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ON $I-$ CONVERGENT TRIPLE SEQUENCE SPACES DEFINED BY A COMPACT OPERATOR AND AN ORLICZ FUNCTION

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ABSTRACT. In this article, we introduce $I-$ convergent triple sequence spaces ${}_3S^I(M)$, ${}_3S_0^I(M)$, ${}_3S_\infty^I(M)$ with the help of a compact operator and an Orlicz function. We study some of their algebraic and topological properties like solidity, monotonicity, convergence free etc. Also, we prove some inclusion relations of these spaces.

Keywords: Compact operator, Orlicz function, Triple sequence, $I-$ convergence field, Ideal, filter.

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1. Introduction and Preliminaries

A triple sequence is a function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}$ or $\mathbb{C}$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers, respectively. Let ${}_3\omega$ denote the class of all complex triple sequences (x_{nkl}), where $n, k, l \in \mathbb{N}$. Then the classes of triple sequences ${}_3l_\infty$, ${}_3c$ and ${}_3c_0$ denote the triple sequence spaces which are bounded in Pringsheim's sense, convergent in Pringsheim's sense and convergent to zero in Pringsheim's sense, respectively, normed by

$$\|x\|_\infty = \sup_{n,k,l} |x_{nkl}|, \quad where \ n, k, l \in \mathbb{N}.$$

The different types of notions of triple sequence spaces was introduced and investigated at the initial stage by Sahiner et. al [20, 21]. Esi [6] examined the triple sequence defined by orlicz function and probabilistic normed spaces. Datta et. al [4] studied the statistical convergence of triple sequences and Debnath et. al [5] analyzed the generalized triple sequence of real numbers. Further, this concept has been studied by many authors (see, [2, 7, 9, 12, 15, 18, 19, 25, 26]).

The idea of statistical convergence was first presented by Fast [8] and Schoenberg [23] independently. Later on, Mursaleen et. al [16, 17] studied the statistical convergence of

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double sequences and A-statistical approximation theorems. The notion of $I-$ convergence is a generalization of statistical convergence which was introduced by Kostyrko et. al [13]. Later on, it was studied by Salat et. al [22], Tripathy [27], Khan et. al [11, **?**], Kara and Ilkhan [30, 31], D$\ddot{u}$ndar and Ulusu [28], Kişi and Erhan [29] and many others.

A family $I \subseteq 2^X$ of subsets of a non-empty set X is said to be an ideal in X if $\emptyset \in I$, $A, B \in I$ implies $A \cup B \in I$, $A \in I, B \subseteq A$ implies $B \in I$. A non-empty family of sets $F \subseteq 2^X$ is a filter on X if and only if $\emptyset \notin F$, $A, B \in F$ implies $A \cap B \in A \in F$ and $A \subseteq B$ implies $B \in F$. An ideal I is called a non-trivial ideal if $I \neq \emptyset$ and $X \notin I$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subseteq I$ maximal if there is not exist any non-trivial ideal $J \neq I$ containing I as a subset. For each non-trivial ideal I there exists a filter $F(I) = \{A : X \setminus A \in I\}$ in X.

Definition 1.1 [20] A triple sequence (x_{nkl}) is said to be convergent to L in Pringsheim's sense if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that

$$|x_{nkl} - L| < \epsilon \text{ whenever } n \geq N, \;\; k \geq N, \;\; l \geq N.$$

Example 1.1:[20] Let

$$x_{nkl} = \begin{cases} kl, & n = 3 \\ nl, & k = 5 \\ nk, & l = 7 \\ 8, & \text{otherwise.} \end{cases}$$

Then $(x_{nkl}) \to 8$ in Pringsheim's sense.

Definition 1.2 [20] A triple sequence (x_{nkl}) is said to be a Cauchy sequence if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that

$$|x_{nkl} - x_{pqr}| < \epsilon \text{ whenever } n \geq p \geq N, \;\; k \geq q \geq N, \;\; l \geq r \geq N.$$

Definition 1.3 [20] A triple sequence (x_{nkl}) is said to be bounded if there exists $M > 0$ such that $|x_{nkl}| < M$ for all n,k,l.

Definition 1.4 [3] A triple sequence (x_{nkl}) is said to be I-convergent to a number L if for every $\epsilon > 0$, such that

$$\Big\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - L| \geq \epsilon\Big\} \in I.$$

in this case we write $I - \lim x_{nkl} = L$.

Definition 1.5 [3] A triple sequence (x_{nkl}) is said to be I-null if $L = 0$. In this case we write $I - \lim x_{nkl} = 0$.

Definition 1.6 [3] A triple sequence (x_{nkl}) is said to be I-Cauchy if for every $\epsilon > 0$, there exists $p = p(\epsilon), q = q(\epsilon)$ and $r = r(\epsilon)$ such that

$$\Big\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - x_{pqr}| \geq \epsilon\Big\} \in I.$$

Definition 1.7 [3] A triple sequence (x_{nkl}) is said to be I-bounded if there exists $K > 0$ such that

$$\Big\{(n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl}| > K\Big\} \in I.$$

Definition 1.8 [3] A triple sequence space E is said to be solid if $(\alpha_{nkl}x_{nkl}) \in E$ whenever $(x_{nkl}) \in E$ and for all sequences (α_{nkl}) of scalars with $|\alpha_{nkl}| \leq 1$, for all $n, k, l \in \mathbb{N}$.

Definition 1.9 [3] A triple sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Definition 1.10 [3] A triple sequence space E is said to be sequence algebra if $(x_{nkl} \star y_{nkl}) \in E$, whenever $(x_{nkl}) \in E$ and $y_{nkl} \in E$.

The following lemmas will be used for establishing some results of this article:

Lemma 1.1 [13] Let E be a sequence space. If E is solid then E is monotone.

Lemma 1.2 [13] If $I \subset 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$, $M \notin I$, then $M \cap \mathbb{N} \notin I$.

An $Orlicz function$ is a function $M : [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

If the convexity of an Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then this function is called Modulus function. If M is an Orlicz function , then $M(\lambda X) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$. An Orlicz function M is said to satisfy $\triangle_2$-condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$.

Lindenstrauss and Tzafriri [32] used the idea of an Orlicz function to construct the sequence space

$$l_M = \{x \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

The space l_M becomes a Banach space with the norm

$$\|x\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_K|}{\rho}\Big) \leq 1\Big\},$$

which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(t) = t^p$ for$1 \leq p < \infty$.

Definition 1.11 [14] Let X and Y be two normed linear spaces. An operator T defined by $T : X \to Y$ is said to be a Compact Linear Operator (completely continous linear operator) if T is linear and T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence. The set of all bounded linear operators $\mathcal{B}$(X,Y) is normed linear space normed by

$$\|T\| = \sup_{x \in X, \|x\|=1} \|Tx\|$$

The set of all compact linear operator $\mathcal{C}(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$ and $\mathcal{C}(X, Y)$ is a Banach space if Y is a Banach space.

Following Başar and Altay [1], and $Seng$önül [24], Das [3] introduced the triple sequence spaces $c^3_{0I}(F)$ and $c^3_I(F)$ with the help of sequence of moduli $F = (f_{nkl})$ as follows:

$$c^3_I(F) \;=\; \{x = (x_{nkl}) \in \; \omega^3 : \; I - \lim f_{nkl}(| \, x_{nkl} - L \, |) = 0, \text{for some } L \in \mathbb{C}\} \in I$$

$$c^3_{0I}(F) \;=\; \{x = (x_{nkl}) \in\; \omega^3 :\; I - \lim f_{nkl}(\mid x_{nkl} \mid) = 0\} \in I.$$

The main aim of this article is to extend the concept of I-convergence from double sequences to triple sequences with the help of a compact linear operator and an orlicz function and establish some useful results.

2. Mean Results

In this section we introduce the following classes of sequence spaces.

$${}_3\mathcal{S}^I(M) = \left\{x = (x_{nkl}) \in\; {}_3l_\infty : I - \lim_{nkl} M(\frac{|T(x_{nkl}) - L|}{\rho}) = 0, \;\text{ for some } L \in \mathbb{C}, \rho > 0\right\},$$

$${}_3\mathcal{S}^I_0(M) = \left\{x = (x_{nkl}) \in\; {}_3l_\infty : I - \lim_{nkl} M(\frac{|T(x_{nkl})|}{\rho}) = 0, \rho > 0\right\},$$

$${}_3\mathcal{S}^I_\infty(M) = \left\{x = (x_{nkl}) \in\; {}_3l_\infty : \exists\; K > 0 \; s.t \; \{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : M(\tfrac{|T(x_{nkl})|}{\rho}) \geq K, \rho > 0\} \in I\right\},$$

$${}_3\mathcal{S}_\infty(M) = \left\{x = (x_{nkl}) : \sup_{nkl} M(\frac{|T(x_{nkl})|}{\rho}) < \infty, \;\text{ for some }\; \rho > 0\right\}.$$

We also denote by,

$${}_3m^I_S(M) = \;{}_3\mathcal{S}^I(M) \cap \;{}_3\mathcal{S}_\infty(M)$$

$${}_3m^I_{S_0}(M) = \;{}_3\mathcal{S}^I_0(M) \cap \;{}_3\mathcal{S}_\infty(M)$$

Theorem 2.1 For an Orlicz function M the classes of triple sequences ${}_3\mathcal{S}^I(M)$, ${}_3\mathcal{S}^I_0(M)$, ${}_3m^I_S(M)$ and ${}_3m^I_{S_0}(M)$ are linear spaces.

Proof. We shall prove the result for the space ${}_3\mathcal{S}^I(M)$. The proof for the other spaces will follow similarly.

Let $(x_{nkl}), (y_{nkl}) \in\; {}_3\mathcal{S}^I(M)$ and let α, β be scalars. Then there exist some positive numbers $L_1, L_2 \in \mathbb{C}$ and $\rho_1, \rho_2 > 0$ such that

$$I - \lim_{nkl} M(\frac{|T(x_{nkl}) - L_1|}{\rho_1}) = 0,$$

and

$$I - \lim_{nkl} M(\frac{|T(y_{nkl}) - L_2|}{\rho_2}) = 0.$$

For any $\epsilon > 0$, the sets

$$A_1 = \{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : M(\frac{|T(x_{nkl}) - L_1|}{\rho_1}) \geq \frac{\epsilon}{2}\} \in I, \tag{1}$$

and

$$A_2 = \{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : M(\frac{|T(y_{nkl}) - L_2|}{\rho_2}) \geq \frac{\epsilon}{2}\} \in I. \tag{2}$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is non-decreasing and convex function, we have

$$\begin{aligned} M(\frac{|T(\alpha x_{nkl}+\beta y_{nkl})-(\alpha L_1+\beta L_2)|}{\rho_3}) &= M(\frac{|\alpha T(x_{nkl})+\beta T(y_{nkl})-\alpha L_1-\beta L_2|}{\rho_3}) \\ &\leq M(\frac{|\alpha||T(x_{nkl})-L_1|}{\rho_3})+M(\frac{|\beta||T(y_{nkl})-L_2|}{\rho_3}) \\ &\leq M(\frac{|T(x_{nkl})-L_1|}{\rho_1})+M(\frac{|T(y_{nkl})-L_2|}{\rho_2}) \\ &\leq M(\frac{|T(x_{nkl})-L_1|}{\rho_1})+M(\frac{|T(y_{nkl})-L_2|}{\rho_2}). \end{aligned} \tag{3}$$

Therefore, from (1), (2) and (3), we have

$$\{(n,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: M(\frac{|T(\alpha x_{nkl}+\beta y_{nkl})-(\alpha L_1+\beta L_2)|}{\rho_3})\geq\epsilon\}\subset(A_1\cup A_2)\in I,$$

this implies that

$$\{(n,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}: M(\frac{|T(\alpha x_{nkl}+\beta y_{nkl})-(\alpha L_1+\beta L_2)|}{\rho_3})\geq\epsilon\}\in I.$$

$$\Rightarrow \lim_{nkl} M(\frac{|T(\alpha x_{nkl}+\beta y_{nkl})-(\alpha L_1+\beta L_2)|}{\rho_3})=0$$

$$\Rightarrow \ \alpha x_{nkl}+\beta y_{nkl}\in\ {}_2\mathcal{S}^I(M).$$

Hence ${}_3\mathcal{S}^I(M)$ is a linear space.

Remark. For an Orlicz function M, the spaces ${}_3m^I_{S_0}(M)$ and ${}_3m^I_S(M)$ are Banach spaces normed by

$$\|x\|=\inf\{\rho>0:\sup_{nkl} M(\frac{|T(x_{nkl})|}{\rho})<1,\rho>0\}.$$

Theorem 2.2 Let M_1, M_2 be two Orlicz functions statisfying $\triangle_2$ condition, then

(a) $X(M_2)\subseteq X(M_1M_2)$

(b) $X(M_1)\cap X(M_2)\subseteq X(M_1+M_2)$ for $X=\ {}_3\mathcal{S}^I,\ {}_3\mathcal{S}^I_0,\ {}_3m^I_S$ and ${}_3m^I_{S_0}$.

Proof. (a) Let $x=(x_{nkl})\in\ {}_3\mathcal{S}^I_0(M_2)$ be an arbitrary element
$\Rightarrow \rho>0$ such that

$$I-\lim_{nkl} M_2(\frac{|T(x_{nkl})|}{\rho})=0. \tag{4}$$

Let $\epsilon>0$ and choose δ with $0<\delta<1$ such that $M_1(t)<\epsilon$ for $0\leq t\leq\delta$.
Put $y_{nkl}=M_2(\frac{|T(x_{nkl})|}{\rho})$ and consider,

$$\lim_{nkl} M_1(y_{nkl})=\lim_{y_{nkl}\leq\delta,\ n,k,l\in\mathbb{N}} M_1(y_{nkl})+\lim_{y_{nkl}>\delta,\ n,k,l\in\mathbb{N}} M_1(y_{nkl}). \tag{5}$$

Now, since M_1 is an Orlicz function so we have $M_1(\lambda x)\leq\lambda M_1(x), 0<\lambda<1$.
Therefore we have,

$$\lim_{y_{nkl}\leq\delta,\ n,k,l\in\mathbb{N}} M_1(y_{nkl})\leq M_1(2)\lim_{y_{nkl}\leq\delta,\ n,k,l\in\mathbb{N}}(y_{nkl}).$$

For $y_{nkl} > \delta$, we have $y_{nkl} < \frac{y_{nkl}}{\delta} < 1 + \frac{y_{nkl}}{\delta}$. Now, since M_1 is non-decreasing and convex, it follows that

$$M_1(y_{nkl}) < M_1(1 + \frac{y_{nkl}}{\delta}) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1(\frac{2y_{nkl}}{\delta}).$$

Since M_1 satisfies the $\triangle_2$- condition we have,

$$\begin{aligned} M_1(y_{nkl}) &< \frac{1}{2}K\frac{y_{nkl}}{\delta}M_1(2) + \frac{1}{2}KM_1(\frac{2y_{nkl}}{\delta}) \\ &< \frac{1}{2}K\frac{y_{nkl}}{\delta}M_1(2) + \frac{1}{2}K\frac{y_{nkl}}{\delta}M_1(2) \\ &= K\frac{y_{nkl}}{\delta}M_1(2). \end{aligned}$$

This implies that

$$M_1(y_{nkl}) < K\frac{y_{nkl}}{\delta}M_1(2).$$

Hence,we have

$$\lim_{y_{nkl}>\delta,\ n,k,l\in\mathbf{N}} M_1(y_{nkl}) \leq max\{1, K\delta^{-1}M_1(2) \lim_{y_{nkl}>\delta,n,k,l\in\mathbf{N}}(y_{nkl})\}.$$

Therefore from (4) and (5) we have

$$I - \lim_{nkl} M_1(y_{nkl}) = 0.$$

$$\Rightarrow I - \lim_{nkl} M_1M_2(\frac{|T(x_{nkl})|}{\rho}) = 0.$$

This implies that $x = (x_{nkl}) \in {}_3\mathcal{S}_0^I(M_1M_2)$. Hence $X(M_2) \subseteq X(M_1M_2)$ for $X = {}_3\mathcal{S}_0^I$. The other cases can be proved in a similar way.

(b) Let $x = (x_{nkl}) \in {}_3\mathcal{S}_0^I(M_1) \cap {}_3\mathcal{S}_0^I(M_2)$. Let $\epsilon > 0$ be given, then $\exists\rho > 0$ such that,

$$I - \lim_{nkl} M_1(\frac{|T(x_{nkl})|}{\rho}) = 0, \tag{6}$$

and

$$I - \lim_{nkl} M_2(\frac{|T(x_{nkl})|}{\rho}) = 0. \tag{7}$$

Therefore

$$I - \lim_{nkl}(M_1 + M_2)(\frac{|T(x_{nkl}|}{\rho}) = I - \lim_{nkl} M_1(\frac{|T(x_{nkl})|}{\rho}) + I - \lim_{nkl} M_2(\frac{|T(x_{nkl})|}{\rho}),$$

from (6)and(7)

$$\Rightarrow I - \lim_{nkl}(M_1 + M_2)(\frac{|T(x_{nkl})|}{\rho}) = 0.$$

we get

$$x = (x_{nkl}) \in {}_3\mathcal{S}_0^I(M_1 + M_2).$$

Hence we get ${}_3\mathcal{S}_0^I(M_1) \cap {}_3\mathcal{S}_0^I(M_2) \subseteq {}_3\mathcal{S}_0^I(M_1 + M_2)$.
For $X = {}_3\mathcal{S}^I,\ {}_3m_S^I,\ {}_3m_{S_0}^I$ the inclusion are similar.

Corollary. $X \subseteq X(M)$ for $X = {}_3\mathcal{S}^I,\ {}_3\mathcal{S}_0^I,\ {}_3m_S^I$ and ${}_3m_{S_0}^I$.

Theorem 2.3 For an Orlicz function M, the spaces ${}_3\mathcal{S}_0^I(M)$ and ${}_3m_{S_0}^I(M)$ are solid and monotone.

Proof. Here we consider ${}_3\mathcal{S}_0^I(M)$ and for ${}_3m_{S_0}^I(M)$ the proof shall be similar.
Let $x = x_{nkl} \in {}_3\mathcal{S}_0^I(M)$ be an arbitrary element , $\Rightarrow \exists \rho > 0$ such that

$$I - \lim_{nkl} M(\frac{|T(x_{nkl})|}{\rho}) = 0.$$

Let (α_{nkl}) be a sequence of scalars with $|\alpha_{nkl}| \leq 1$ for $n, k, l \in \mathbb{N}$.
Now, M is an Orlicz function and for $\epsilon > 0$, the results follows from the following inclusion

$$\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : M(\frac{|T(\alpha_{nkl}x_{nkl})|}{\rho}) \geq \epsilon\} \subseteq \{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : M(\frac{|T(x_{nkl})|}{\rho}) \geq \epsilon\}.$$

This implies that,

$$I - \lim_{nkl} M(\frac{|T(\alpha_{nkl}x_{nkl})|}{\rho}) = 0.$$

Thus we have $(\alpha_{nkl}x_{nkl}) \in {}_3\mathcal{S}_0^I(M)$. Hence ${}_3\mathcal{S}_0^I(M)$ is solid. Therefore ${}_3\mathcal{S}_0^I(M)$ is monotone. Since every solid sequence space is monotone. For ${}_3m_{S_0}^I(M)$ the proof shall be similar.

Theorem 2.4 For an Orlicz function M,the space ${}_3\mathcal{S}^I(M)$ and ${}_3m_S^I(M)$ are neither solid nor monotone in general.

Proof. Here we give counter example for establishment of this result. Let $X = {}_3\mathcal{S}^I$ and ${}_3m_S^I$. Let us consider $I = I_\delta$ and $M(x) = x^2$, for all $x = x_{nkl} \in [0, \infty)$ and T is an identity operator on $\mathbb{R}$. Consider,the K-step space $X_K(M)$ of $X(M)$ defined as follows:
Let $x = (x_{nkl}) \in X(M)$ and $y = (y_{nkl}) \in X_K(M)$ be such that

$$y_{nkl} = \begin{cases} x_{nkl}, & \text{if n+k+l is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_{nkl}) defined by $(x_{nkl}) = 1$ for all $n, k, l \in \mathbb{N}$.
Then $x = (x_{nkl}) \in {}_3\mathcal{S}^I(M)$ and ${}_3m_S^I(M)$, but K-step space preimage does not belong to ${}_3\mathcal{S}^I(M)$ and ${}_3m_S^I(M)$. Thus ${}_3\mathcal{S}^I(M)$ and ${}_3m_S^I(M)$ are not monotone and hence they are not solid.

Theorem 2.5 For an Orlicz function M and an identity operator T on $\mathbb{R}$, the spaces ${}_3\mathcal{S}_0^I(M)$ and ${}_3\mathcal{S}^I(M)$ are sequence algebra.

Proof. Here we consider ${}_3\mathcal{S}_0^I(M)$. Let $(x_{nkl}), (y_{nkl}) \in {}_3\mathcal{S}^I(M)$ be any two arbitrary elements. $\Rightarrow \exists \rho_1, \rho_2 > 0$ such that,

$$I - \lim_{nkl} M(\frac{|T(x_{nkl})|}{\rho_1}) = 0,$$

and

$$I - \lim_{nkl} M(\frac{|T(y_{nkl})|}{\rho_2}) = 0.$$

Let $\rho = \rho_1\rho_2 > 0$. Then

$$M(\frac{|T(x_{nkl})\ T(y_{nkl})|}{\rho}) = M(\frac{|T(x_{nkl})|}{\rho_1})M(\frac{|T(y_{nkl})|}{\rho_2})$$

$$\Rightarrow I - \lim_{nkl} M(\frac{|T(x_{nkl})\ T(y_{nkl})|}{\rho}) = 0.$$

Therefore we have $(x_{nkl}y_{nkl}) \in {}_3\mathcal{S}_0^I(M)$.
Hence ${}_3\mathcal{S}_0^I(M)$ is sequence algebra.

Theorem 2.6 Let M be an Orlicz function. Then

$${}_3\mathcal{S}_0^I(M) \subset {}_3\mathcal{S}^I(M) \subset {}_3\mathcal{S}_\infty^I(M).$$

Proof. Let M be an Orlicz function. Then, we have to show that

$${}_3\mathcal{S}_0^I(M) \subset {}_3\mathcal{S}^I(M) \subset {}_3\mathcal{S}_\infty^I(M).$$

Firstly, ${}_3\mathcal{S}_0^I(M) \subset {}_3\mathcal{S}^I(M)$ is obvious. Now, let $x = (x_{nkl}) \in {}_3S^I(M)$ be any arbitrary element $\Rightarrow \exists\rho > 0$ such that $I - \lim_{nkl} M(\frac{|T(x_{nkl})-L|}{\rho}) = 0$ for some $L \in \mathbb{C}$. Now, $M(\frac{|T(x_{nkl})|}{2\rho}) \leq \frac{1}{2}M(\frac{|T(x_{nkl})-L|}{\rho}) + \frac{1}{2}M(\frac{|L|}{\rho})$. Taking supremum over n,k,l on both sides, we have $x = (x_{nkl}) \in {}_3\mathcal{S}_\infty^I(M)$. Thus ${}_3\mathcal{S}_0^I(M) \subset {}_3\mathcal{S}^I(M) \subset {}_3\mathcal{S}_\infty^I(M)$.

Theorem 2.7 The set ${}_3m_S^I(M)$ is closed subspace of ${}_3\mathcal{S}_\infty(M)$.

Proof. Let $(x_{nkl}^{(pqr)})$ be a Cauchy sequence in ${}_3m_S^I(M)$ such that $x^{(pqr)} \to x$. We show that $x \in {}_3m_S^I(M)$. Since, $(x_{nkl}^{(pqr)}) \in {}_3m_S^I(M)$ then there exists a_{pqr}, and $\rho > 0$ such that

$$\{n, k, l \in \mathbb{N} : M(\frac{|T(x_{nkl}^{(pqr)}) - a_{pqr}|}{\rho}) \geq \epsilon\} \in I.$$

We need to show that
(1) (a_{pqr}) converges to a.
(2) If $U = \{n, k, l \in \mathbb{N} : M(\frac{|T(x_{nkl}^{(pqr)})-a|}{\rho}) \geq \epsilon\}$, then $U^c \in I$.
Since $(x_{nkl}^{(pqr)})$ is a Cauchy sequence in ${}_3m_S^I(M)$ then for a given $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\sup_{nkl} M(\frac{|T(x_{nkl}^{(pqr)}) - T(x_{nkl}^{(p'q'r')})|}{\rho}) < \frac{\epsilon}{3}, \text{ for all } p, q, r \geq k_0 \ \ and \ \ p', q', r' \geq k_0.$$

For a given $\epsilon > 0$, we have

$$B_{pqrp'q'r'} = \{n, k, l \in \mathbb{N} : M(\frac{|T(x_{nkl}^{(pqr)}) - T(x_{nkl}^{(p'q'r')})|}{\rho}) < \frac{\epsilon}{3}\},$$

$$B_{pqr} = \{n, k, l \in \mathbb{N} : M(\frac{|T(x_{nkl}^{(pqr)}) - a_{pq}|}{\rho}) < \frac{\epsilon}{3}\},$$

$$B_{p'q'r'} = \{n, k, l \in \mathbb{N} : M(\frac{|T(x_{nkl}^{(p'q'r')}) - a_{rs}|}{\rho}) < \frac{\epsilon}{3}\}.$$

Then $B^c_{pqrp'q'r'}$, B^c_{pqr}, $B^c_{p'q'r'} \in I$. Let $B^c = B^c_{pqrp'q'r'} \cap B^c_{pqr} \cap B^c_{p'q'r'}$,
where $B = \{n, k, l \in \mathbb{N} : M(\frac{|a_{pqr}-a_{p'q'r'}|}{\rho}) < \epsilon\}$, then $B^c \in I$. We choose $k_0 \in B^c$, then for each $p, q, r, \geq k_0$ and $p', q', r' \geq k_0$ we have

$$\{n, k, l \in \mathbb{N} : M(\frac{|a_{pqr} - a_{p'q'r'}|}{\rho}) < \epsilon\} \supseteq \Big[\{n, k, l \in \mathbb{N} : M(\frac{|T(x_{nkl}^{(pqr)}) - a_{pqr}|}{\rho}) < \frac{\epsilon}{3}\}$$

$$\cap\{n, k, l \in \mathbb{N} : M(\frac{|T(x_{nkl}^{(pqr)}) - T(x_{nkl}^{(p'q'r')})|}{\rho}) < \frac{\epsilon}{3}\}$$

$$\cap\{n,k,l\in\mathbb{N}: M(\frac{|T(x_{nkl}^{(p'q'r')})-a_{p'q'r'}|}{\rho})<\frac{\epsilon}{3}\}\Big].$$

Then (a_{pqr}) is a Cauchy sequence in $\mathbb{C}$. So, there exists a scalar $a\in\mathbb{C}$ such that $(a_{pqr})\rightarrow a$, as $p,q,r\rightarrow\infty$.
(2) For the next step, let $0<\delta<1$ be given. Then, we show that if,

$$U=\{n,k,l\in\mathbb{N}: M(\frac{|T(x_{nkl}^{(pqr)})-a|}{\rho})<\delta\}$$

then $U^c\in I$. Since $x_{nkl}^{(pqr)}\rightarrow x$, then there exists $p_0,q_0,r_0\in\mathbb{N}$ such that,

$$P=\{n,k,l\in\mathbb{N}: M(\frac{|T(x_{nkl}^{(p_0q_0r_0)})-T(x)|}{\rho})<\frac{\delta}{3}\}$$

implies $P^c\in I$. The numbers p_0,q_0,r_0 be so choosen such that we have

$$Q=\{n,k,l\in\mathbb{N}: M(\frac{|a_{p_0q_0r_0}-a|}{\rho})<\frac{\delta}{3}\}$$

such that $Q^c\in I$. Since $(x_{nkl}^{(pqr)})\in\ {}_3m_S^I(M)$.
We have

$$\{n,k,l\in\mathbb{N}: M(\frac{|T(x_{nkl}^{(p_0q_0r_0)})-a_{p_0q_0r_0}|}{\rho})\geq\delta\}\in I.$$

Then we have a subset S of $\mathbb{N}$ such that $S^c\in I$, where

$$S=\{n,k,l\in\mathbb{N}: M(\frac{|T(x_{nkl}^{(p_0q_0r_0)})-a_{p_0q_0r_0}|}{\rho})<\frac{\delta}{3}\}.$$

Let $U^c=P^c\cup Q^c\cup S^c$, where

$$U=\{n,k,l\in\mathbb{N}: M(\frac{|T(x)-a|}{\rho})<\delta\}$$

Therefore, for each $n,k,l\in U^c$ we have

$$\{n,k,l\in\mathbb{N}: M(\frac{|T(x)-a|}{\rho})<\delta\}\supseteq\Big[\{n,k,l\in\mathbb{N}: M(\frac{|T(x_{nkl}^{(p_0q_0r_0)})-T(x)|}{\rho})<\frac{\delta}{3}\}$$

$$\cap\{n,k,l\in\mathbb{N}: M(\frac{|a_{p_0q_0r_0}-a|}{\rho})<\frac{\delta}{3}\}$$

$$\cap\{n,k,l\in\mathbb{N}: M(\frac{|T(x_{nkl}^{(p_0q_0r_0)})-a_{p_0q_0r_0}|}{\rho})<\frac{\delta}{3}\}\Big].$$

Hence the result ${}_3m_S^I(M)\subset\ {}_3\mathcal{S}_\infty(M)$ follows.

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