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FUZZY INJECTIVE S-ACT ON MONOIDS

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Abstract. The fuzzy injectivity and their generalizations on R-modules were studied by many authors. It is well known that the act of semigroups or monoids is a generalization of R-modules, here we will define the fuzzy injective S-acts or S-systems and studied some of their properties first we discuss when the subacts of fuzzy S-acts is fuzzy injective S-act , we studied the retracts ,the factor fuzzy injective S-act and finally we studied their direct product.

Keywords: Semigroups, S-acts, injectivity, fuzzy subsets, fuzzy injectivity.

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1. Introduction

Action of monoids on sets are very powerful , useful and used in a very wide range in mathematics and computer sciences as in: Conjugation ,translation as well as in Automata theory and pattern recognition , In the other hand fuzzy action was studied and used in many applications, by many authors as on rings and modules [16, 17] and the fuzzy action on semigroups or monoids see [7, 8, 12, 3].

In this paper we will study the action of monoid on a set for this we recall that a semigroup is a non-empty set with associative binary operation , If the semigroup S has an identity then it is called a monoid [4] . Also A right S-act A_S over a monoid S is a non-empty set , with a function $A \times S \to A$, that is $(a, s) \to (as)$ such that the following properties hold : $(as)r = a(sr)$ for all $a \in A$ and $s, r \in S$. And $ae = a$ where e is the identity of S .

If S is an S-act over itself then we denoted by S_S [11].

So it is clear from the definition that the theory of monoid and S-act are generalizations of the theory of rings and modules since every R-module is an S-act but the converse is not true in general. The injectivity is one of the important concepts of the action in the module theory and semigroup theory many authors has been studied the fuzzy injectivity on R-modules [1, 5]. And the injectivity of S-act, See on injectivity on S-act [14, 6] ,and [10, 18].

Here we will give the concept of fuzzy injective S-act, Fuzzy injective S-subact and some

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of their properties , This has many application in Automata theory and fuzzy languages [15].

2. Preliminaries

We must mention that the theory of the S-act the following names are also used for the same concept :
S-sets, S-operands, S-polygons, S-systems, transition systems, S-automata.
In this section first we will recall some of the needed concepts in the semigroup theory

Definition 2.1 ([10]). *A non-empty subset C of an S-act A_S such that $cs \in C$ for all $c \in C$ and $s \in S$ then C is called an S-subact of A_S .*

Definition 2.2 ([10]). *A map f is said to be S-homomorphism from an S-act A_S into the S-act B_S if $f(as) = f(a)s$ $\forall a \in A_S$ and $s \in S$.*

Definition 2.3 ([10]). *An S-homomorphism $f : A_S \to B_S$ is called a retraction if there exist an S-homomorphism $h : B_S \to A_S$ such that $f \circ h = I_{B_S}$ and B_S said to be a retract of A_S.*

Definition 2.4 ([2]). *Let A_S and B_S be two S-acts then we say that A_S is B_S-injective S-act if for any S-monomorphism g from an S-act N_S into B_S and any S-homomorphism f from N_S into A_S , there exists a homomorphism h from B_S into A_S such that $h \circ g = f$. furthermore we said that A_S is injective if and only if A_S is B_S-injective for all S-act B_S.*

Definition 2.5 ([9]). *A congruence R on an S-act A_S is an equivalent relation on A_S such that if $(x, y) \in R$ then $(xs, ys) \in R$ $\forall s \in S$, the identity congruence of A_S is denoted by I_R .*

Definition 2.6 ([10]). *Let R be congruence on an S-act A_S define the factor set on R by $A_S/R = \{[a]_R | a \in A_S\}$ and the multiplication by an element of S on the factor set A_S/R by: $[a]_R s = [as]_R$ $\forall s \in S$ Then A_S/R becomes an S-act called the factor S-act of A_S by R*

Definition 2.7 ([10]). *Define the Cartesian product of a family $\{X_i\}_{i \in I}$ of S-acts by of $\prod_{i \in I} X_i = \times_{i \in I} X_i$. Also one can define the projections $p_i : \times_{i \in I} X_i \to X_i, i \in I$ by $p_j(x_1, \ldots, x_j, \ldots) = x_j$.*
Secondly we will recall some needed concepts in the fuzzy set on semigroups.

Definition 2.8 ([12]). *Let S be a semigroup a fuzzy subset $\alpha : S \to [0, 1]$ is called fuzzy semigroup of S if $\alpha(xy) \geq \alpha(x) \wedge \alpha(y)$ for each $x, y \in S$.*

Definition 2.9 ([13]). *Let (S, *) be a monoid and let α be a fuzzy subset of S then α is called a fuzzy monoid of S if $\alpha(xy) \geq \alpha(x) \wedge \alpha(y)$ for each $x, y \in S$ and $\forall a \in S \alpha(e) \geq \alpha(a)$ where e is the identity of S .*

Definition 2.10 ([3]). *Let (A, σ_A) and (B, σ_B) be two fuzzy S-acts then an S-homomorphism f from A into B is called fuzzy S-homomorphism from $(A, \sigma_A) \to (B, \sigma_B)$ if*
$\sigma_B(f(a)) \geq \sigma_A(a)$ $\forall$ $a \in A$

Definition 2.11 ([7, 3]). *Let A_S be an S-act on S and let $\partial : A_S \to [0, 1]$ be a fuzzy subset on A_S we said that ∂ is a fuzzy subact of A_S if $\partial(as) \geq \partial(a)$for all $a \in A_S$ and $s \in S$. Then (A_S, ∂) called fuzzy S-act or fuzzy S-system.*

Lemma 2.1 ([3]). *Let A_S be an S-act and C be a subset of A_S. Then the characteristic function ∂_C of C is a fuzzy subact of A_S if and only if C is an S-subact of A_S.*

Definition 2.12 ([3]). *A fuzzy S-act (A, σ_A) is called retract of a fuzzy S-act (B, σ_B) if there exist a fuzzy S-homomorphisms $\tau : (B, \alpha_B) \to (A, \alpha_A)$ and $\vartheta : (A, \alpha_A) \to (B, \alpha_B)$ such that $\vartheta \circ \tau = I_A$*

Definition 2.13 ([8]). *By an n-array fuzzy operation τ in fuzzy set A^τ we mean a fuzzy function from A^n to A such that for each $a_1, \ldots, a_n \in A$*
$\wedge_{i=1,\ldots,n}\ \mu(a_i) \leq \mu(\tau(a_1, \ldots, a_n))$ *Where $\mu : A \to [0, 1]$*

3. Fuzzy injective S-act over monoid

In this section we will introduce the definition of fuzzy injective S. act over monoid.

Definition 3.1. *Let (A, σ_A) and (B, σ_B) be two fuzzy S-acts over the monoid S then we say that σ_A is σ_B-injective if:*
1. *A is $aB - injective$.*
2. *$\sigma_B(b) \leq \sigma_A(\theta(b))\ \forall\ \theta \in Hom(a, B)$ and $b \in B$.*

Furthermore we said that (A, σ_A) fuzzy injective S-act if it is σ_B-injective for each S-act (B, σ_B).

In [10] they investigate the injectivity of S-act and retract of injective S-act :

Theorem 3.1 ([10]). *A retract of an injective S-act is injective.*

Proposition 3.1 ([10]). *If an injective S-act Q_s is an S-subact of an S-act A_s, then it is a retract Of A_s.*

Here give similar results of 3.2 and 3.3 for the fuzzy injective S-acts.

Theorem 3.2. *A retract of fuzzy injective S-act is a fuzzy injective S-act.*

Proof. Let (A, σ_A) and (B, σ_B) to be two fuzzy S-acts such that (B, σ_B) is a retract of (A, σ_A), Let (A, σ_A) be a fuzzy injective S-act so A is injective so

$$\sigma_Q(q) \leq \sigma_A(h(q))\ for\ any\ S - act\ Q, q \in Q, f \in Hom(Q, A) \tag{1}$$

But B is a retract of A so by theorem (3.2) B is an injective S-act.
To show that $\sigma_Q(q) \leq \sigma_B(t(q))\ for\ any\ S - act\ Q, q \in Q, t \in Hom(Q, B)$
By definition of the fuzzy retract for any fuzzy S-homomorphism
$f : (A, \sigma_A) \to (B, \sigma_B)$ that is $\sigma_B(f(A)) \geq \sigma_A(a)$ there exist a fuzzy S-homomorphism

$$g : (B, \sigma_B) \to (A, \sigma_A) \text{ that is } \sigma_A(g(b)) \geq \sigma_B(b) \tag{2}$$

Such that

$$f \circ g = I_A \tag{3}$$

Now by (1), (2), (3) we will get $\sigma_Q(q) \leq \sigma_A(h(q))$

$$\begin{aligned} &= \sigma_A(g \circ f(h(q))) \\ &= \sigma_A(g(f(h(q)))) \\ &\geq \sigma_B(f(h(q))) \\ \sigma_Q(q) &= \sigma_B(f \circ h(q)), \end{aligned}$$

So B is fuzzy injective S-act as in figure (1).

$$
\begin{tikzcd}
H \arrow[rr] \arrow[dd] & & Q \arrow[dl, "h"'] \\
& A \arrow[dl, "g"'] & \\
B \arrow[ur, bend right, "f"'] & &
\end{tikzcd}
$$

Figure (1).

□

Proposition 3.2. *If an injective fuzzy S- act (C, σ_C) is a subact of a fuzzy S-act (A, σ_A) then it is a fuzzy retract of (A, σ_A)*

Proof. Let (C, σ_C) be a fuzzy injective S-subact of (A, σ_A) and $id : C \to C$ so by the injectivity of C there exist an extension f of id (i.e. $f \circ inc = id$) and $\sigma_A(a) \leq \sigma_C(f(a))$ so f is a fuzzy S-epiomorphism from A onto C hence $f(c) = c \ \forall c \in C \xrightarrow{yields} \exists\, inc : C \to A$ which is clearly fuzzy S-homomorphism $f \circ inc(c) = c$,Hence (C, σ_C) is a fuzzy retract of (A, σ_A) .(see figure (2))

$$
\begin{tikzcd}
C \arrow[r, "\text{inc}"] \arrow[d, "\text{id}"'] & A \arrow[dl, "f"] \\
C &
\end{tikzcd}
$$

Figure (2)

□

To study the factor fuzzy injective S-act in theorem (3.8) we must first prove the following two propositions:

Proposition 3.3. *Let A and B be two S-acts and let the quotient S-act A/R if B is A-injective then B is A/R-injective .*

Proof. let C/R be a subact of A/R and $f : C/R \to B$ be an S-homomorphism ,
$\pi_C : C \to C/R$ so $\pi_C \circ f : C \to B$ but B is A $-$ injective ,
There exist $g : A \to B$ which is an extension of $\pi_C \circ f$ see figure (3)

$$\begin{tikzcd}
C \arrow[rr] \arrow[d, "\pi_C"'] & & A \arrow[dl, "\pi_A"'] \arrow[ddll, bend left, "\boldsymbol{g}"] \\
C/R \arrow[r, "i_{C/R}"] \arrow[d, "f"'] & A/R \arrow[dl, "? = \grave{g}"'] & \\
B & &
\end{tikzcd}$$

Figure (3).

Define $\grave{g} : A/R \to B$ by $\grave{g}([a]_R) = g(a)\ \forall a \in A$ then $\grave{g}$ is well defined :
Let $[a_1]_R = [a_2]_R$ for each $a_1, a_2 \in A$,
$\to [a_1]_R = i_{C/R} \circ \pi_C(x_1)$ *and* $[a_2]_R = i_{(C/R)} \circ \pi_C(x_2)\ \forall x_1, x_2 \in C$
$\to i_{(C/R)} \circ \pi_C(x_1) = i_{(C/R)} \circ \pi_C(x_2) \to \pi_C(x_1) = \pi_C(x_2) \to f(\pi_C(x_1)) = f(\pi_C(x_2))$
$\to g(a_1) = g(a_2)$ since f is well defined.
$\to \grave{g}([a_1]_R) = \grave{g}([a_2]_R)$.
$\grave{g}$ is S-homomorphism and an extension of f
: $\forall [a]_R \in A/R, s \in S$ we have $\grave{g}(s[a]_R) = sg(a) = g(sm) = \grave{g}([sa]_R)$
Now $\forall [x]_R \in C/R, i_{(C/R)}([x]_R) \in A/R$ we have $\grave{g}(i_{(C/R)}([x]_R)) \in B$,
$\to \grave{g}([x]_R) = g(x) = f(\pi_C(x)) = f([x]_R)$.
Hence B is A/R-injective . □

Proposition 3.4. *Let A be an S – act and C be a subact of A if A has fuzzy subact then C and A/R has fuzzy subacts.*

Proof. Let σ_A be a fuzzy subact of A then $\sigma_C = \sigma_A|_C$ is the fuzzy subact of C.
For the second part of the proof define $\vartheta : A/R \to [0, 1]$ by $\vartheta([a]_R) = \sigma_A(a)\ \forall a \in A$ to show that ϑ is a fuzzy subact of A/R,
by $\vartheta([sa]_R) = s\vartheta([a]_R) = s\sigma_A(a) = \sigma_A(sa) \geq \sigma_A(a) = \vartheta([a]_R)$
that is $\vartheta([sa]_R) \geq \vartheta([a]_R)$ so ϑ is a fuzzy subact of A/R.

□

Now we are ready to prove the following theorem:

Theorem 3.3. *Let (A, σ_A) and (B, σ_B) be two fuzzy S-acts and let $\sigma_{B_{/R}}$ be the fuzzy subact of B/R where R is the congruence on B then if σ_A is $\sigma_B - injective$ then σ_A is $\sigma_{B_{/R}} - injective$.*

Proof. Let σ_A is $\sigma_B - injective$ so A is $B - injective$ and ,
$\sigma_B(b) \leq \sigma_A(f(b))\ \forall f \in Hom(A, B)$
By proposition (3.6) we get A is $B/R - injective$ also by proposition (3.7) there exist a fuzzy subact of B/R defined by $\delta([x]_R) = \sigma_B(x)\ \forall x \in B$
to prove $\delta([x]_R) \leq \sigma_A(g([x]_R)\ \forall g \in Hom(B/R, A)$ see figure (4) .
$\delta([x]_R) = \sigma_B(x) \leq \sigma_A(f(b))$ but $f = go\pi_B$
$= \sigma_A(go\pi_B) = \sigma_A(g[x]_R)$

$$\begin{tikzcd}
C \arrow[rr] \arrow[dd] & & B \arrow[dl, "\pi_b"'] \arrow[ddll, bend left, "f"] \\
& B/R \arrow[dl, "g"'] & \\
A & &
\end{tikzcd}$$

Figure (4).

□

The direct product of an injective S-act is again an injective S-act and vice versa [10] here we will investigate the direct product of the fuzzy injective S-act.

Proposition 3.5 ([10]). *Take $Q_s, (q_i)_{i\in I})$ with $Q_s = \prod_{i\in I} Q_i$ where $Q_i, i \in I$ are S-acts .Then Q_s is injective if and only if Q_i are injective for all $i \in I$.*

Lemma 3.1. *Let A be an S-act over a monoid S and C be an S-subact of A if there exist a fuzzy subact of A then there exist a fuzzy subact of C.*

Proof. Let σ_A be a fuzzy subact of A then we can put $\sigma_C = \sigma_A|_C$ then it easy to see that σ_C is a fuzzy subact of C.

□

Proposition 3.6. *let $(A, \sigma_A), (B, \sigma_B)$ be two fuzzy S-acts, C subset of A and σ_C a fuzzy subact on C such that σ_A is $\sigma_B - injective$ then σ_A is $\sigma_C - injective$ if $\sigma_C \subset \sigma_B|_C$.*

Proof. Since σ_A is $\sigma_B - injective$ then A is B-injective and $\sigma_B(b) \leq \sigma_A(\alpha(b))$ for each $\alpha \in Hom(A, B)$ *and* $b \in B$, Now for each function f from a subset of C into A there exist $\alpha \in Hom(A, B)$ hence we can take $\beta = \alpha|_C \in Hom(C, A)$, So A is C-injective, for the second part of the fuzzy injectivity, Since $\sigma_B(b) \leq \sigma_A(\alpha(b))$ and $C \subset B$ so $\sigma_B(c) \leq \sigma_A(\alpha(c))$ $\forall c \in C$, By hypothesis we will get

$$\sigma_C(c) \leq \sigma_B(c) \leq \sigma_A(\alpha(c)) \xrightarrow{yields} \alpha_C(c) \leq \sigma_A(\alpha(c))$$

So σ_A *is* $\sigma_C - injective$. □

Theorem 3.4. *Let (A, μ) be a fuzzy S-act ,where $A = \prod_{i\in I} A_i, I = 1, \ldots, n$ direct product of S-acts , σ_{A_i} fuzzy subact of A_i and $\sigma_A = \prod_{i\in I} |\sigma_{A_i}$ direct product of fuzzy subacts of A_i then μ is $\sigma_{A_i} - injective$ if and only if μ is $\sigma_A - injective$*

Proof. Let μ is $\sigma_A - injective$ so A is A-injective and

$$\sigma_A(a) \leq \mu(f(a)) foreach f \in Hom(A, A) \tag{4}$$

To prove μ is $\sigma_{A_i} - injective$ (i.e. to prove A is $A_i - injective$ and $\sigma_{A_i}(a_i) \leq \mu(g(a_i))$ for each $g \in Hom(A_i, A)$)
For the first part of the prove we get it from proposition (3.11) For the second part :

$$Let\ a_i = (0, \ldots, a_i, \ldots, 0)\ then\ \sigma_A(a_i) = \sigma_(A_1)(0) \wedge \ldots \wedge \sigma_{A_i}(a_i) \wedge \ldots \wedge \sigma_{A_n}(0) = \sigma_{A_i}(a_i) \tag{5}$$

$$By\ (4)\ and\ (5)\ we\ have : \sigma_{A_i}(a_i) = \sigma_A(a_i) \leq \mu(f(a_i)) \tag{6}$$

$$But\ f\ is\ an\ extension\ of\ g\ so\ f(a_i) = g(a_i) \tag{7}$$

By (6) and (7) we get $\sigma_{A_i}(a_i) = \sigma_A(a_i) \leq \mu(g(a_i))$
Hence μ is $\sigma_{A_i} - injective.$
Conversely to prove μ is $\sigma_A - injective$ (i.e. to show that A is A-injective and $\sigma_A(a) \leq \mu(f(a))\ \forall f \in Hom(A, A)\ and\ a \in A$
Since μ is $\sigma_{A_i} - injective$ so A is $A_i - injective$ and $\sigma_{A_i}(a_i) \leq \mu(g(a_i)) \forall g \in Hom(A_i, A)$ we can defined f from A into A such that f is an extension of g that is $f|_{A_i} = g$ which is an S-homomorphism by proposition (3.9) hence A is A-injective.
For each $a \in A = \prod_{i \in I} A_i \to a = (a_1, a_2, \ldots, a_n)$

$$\begin{aligned}\sigma_A(a) &= \sigma_A(a_1, a_2, \ldots a_n) \\ &= \sigma_{A_1}(a_1) \wedge \ldots \wedge \sigma_{A_i}(a_i) \wedge \ldots \wedge \sigma_{A_n}(a_n) \\ &\leq \sigma_{A_i}(a_i) \\ &\leq \mu(g(a_i)) = \mu(f(a_i)) \text{ for each } i\end{aligned}$$

This induces that $\sigma_A(a) \leq \wedge\{\mu(f(a_i))\}\ for\ each\ i$ but μ is fuzzy S-act so by definition 2.14

$$\leq \mu(f(a_1, a_2, \ldots a_n)) = \mu(f(a)).$$

□

4. Conclusion

In this work, we have defined the fuzzy injective S-act where S is a monoid and studied its properties for S-subact , retract ,the direct product of fuzzy injective S-act .
This paper can be extended and generalized for fuzzy principal injective, weak principal injective and quasi-injective S-acts.

References

[1] A. Abdul-Kareem, (2019), Intuitionistic fuzzy semi-injective modules, vol.17, pp.119–132, Journal of Algebra and Applied Mathematics.

[2] J. Ahsan, (1987), Monoids characterized by their quasi-injective s-systems, Semigroup Forum, vol.36, pp.285–292, Springer.

[3] M. S. Javed Ahsan, (1987), Fuzzy divisible semigroups, vol. 36, no. 1, pp. 285–292.

[4] M. Roueentan and M. Ershad, (2014), Strongly duo and duo right s-acts, Italian J. Pure Appl. Math, vol. 32, pp. 143–154.

[5] S. Fernandez, (2014), A study of fuzzy G -modules, PhD thesis.

[6] J. Hussain and M. Shabir, (2018), On m-principally injective and projective s-acts, European Journal of Pure and Applied Mathematics, vol. 11, no. 2, pp. 431–443.

[7] M. Haddadi, (2013), Some algebraic properties of fuzzy s-acts, Ratio Mathematica, vol. 24, p. 53.

[8] M. Haddadi, (2014), Fuzzy acts over fuzzy semigroups and sheaves, Iranian Journal of Fuzzy Systems, vol. 11, no. 4, pp. 61–73.

[9] J. Kim, (1991), Injective s-systems and regular semigroups, Journal of the Chungcheong Mathematical Society, vol. 4, no. 1, pp. 1–9.

[10] M. Kilp, U. Knauer, and A. V. Mikhalev, (2011), Monoids, Acts and Categories: With Applications to Wreath Products and Graphs. A Handbook for Students and Researchers, vol. 29, Walter de Gruyter.

[11] A. M. Lopez Jr and J. K. Luedeman, (1979), Quasi-injective *s*-systems and their *s*-endomorphism semigroup, Czechoslovak Mathematical Journal, vol. 29, no. 1, pp. 97–104.

[12] J. N. Mordeson, D. S. Malik, and N. Kuroki, (2012), Fuzzy semigroups, vol. 131, Springer.

[13] D. S. M. J. N. Mordeson and N. Kuroki, (2011), Fuzzy semigroups,studies in fuzziness and soft computing, Springer-Verlag, Berlin.

[14] G. Moghaddasi, M. Haddadi, and S. Delavari, (2017), On injectivity of acts, University Politechnica of Bucharest scientific Bulletin Series, Applied Mathemathics and Physics, vol. 79, no. 4, pp. 189–198.

[15] S. Pamy, (2013), A study on semigroup action on fuzzy subsets inverse fuzzy automata and related topics.

[16] P. Sharma, (2016), Reducibility and complete reducibility, Annals of Fuzzy Mathematics and Informatics, vol. 11, no. 6, pp. 885–898.
[17] P. Sharma and T. Kaur, (2016), On intuitionistic fuzzy representation of intuitionistic fuzzy g-modules, Annals of Fuzzy Mathematics and Informatics, vol. 11, no. 4, pp. 557–569.
[18] X. Zhang and J. Paseka, (2019), On injective constructions of s-semigroups, Fuzzy Sets and Systems.

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