorems.

Theorem 2.1. Each extreme vertex of a connected graph G belongs to every minimal connected detour monophonic set of G.

Proof. This follows from Theorem 1.1. □

Corollary 2.1. *For the complete graph K_p, $dm_c^+(K_p) = p$.*

Theorem 2.2. Let G be a connected graph with cutvertices and let S be a minimal connected detour monophonic set of G. If v is a cutvertex of G, then every component of $G - v$ contains an element of S.

Proof. This follows from Theorem 1.2. □

Theorem 2.3. Every cutvertex of a connected graph G belongs to every minimal connected detour monophonic set of G.

Proof. This follows from Theorem 1.3. □

Corollary 2.2. *For any non-trivial tree T of order p, $dm_c(T) = dm_c^+(T) = p$.*

Proof. This follows from Theorems 1.1, 1.3, 2.1 and 2.3. □

Theorem 2.4. For the complete bipartite graph $G = K_{m,n}$,
(i) $dm_c^+(G) = 2$ if $m = n = 1$.
(ii) $dm_c^+(G) = n + 1$ if $m = 1, n \geq 2$.
(iii) $dm_c^+(G) = max\{m, n\} + 1$ if $m, n \geq 2$.

Proof. (i) and (ii) follow from Corollary 2.2.
(iii) Let $m, n \geq 2$. Assume without loss of generality that $m \leq n$. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of G. Clearly $S = Y$ is a minimal detour monophonic set of G. Since $G[S]$ is not connected, and since $S' = S \cup \{x_i\}$ is a minimal connected detour monophonic set of G for any $i(1 \leq i \leq m)$, we have $dm_c^+(G) \geq n + 1$.

Let S_1 be any minimal connected detour monophonic set of G such that $|S_1| \geq n + 2$. Since any vertex $y_i(1 \leq i \leq n)$ lies on the detour monophonic path x_j, y_i, x_k for $j \neq k$, it follows that $X \cup \{y_i\}$ for some i, is a connected detour monophonic set of G. Hence S_1 cannot contain $X \cup \{y_i\}$ for some i. Similarly, since $Y \cup \{x_j\}$ for some j, is a connected detour monophonic set of G, S_1 cannot contain $Y \cup \{x_j\}$ for some j. Hence $S_1 \subset X' \cup Y'$, where $X' \subset X$ and $Y' \subset Y$. Hence there exists a vertex $x_i \in X(1 \leq i \leq m)$ and a vertex $y_j \in Y(1 \leq j \leq n)$ such that $x_i, y_j \notin S_1$.

Suppose that S_1 contains exactly one vertex from X', then the vertex $y_j\,(1 \leq j \leq n)$ is not an internal vertex of any $x - y$ detour monophonic path for some $x, y \in S_1$ and so S_1 is not a minimal connected detour monophonic set of G. Suppose that S_1 contains exactly one vertex from Y', then the vertex $x_i\,(1 \leq i \leq m)$ is not an internal vertex of any $x - y$ detour monophonic path for some $x, y \in S_1$ and so S_1 is not a minimal connected detour monophonic set of G. If S_1 contains more than one vertex from both X' and Y', then $S' = \{x_{i'}, x_{j'}, y_l, y_k\}$ is a connected detour monophonic set of G for some $x_{i'}, x_{j'} \in X'$ and $y_l, y_k \in Y'$. It follows that S_1 is not a minimal connected detour monophonic set of G, which is a contradiction. Thus any minimal connected detour monophonic set of G contains at most $n + 1$ elements so that $dm_c^+(G) \leq n + 1$. Hence $dm_c^+(G) = n + 1$. □

Theorem 2.5. For any connected graph G of order $p \geq 2$, $2 \leq dm_c(G) \leq dm_c^+(G) \leq p$.

Proof. Any connected detour monophonic set needs at least two vertices and so $dm_c(G) \geq 2$. Since every minimal connected detour monophonic set of G is also a connected detour monophonic set of G, it follows that $dm_c(G) \leq dm_c^+(G)$. Also, since $V(G)$ induces a connected detour monophonic set of G, it is clear that $dm_c^+(G) \leq p$. □

Remark 2.1. The bounds in Theorem 2.5 are sharp. For the complete graph K_2, $dm_c(K_2) = dm_c^+(K_2) = 2$ and if G is a non-trivial tree of order p, then $dm_c(G) = dm_c^+(G) = p$. All the inequalities in Theorem 2.5 are strict. For graph G given in Figure 2.1, $dm_c(G) = 3$, $dm_c^+(G) = 4$ and $p = 5$. Thus we have $2 < dm_c(G) < dm_c^+(G) < p$.

Theorem 2.6. Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $dm_c^+(G) = 2$.

Proof. This follows from Theorems 1.4 and 2.5. □

Theorem 2.7. Let G be a connected graph of order p with every vertex of G is either a cutvertex or an extreme vertex. Then $dm_c^+(G) = p$.

Proof. Let G be a connected graph with every vertex of G is either a cutvertex or an extreme vertex. Then by Theorems 2.1 and 2.3, we have $dm_c^+(G) = p$. □

Remark 2.2. The converse of Theorem 2.7 is not true. For the graph G given in Figure 2.2, $dm_c^+(G) = p$, but the vertex x is neither a cutvertex nor an extreme vertex of G.

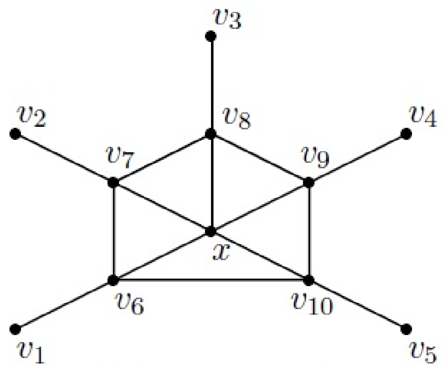

Figure 2.2: G

Theorem 2.8. For a connected graph G, $dm_c^+(G) = p$ if and only if $dm_c(G) = p$.

Proof. Let $dm_c^+(G) = p$. Then $S = V(G)$ is the unique minimal connected detour monophonic set of G. Since no proper subset of S is a connected detour monophonic set of G, it is clear that S is the unique minimum connected detour monophonic set of G and so $dm_c(G) = p$. The converse follows from Theorem 2.5. □

Theorem 2.9. If G is a connected graph of order p with $dm_c(G) = p-1$, then $dm_c^+(G) = p-1$.

Proof. Let $dm_c(G) = p-1$. Then by Theorem 2.5, we have $dm_c^+(G) = p$ or $p-1$. If $dm_c^+(G) = p$, then by Theorem 2.8, $dm_c(G) = p$, which is a contradiction. Hence $dm_c^+(G) = p-1$. □

Remark 2.3. The converse of Theorem 2.9 is not true. For example, consider the graph G given in Figure 2.1. It is clear that $dm_c^+(G) = p-1$ and $dm_c(G) = p-2$.

We leave the following problem as an open question.

Problem 2.1. Characterize graphs G for which $dm_c(G) = dm_c^+(G)$.

3. Realization results for $dm_c^+(G)$

In view of Theorem 2.5, we have the following realization theorem.

Theorem 3.1. For every pair a, b of positive integers with $4 \le a \le b$, there is a connected graph G with $dm_c(G) = a$ and $dm_c^+(G) = b$.

Proof. Case 1. $a = b$. Let G be any tree having a endvertices. Then by Corollary 2.2, G has the desired property.

Case 2. $a < b$. Let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained by adding $b-2$ new vertices $v_1, v_2, \dots, v_{a-3}, w_1, w_2, \dots, w_{b-a+1}$ to P_3 and joining each $w_i (1 \le i \le b-a+1)$ to both x, z; and also joining each $v_i (1 \le i \le a-3)$ to x. The graph G is shown in Figure 3.1. By Theorems 1.1, 1.3, 2.1 and 2.3, $S = \{v_1, v_2, \dots, v_{a-3}, x\}$ is contained in every connected detour monophonic set and every minimal connected detour monophonic set of G. It is clear that S is not a detour monophonic set of G. It is easily verified that $S' = S \cup \{z\}$ is a detour monophonic set of G. Since the induced subgraph $G[S']$ is not connected, S' is not a connected detour monophonic set of G. Now, for any vertex $v \in \{y, w_1, w_2, \dots, w_{b-a+1}\}$, it is clear that $S' \cup \{v\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = a$.

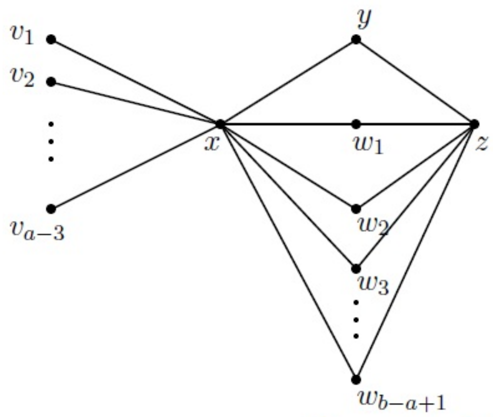

Figure 3.1: G

Next we show that $dm_c^+(G) = b$. Clearly, $T = S \cup \{y, w_1, w_2, \dots, w_{b-a+1}\}$ is a connected detour monophonic set of G. We claim that T is a minimal connected detour monophonic set of G. Assume, to the contrary, that T is not a minimal connected detour monophonic set of G. Then there is a proper subset W of T such that W is a connected detour monophonic set of G. Hence there exists a vertex, say v, such that $v \in T$ and $v \notin W$. By Theorems 2.1 and 2.3, $v \in \{y, w_1, w_2, \dots, w_{b-a+1}\}$. It is easily verified that v is not an internal vertex of any $x - y$ detour monophonic path for some $x, y \in W$, and it follows that W is not a connected detour monophonic set of G, which is a contradiction. Hence T is a minimal connected detour monophonic set of G and so $dm_c^+(G) \ge b$.

Now, we prove that $dm_c^+(G) = b$. Suppose that $dm_c^+(G) > b$. Let N be a minimal connected detour monophonic set of G with $|N| > b$. Then $N = V(G)$. Since T is a proper subset of N, N is not a minimal connected detour monophonic set of G. Therefore $dm_c^+(G) = b$. □

Remark 3.1. The graph G in Figure 3.1 contains exactly $b - a + 3$ minimal connected detour monophonic sets, namely $S \cup \{y, z\}$, $S \cup \{w_i, z\} (1 \le i \le b - a + 1)$ and $S \cup \{y, w_1, w_2, \dots, w_{b-a+1}\}$. Therefore $dm_c(G) = a$ and $dm_c^+(G) = b$. If $b - a \ge 2$, then this example shows that there is no "Intermediate Value Theorem" for minimal connected detour monophonic sets, that is, if n is an integer such that $dm_c(G) < n < dm_c^+(G)$, then there need not exist a minimal connected detour monophonic set of cardinality n in G.

Theorem 3.2. For any three positive integers a, b and n with $6 \leq a \leq n \leq b$, there exists a connected graph G with $dm_c(G) = a$, $dm_c^+(G) = b$ and a minimal connected detour monophonic set of cardinality n.

Proof. We prove this theorem by considering four cases.

Case 1. $a = n = b$. Let G be any tree with number of endvertices a. Then by Corollary 2.2, G has the desired property.

Case 2. $a = n < b$. For the graph G given in Figure 3.1 of Theorem 3.1, it is proved that $dm_c(G) = a, dm_c^+(G) = b$ and $S = \{v_1, v_2, \ldots, v_{a-3}, x, y, z\}$ is a minimal connected detour monophonic set of cardinality n.

Case 3. $a < n = b$. For the graph G given in Figure 3.1 of Theorem 3.1, it is proved that $dm_c(G) = a, dm_c^+(G) = b$ and $S = \{v_1, v_2, \ldots, v_{a-3}, x, y, w_1, w_2, \ldots, w_{b-a+1}\}$ is a minimal connected detour monophonic set of cardinality n.

Case 4. $a < n < b$. Let $l = n - a + 2$ and $m = b - n + 2$. Let $F_1 = mK_1 + \overline{K_2}$, where $U_1 = V(\overline{K_2}) = \{x, u_1\}$ and $X = V(mK_1) = \{x_1, x_2, \ldots, x_m\}$. Similarly, $F_2 = lK_1 + \overline{K_2}$, where $U_2 = V(\overline{K_2}) = \{u_2, y\}$ and $Y = V(lK_1) = \{y_1, y_2, \ldots, y_l\}$. Let $K_{1,a-5}$ be the star at the vertex u and let $S = \{w_1, w_2, \ldots, w_{a-5}\}$ be the set of endvertices of $K_{1,a-5}$. Let G be the graph obtained from F_1, F_2 and $K_{1,a-5}$ by identifying the vertices u_1 from F_1, u_2 from F_2 and u from $K_{1,a-5}$. The graph G is shown in Figure 3.2. It follows from Theorems 1.1, 1.3, 2.1 and 2.3, every connected detour monophonic set and every minimal connected detour monophonic set of G contains $S' = S \cup \{u\}$.

First we show that $dm_c(G) = a$. It is clear that S' is not a detour monophonic set of G. Also, for any vertex $v \in V(G) - S'$, $S' \cup \{v\}$ is not a detour monophonic set of G. Let $S'' = S' \cup \{x, y\}$. It is easily verified that S'' is a minimum detour monophonic set of G, which is not connected. For any vertex $w \in \{x_1, x_2, \ldots, x_m\}$ and $v \in \{y_1, y_2, \ldots, y_l\}$, it is clear that $S'' \cup \{v, w\}$ is a minimum connected detour monophonic set of G and so $dm_c(G) = a$.

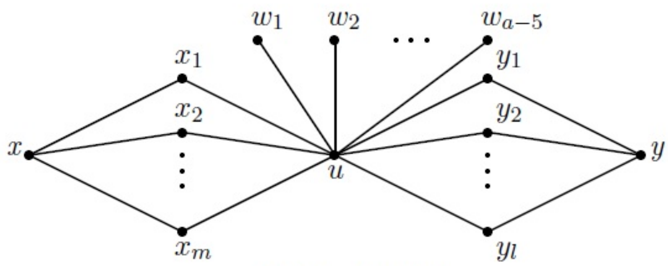

Figure 3.2: G

Next, we show that $dm_c^+(G) = b$. Let $T = S' \cup X \cup Y$. It is clear that T is a connected detour monophonic set of G. We claim that T is a minimal connected detour monophonic set of G. Assume, to the contrary, that T is not a minimal connected detour monophonic set of G. Then there is a proper subset W of T such that W is a connected detour monophonic set of G. Hence there exists a vertex, say $v \in T$, such that $v \notin W$. Assume first that $v = x_i$ for some $i (1 \leq i \leq m)$ or $v = y_j$ for some $j (1 \leq j \leq l)$. Then the vertex v is not an internal vertex of any detour monophonic path joining a pair of vertices in W. If $v = w_i$ for some $i (1 \leq i \leq a-5)$, then the vertex v is not an internal vertex of any $x - y$ detour monophonic path for some $x, y \in W$. If $v = u$, then the vertex v is an internal vertex of any $x_i - y_j$ detour monophonic path for some i, j. It is easily verified that W is a detour monophonic set of G, which is not connected. Hence T is a minimal connected detour monophonic set of G so that $dm_c^+(G) \geq b$.

Now, we prove that $dm_c^+(G) = b$. Suppose that $dm_c^+(G) > b$. Let T' be a minimal connected detour monophonic set of G with $|T'| > b$. Then there exists at least one vertex, say $v \in T'$ such that $v \notin T$. Clearly, $v \in \{x, y\}$. If $v = x$, then $(T' - X) \cup \{x_1\}$ is a connected detour monophonic set of G and it is a proper subset of T', which is a contradiction to T' a minimal connected detour monophonic set of G. Similarly, if $v = y$, then $(T' - Y) \cup \{y_1\}$ is a connected detour monophonic set of G and it is a proper subset of T', which is a contradiction. Hence $dm_c^+(G) = b$.

Finally, we show that there is a minimal connected detour monophonic set of cardinality n. Let $P = S \cup Y \cup \{u, x, x_1\}$. It is clear that P is a connected detour monophonic set of G. We claim that P is a minimal connected detour monophonic set of G. Assume, to the contrary, that P is not a minimal connected detour monophonic set. Then there is a proper subset P' of P such that P' is a connected detour monophonic set of G. Let $v \in P$ and $v \notin P'$. By Theorems 1.1 and 1.3, clearly $v = x$, $v = x_1$ or $v = y_i$ for some $i = 1, 2, \ldots, l$. If $v = x$, then the vertex v is not an internal vertex of any $s - t$ detour monophonic path for some $s, t \in P'$. If $v = y_i$ for some $i = 1, 2, \ldots l$, then the vertex v is not an internal vertex of any $s - t$ detour monophonic path for some $s, t \in P'$. If $v = x_1$, then the vertex v is an internal vertex of any $x - w_i$ detour monophonic path for some i. It is easily verified that P' is a detour monophonic set of G, which is not connected. Thus P is a minimal connected detour monophonic set of G with $|P| = n$. Hence the theorem. □

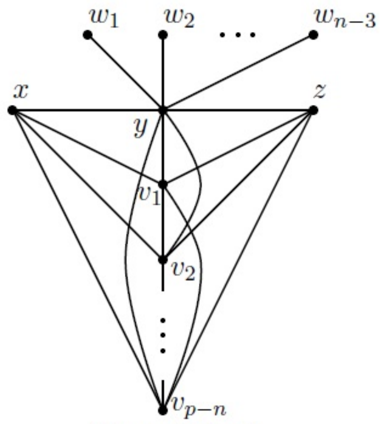

Figure 3.3: G

Theorem 3.3. If p, d and n are positive integers such that $2 \leq d \leq p - 2$, $4 \leq n \leq p$ and $p - d - n + 1 \geq 0$, then there exists a connected graph G of order p, monophonic diameter d and $dm_c^+(G) = n$.

Proof. We prove this theorem by considering three cases.

Case 1. $d = 2$. First, let $n = p$. Then the star $K_{1,n-1}$ has the desired property. Now, let $4 \leq n < p$. Let $P_3 : x, y, z$ be a path of order 3. Let G be the graph obtained by adding $p-3$ new vertices $v_1, v_2, \ldots, v_{p-n}, w_1, w_2, \ldots, w_{n-3}$ to P_3 and joining each $w_i (1 \leq i \leq n-3)$ to y; joining each $v_i (1 \leq i \leq p - n)$ with x, y and z; and joining each $v_i (1 \leq i \leq p - n - 1)$ with $v_j (i + 1 \leq j \leq p - n)$. The graph G is shown in Figure 3.3. Then G has order p and monophonic diameter $d = 2$. By Theorems 2.1 and 2.3, every minimal connected detour monophonic set of G contains $S = \{w_1, w_2, w_3, \ldots, w_{n-3}, x, z, y\}$. It is easily verified that S is the unique minimal connected detour monophonic set of G and so $dm_c^+(G) = n$.

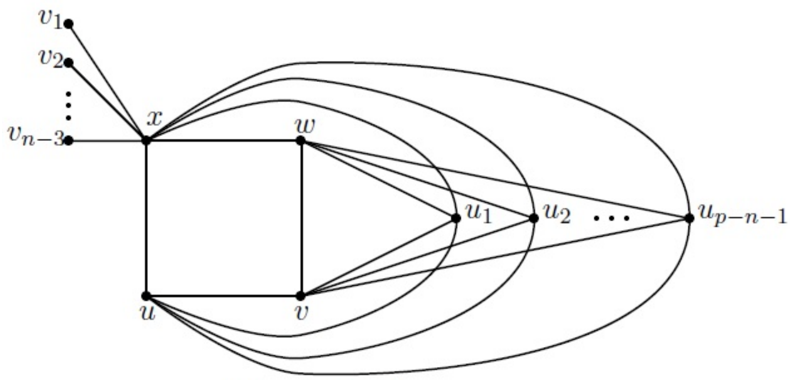

Figure 3.4: G

Case 2. $d = 3$. Let $C_4 : u, v, w, x, u$ be a cycle of order 4. Now, let $H = C_4 + (p-n-1)K_1$, where $V((p-n-1)K_1) = \{u_1, u_2, \ldots, u_{p-n-1}\}$. Let G be the graph obtained from H by adding $n-3$ new vertices $v_1, v_2, \ldots, v_{n-3}$ and joining each $v_i (1 \leq i \leq n-3)$ to x in H. The graph G is shown in Figure 3.4. Then G has order p and monophonic diameter $d = 3$. By Theorems 2.1 and 2.3, every minimal connected detour monophonic set of G contains $S = \{v_1, v_2, v_3, \ldots, v_{n-3}, x\}$. Clearly, S is not a connected detour monophonic set of G. Also, for any vertex $y \in V(G) - S$, $S \cup \{y\}$ is not a connected detour monophonic set of G. It is easily verified that $S_1 = S \cup \{u, w\}, S_2 = S \cup \{u, v\}$ and $S_3 = S \cup \{v, w\}$ are the minimal connected detour monophonic sets of G and so $dm_c^+(G) = n$.

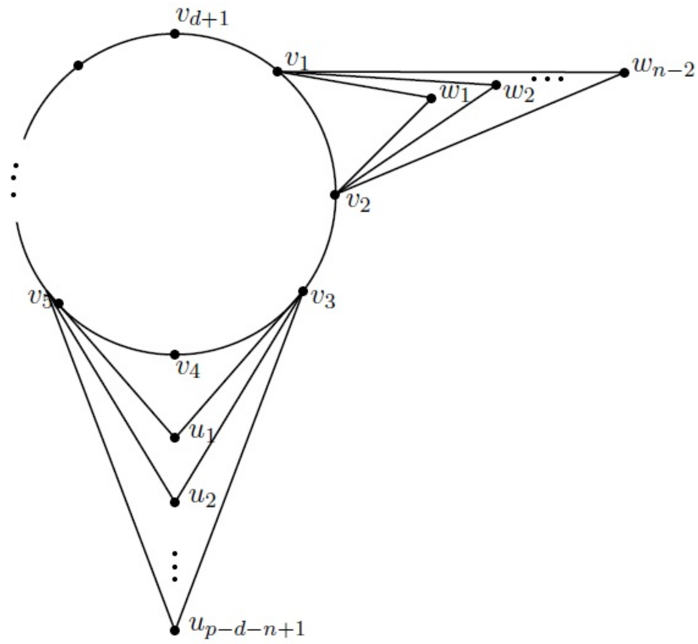

Figure 3.5: G

Case 3. $4 \leq d \leq p-2$. Let $C_{d+1} : v_1, v_2, ..., v_{d+1}, v_1$ be the cycle of order $d+1$. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{n-2}, u_1, u_2, \ldots, u_{p-d-n+1}$ to C_{d+1} and join each vertex $w_i (1 \leq i \leq n-2)$ to both v_1 and v_2; and join each vertex $u_j (1 \leq j \leq p-d-n+1)$ to both v_3 and v_5, thereby producing the graph G of Figure 3.5. Then G has order p and monophonic diameter d. Let $S = \{w_1, w_2, \ldots, w_{n-2}\}$ be the set of all extreme vertices of G. Then by Theorem 2.1, S is contained in every minimal connected detour monophonic set of G. It is clear that $S_1 = S \cup \{v_2, v_3\}$ and $S_2 = S \cup \{v_1, v_{d+1}\}$ are the only minimal connected detour monophonic sets of G and so $dm_c^+(G) = n$. $\square$

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