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SOME OPERATORS ON PYTHAGOREAN FUZZY SOFT MATRICES

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Abstract. In this paper, we define modal operators of Pythagorean fuzzy soft matrices and establish their algebraic properties. Also, we define implication operators on Pythagorean fuzzy soft matrices and discuss some properties on implication compained with max-min and min-max compositions of Pythagorean fuzzy soft matrices.

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1. Introduction

The concept of fuzzy set theory was first introduced Zadeh[16] in 1965. An intuitionistic fuzzy set was introduced in 1983, by Atanassov[2] as an extension of Zadeh's fuzzy set. In 1999, Molodtsov[8] introduced soft set theory-first result. In 2001, Maji et al.[6] defined Intuitionistic fuzzy soft sets. In 2010, Meenakshi and Gandhimathi[7] studied Intuitionistic fuzzy relational equations. In 2012, Chetia and Das[4] characterized intuitionistic fuzzy soft matrices and their operations which are more functional to make theoretical studies in the intuitionistic fuzzy soft set theory. In 2013, Mondal and Roy[9] studied the properties of necessity and possibility operators of Intuitionistic fuzzy soft matrices. In 2013, Yager and Abbasov[15] developed Pythagorean fuzzy set (PFS) characterized by a membership degree and non-membership degree, which satisfies the condition that the square sum of its membership degree and non-membership degree is less than or equal to 1.

In 2014, Zhang and Xu[17] proposed an extension of technique for order preference by similarity to an ideal solution (TOPSIS) to solve multiple attribute decision making problem with Pythagorean fuzzy information. In 2014, Murugadas et al.[10] defined Model operators in intuitionistic fuzzy matrices. In 2014, Sarala and Rajkumari[12] establish necessity and possibility operation of Intuitionistic fuzzy soft matrices. In 2019, Boobalan[3] studied their certain results on necessity and possibility operations of Intuitionistic fuzzy matrices. In 2015, Peng and Yang[11] introduced the Pythagorean Fuzzy Soft Set (PFSS) and studied various binary operations over it. In 2018, Silambarasan

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and Sriram[13] introduced Algebraic operations on Pythagorean fuzzy matrices. Venkatesan and Sriram[14] studied their commutative monoid and monoid homomorphism on Lukasiewicz disjunction and conjunction of Pythagorean fuzzy matrices and established their algebraic properties. In 2018, Guleria and Bajaj[5] introduced modal operators on Pythagorean fuzzy soft matrices (PFSMs) and studied their applications in decision making and medical diagnosis. In 2019, Arikrishnan and Sriram[1] defined necessity and possibility operators on Pythagorean fuzzy soft matrices. In this paper, we defined modal operators of Pythagorean fuzzy soft matrices and established their algebraic properties. Finally, we discussed some properties on implication compained with max-min and min-max compositions of Pythagorean fuzzy soft matrices.

2. Preliminaries

In this section, we recall some fundamental concepts in connection with the Pythagorean fuzzy set, which are well known in literature.

Definition 2.1. [2] *An Intuitionistic fuzzy set (IFS) I in X (universe of discourse) is given by* $I = \{< x, \mu_I(x), \nu_I(x) > | x \in X\}$*; where* $\mu_I : X \to [0,1]$ *and* $\nu_I : X \to [0,1]$ *denote the degree of membership and degree of non-membership, respectively, and for every* $x \in X$ *satisfy the condition* $0 \le \mu_I(x) + \nu_I(x) \le 1$ *and the degree of indeterminacy for any IFS I and* $x \in X$ *is given by* $\pi_I(x) = 1 - \mu_I(x) - \nu_I(x)$.

Definition 2.2. [15] *A Pythagorean fuzzy set (PFS) M in X (universe of discourse) is given by* $M = \{< x, \mu_M(x), \nu_M(x) > | x \in X\}$*; where* $\mu_M : X \to [0,1]$ *and* $\nu_M : X \to [0,1]$ *denote the degree of membership and degree of non-membership, respectively, and for every* $x \in X$ *satisfy the condition* $0 \le \mu_M^2(x) + \nu_M^2(x) \le 1$ *and the degree of indeterminacy for any IFS M and* $x \in X$ *is given by* $\pi_M(x) = \sqrt{1 - \mu_M^2(x) - \nu_M^2(x)}$*. In case of PES, the restriction corresponding to the degree of membership* $\mu_M(x)$ *and the degree of non membership* ν_M *is* $0 \le \mu_I^2(x) + \nu_I^2(x) \le 1$*, where as the condition in case of IFS is* $0 \le \mu_I(x) + \nu_I(x) \le 1$ *for* $\mu_M(x), \nu_M(x) \in [0,1]$.

Definition 2.3. [8] *Suppose that U is an initial universe set and E is a set of parameters, let* $P(U)$ *denotes the power set of U. Let* $A \subseteq E$*. A pair* (F_A, E) *is called a soft set over U, where* F_A *is a mapping given by* $F_A : E \longrightarrow P(U)$*. Such that* $F_A = \phi$ *if* $e \notin A$*. Here* F_A *is called approximate function of soft set* (F_A, E) *The set* $F_A(e)$ *is called e-approximate value set which consist of related objects of the parameter* $E \in e$*. In other words, a soft set over U is a parametrized family of subsets of the universe U.*

Definition 2.4. [4] *Let* (F_A, E) *be a soft set over U. Then a subset of* $U \times E$ *is uniquely defined by* $R_A = (u, e); e \in A, u \in F_A(e)$ *which is called a* χ_{R_A} *relation form of* (F_A, E)*. The characteristic function of* R_A *is written by* $\mu_{R_A} : U \times E \to [0,1]$*, where* $\mu_{R_A}(u, e) \in [0,1]$ *is the membership value of* $u \in U$ *for each* $e \in E$*. If* $\mu_{ij} = \mu_{R_A}(u_i, e_j)$*, we can define a matrix*

$$[\mu_{i,j}]_{m\times n} = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mn} \end{bmatrix}$$

which is called an $m \times n$ *soft matrix of the soft set* (F_A, E) *over U. therefore, we can say that a fuzzy soft set* (F_A, E) *is uniquely characterized by the matrix* $[\mu_{ij}]_{m\times n}$ *and both concept are interchangeable. The set of all* $m \times n$ *fuzzy soft matrices over U will be denoted by* $FSM_{m\times n}$.

Definition 2.5. [11] *If (F_A, E) be a Pythagorean fuzzy soft set over X, then the subset, $X \times E$ is uniquely defined by $R_A = (x, e), e \in A, x \in F_A(e)$. The R_A can be characterized by its membership function and non-membership function given by $\mu_R : X \times E \to [0,1]$ and $\nu_R : X \times E \to [0,1]$, respectively. If $(\mu_{i,j}, \nu_{i,j}) = (\mu_{R_A}(x_i, e_j), \nu_{R_A}(x_i, ej))$, where $\mu_{R_A}(x_i, e_j)$ is the membership of x_i in the Pythagorean fuzzy set $F(e_j)$ and $\nu_{R_A}(x_i, e_j)$ is the non-membership of x_i in the Pythagorean fuzzy set $F(e_j)$, respectively, then we define a matrix given by*

$$[M] = [m_{i,j}]_{m\times n} = [(\mu^M_{i,j}, \nu^M_{i,j})]_{m\times n} = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{21}, \nu_{21}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (\mu_{m1}, \nu_{m1}) & (\mu_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}) \end{bmatrix}$$

*which is called **Pythagorean fuzzy soft matrix** of order $m \times n$ over X.*

Definition 2.6. [5] *Let $A = [(\mu^A_{ij}, \nu^A_{ij})], B = [(\mu^B_{ij}, \nu^B_{ij})] \in PFSM_{m\times n}$. Then*
*(i) **Sub matrix:** $A \subseteq B$ if $\mu^A_{ij} \leq \mu^B_{ij}$ and $\nu^A_{ij} \geq \nu^B_{ij}$ $\forall$ i and j.*
*(ii) **Super matrix:** $A \supseteq B$ if $\mu^A_{ij} \geq \mu^B_{ij}$ and $\nu^A_{ij} \leq \nu^B_{ij}$ $\forall$ i and j.*
*(iii) **Equal matrix:** $A = B$ if $\mu^A_{ij} = \mu^B_{ij}$ and $\nu^A_{ij} = \nu^B_{ij}$ $\forall$ i and j.*

Definition 2.7. [5] *Let $A = [(\mu^A_{ij}, \nu^A_{ij})], B = [(\mu^B_{ij}, \nu^B_{ij})] \in PFSM_{m\times n}$. Then*
*(i) **Complement:** A^0=$[(\nu^A_{ij}, \mu^A_{ij})]$ $\forall$ i and j.*
*(ii) **Union:** $A \cup B$=$[\max(\mu^A_{ij}, \mu^B_{ij}), \min(\nu^A_{ij}, \nu^B_{ij})]$ $\forall$ i and j.*
*(iii) **Intersection:** $A \cap B$=$[\min(\mu^A_{ij}, \mu^B_{ij}), \max(\nu^A_{ij}, \nu^B_{ij})]$ $\forall$ i and j*

Definition 2.8. [5] *Let $A = [(\mu^A_{ij}, \nu^A_{ij})], B = [(\mu^B_{ij}, \nu^B_{ij})] \in PFSM_{m\times n}$. Then*
(i) $A + B$=$[(\mu^A_{ij} + \mu^B_{ij} - \nu^A_{ij} \cdot \nu^B_{ij}, \nu^A_{ij} \cdot \nu^B_{ij})]$ $\forall$ i and j.
(ii) $A \cdot B$=$[(\mu^A_{ij} \cdot \mu^B_{ij}, \nu^A_{ij} + \nu^B_{ij} - \nu^A_{ij} \cdot \nu^B_{ij})]$ $\forall$ i and j.
(iii) $A \oplus B = \left[\left(\sqrt{(\mu^A_{ij})^2 + (\mu^B_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2}, \nu^A_{ij} \cdot \nu^B_{ij}\right)\right]$ $\forall$ i and j.
(iv) $A \otimes B = \left[\left(\mu^A_{ij} \cdot \mu^B_{ij}, \sqrt{(\nu^A_{ij})^2 + (\nu^B_{ij})^2 - (\nu^A_{ij})^2 \cdot (\nu^B_{ij})^2}\right)\right]$ $\forall$ i and j.

3. Modal operators on Pythagorean fuzzy soft matrices

In this section, we define on modal operators of a Pythagorean fuzzy soft matrix and discuss the in algebraic properties. Also we define an implication of Pythagorean fuzzy soft matrices and studied their related properties.

Definition 3.1. *Let $A = [(\mu^A_{ij}, \nu^A_{ij})] \in PFSM_{m\times n}$. Then*
*(i) The **Necessity operators** of A is denoted by $\Box A$ and defined as*
$\Box A = \left[\left(\mu^A_{ij}, \sqrt{1 - \mu^{A^2}_{ij}}\right)\right]$ for all i and j.
*(ii) The **Possibility operators** of A is denoted by $\Diamond A$ and defined as*
$\Diamond A = \left[\left(\sqrt{(1 - \nu^{A^2}_{ij}}, \nu^A_{ij}\right)\right]$ for all i and j.

In the following we have to prove some new results related to the necessity and possibility operators of Pythagorean fuzzy soft matrices.

Theorem 3.1. *Let $A = [(\mu^A_{ij}, \nu^A_{ij})], B = [(\mu^B_{ij}, \nu^B_{ij})]$, $C = [(\mu^C_{ij}, \nu^C_{ij})] \in PFSM_{m\times n}$. Then*
(i) $\Box((A \oplus B) \otimes C) = (\Box A \oplus \Box B) \otimes \Box C$.
(ii) $\Box((A \otimes B) \oplus C) = (\Box A \otimes \Box B) \oplus \Box C$.

Proof. Let $A = [(\mu^A_{ij}, \nu^A_{ij})], B = [(\mu^B_{ij}, \nu^B_{ij})], C = [(\mu^C_{ij}, \nu^C_{ij})] \in PFSM_{m\times n}$. for all i and j,

(i) $((A \oplus B) \otimes C) = \left[\left(\sqrt{(\mu^A_{ij})^2 + (\mu^B_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2}, \nu^A_{ij} \cdot \nu^B_{ij} \right) \right] \otimes \left[\left(\mu^C_{ij}, \nu^C_{ij} \right) \right]$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 + (\mu^B_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2} \cdot \mu^C_{ij}, \sqrt{(\nu^A_{ij} \cdot \nu^B_{ij})^2 + (\nu^C_{ij})^2 - (\nu^A_{ij} \cdot \nu^B_{ij})^2 \cdot (\nu^C_{ij})^2} \right)\right]$$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\mu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij})}, \sqrt{(\nu^A_{ij} \cdot \nu^B_{ij})^2 + (\nu^C_{ij})^2 - (\nu^A_{ij} \cdot \nu^B_{ij})^2 \cdot (\nu^C_{ij})^2} \right)\right].$$

$\square\, ((A \oplus B) \otimes C)$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\mu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij})}, \sqrt{1 - ((\mu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\mu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij}))} \right)\right] \tag{3.1}$$

$\square A \oplus \square B = \left[\left(\mu^A_{ij}, \sqrt{1 - \mu^{A^2}_{ij}} \right) \right] \oplus \left[\left(\mu^B_{ij}, \sqrt{1 - \mu^{B^2}_{ij}} \right) \right]$

$= \left[\left(\sqrt{(\mu^A_{ij})^2 + (\mu^B_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2}, \sqrt{1 - ((\mu^A_{ij})^2 + (\mu^B_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2)} \right) \right]$

$(\square A \oplus \square B) \otimes \square C$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 + (\mu^B_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2}, \sqrt{(1 - \mu^A_{ij})^2 \cdot (1 - \mu^B_{ij})^2} \right)\right] \otimes \left[\left(\mu^C_{ij}, \sqrt{1 - \mu^{C^2}_{ij}} \right) \right]$$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\mu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij})}, \sqrt{(1 - \mu^A_{ij})^2 \cdot (1 - \mu^B_{ij})^2 + (1 - \mu^C_{ij})^2 - (1 - \mu^B_{ij})^2 \cdot (1 - \mu^C_{ij})^2} \right)\right]$$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\mu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij})}, \sqrt{(1 - \mu^A_{ij})^2 \cdot (1 - \mu^B_{ij})^2 (\mu^C_{ij}) + 1 - \mu^C_{ij}} \right)\right]$$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\mu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij})}, \sqrt{1 - ((\mu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\mu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij}))} \right)\right] \tag{3.2}$$

$(\square A \oplus \square B) \otimes \square C$.

from (3.1) and (3.2), (i) is true.

(ii) $\square((A \otimes B) \oplus C) = \left[\left(\mu^A_{ij} \cdot \mu^B_{ij}, \sqrt{(\nu^A_{ij})^2 + (\nu^B_{ij})^2 - (\nu^A_{ij})^2 \cdot (\nu^B_{ij})^2} \right) \right] \oplus \left[\left(\mu^C_{ij}, \nu^C_{ij} \right) \right]$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 + (\mu^C_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij})^2} \right), \left(\sqrt{(\nu^A_{ij})^2 + (\nu^B_{ij})^2 - (\mu^A_{ij})^2 \cdot (\nu^B_{ij})^2} \cdot (\mu^C_{ij}) \right) \right]$$

$$= \left[\left(\sqrt{(\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 + (\mu^C_{ij})^2 - (\mu^A_{ij})^2 \cdot (\mu^B_{ij})^2 \cdot (\mu^C_{ij})^2} \right), \left(\sqrt{(\nu^A_{ij})^2 \cdot (\mu^C_{ij}) + (\nu^B_{ij})^2 \cdot (\mu^C_{ij}) - (\mu^A_{ij})^2 \cdot (\nu^B_{ij})^2 \cdot (\mu^C_{ij})} \right) \right]$$

$\Box((A \otimes B) \oplus C)$

$$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 + (\mu_{ij}^{C})^2 - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 \cdot (\mu_{ij}^{C})^2},$$

$$\sqrt{1 - ((\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 + (\mu_{ij}^{C})^2 - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 \cdot (\mu_{ij}^{C})^2)}\Big)\Big] \tag{3.3}$$

$(\Box A \otimes \Box B) = \Big[\Big(\mu_{ij}^{A}, \sqrt{1-\mu_{ij}^{A^2}}\Big) \otimes \Big(\mu_{ij}^{B}, \sqrt{1-\mu_{ij}^{B^2}}\Big)\Big]$

$= \Big[\Big(\mu_{ij}^{A} \cdot \mu_{ij}^{B}, \sqrt{(1-\mu_{ij}^{A})^2 + (1-\mu_{ij}^{B})^2 - (1-\mu_{ij}^{A})^2 \cdot (1-\mu_{ij}^{B})^2}\Big)\Big]$

$(\Box A \otimes \Box B) \oplus \Box C.$

$= \Big[\Big(\mu_{ij}^{A} \cdot \mu_{ij}^{B}, \sqrt{(1-\mu_{ij}^{A})^2 + (1-\mu_{ij}^{B})^2 - (1-\mu_{ij}^{A})^2 \cdot (1-\mu_{ij}^{B})^2}\Big) \oplus \Big(\mu_{ij}^{C}, \sqrt{(1-\mu_{ij}^{A})^2}\Big)\Big]$

$$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 + (\mu_{ij}^{C})^2 - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 \cdot (\mu_{ij}^{C})^2},$$

$$\sqrt{(1-\mu_{ij}^{A})^2 + (1-\mu_{ij}^{B})^2 - (1-\mu_{ij}^{A})^2 \cdot (1-\mu_{ij}^{B})^2} \cdot (1-\mu_{ij}^{C})^2\Big)\Big]$$

$$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 + (\mu_{ij}^{C})^2 - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 \cdot (\mu_{ij}^{C})^2},$$

$$\sqrt{1 - ((\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 + (\mu_{ij}^{C})^2 - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 \cdot (\mu_{ij}^{C})^2)}\Big)\Big] \tag{3.4}$$

from (3.3) and (3.4), (ii) is true. ∎

Theorem 3.2. *Let $A, B, C \in PFSM_{m\times n}$. Then*
(*i*) $\Diamond((A \oplus B) \otimes C) = (\Diamond A \oplus \Diamond B) \otimes \Diamond C.$
(*ii*) $\Diamond((A \otimes B) \oplus C) = (\Diamond A \otimes \Diamond B) \oplus \Diamond C.$

Proof. (*i*) $((A \oplus B) \otimes C) = \Big[\Big(\sqrt{(\mu_{ij}^{A})^2 + (\mu_{ij}^{B})^2 - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2}, \nu_{ij}^{A} \cdot \nu_{ij}^{B}\Big) \otimes \Big(\mu_{ij}^{C}, \nu_{ij}^{C}\Big)\Big]$

$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2 + (\mu_{ij}^{B})^2 - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2} \cdot \mu_{ij}^{C}, \sqrt{(\nu_{ij}^{A} \cdot \nu_{ij}^{B})^2 + (\nu_{ij}^{C})^2 - (\nu_{ij}^{A} \cdot \nu_{ij}^{B})^2 \cdot (\nu_{ij}^{C})^2}\Big)\Big]$

$$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{C}) + (\mu_{ij}^{B})^2 \cdot (\mu_{ij}^{C}) - (\mu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 \cdot (\mu_{ij}^{C})},$$

$$\sqrt{(\nu_{ij}^{A} \cdot \nu_{ij}^{B})^2 + (\nu_{ij}^{C})^2 - (\nu_{ij}^{A} \cdot \nu_{ij}^{B})^2 \cdot (\nu_{ij}^{C})^2}\Big)\Big]$$

$\Diamond\Big((A \oplus B) \otimes C\Big)$

$$= \Big[\Big(\sqrt{1 - ((\nu_{ij}^{A})^2 \cdot (\nu_{ij}^{B})^2 + (\nu_{ij}^{C})^2 - (\nu_{ij}^{A})^2 \cdot (\nu_{ij}^{B})^2 \cdot (\mu_{ij}^{C})^2)},$$

$$\sqrt{(\nu_{ij}^{A})^2 \cdot (\mu_{ij}^{B})^2 + (\mu_{ij}^{C})^2 - (\nu_{ij}^{A})^2 \cdot (\nu_{ij}^{B})^2 \cdot (\nu_{ij}^{C})^2}\Big)\Big] \tag{3.5}$$

$(\Diamond A \oplus \Diamond B) = \Big[\Big(\sqrt{1-\nu_{ij}^{A^2}}, \nu_{ij}^{A}\Big) \oplus \Big(\sqrt{1-\nu_{ij}^{B^2}}, \nu_{ij}^{B}\Big)\Big]$

$= \Big[\sqrt{(1-\nu_{ij}^{A})^2 + (1-\nu_{ij}^{B})^2 - (1-\nu_{ij}^{A})^2 \cdot (1-\nu_{ij}^{B})^2}, \nu_{ij}^{A} \cdot \nu_{ij}^{B}\Big]$

$(\Diamond A \oplus \Diamond B) \otimes \Diamond C.$

$= \Big[\sqrt{(1-\nu_{ij}^{A})^2 + (1-\nu_{ij}^{B})^2 - (1-\nu_{ij}^{A})^2 \cdot (1-\nu_{ij}^{B})^2}, \nu_{ij}^{A} \cdot \nu_{ij}^{B}\Big] \otimes \Big[\Big(\sqrt{1-\nu_{ij}^{C^2}}\Big)\Big]$

$$= \Big[\Big(\sqrt{(1-\nu_{ij}^{A})^2 + (1-\nu_{ij}^{B})^2 - (1-\nu_{ij}^{A})^2 \cdot (1-\nu_{ij}^{B})^2} \cdot \sqrt{(1-\nu_{ij}^{C})^2},$$

$$\sqrt{(\nu_{ij}^{A})^2 \cdot (\nu_{ij}^{B})^2 + (\nu_{ij}^{C})^2 - (\nu_{ij}^{A})^2 \cdot (\nu_{ij}^{B})^2 \cdot (\nu_{ij}^{C})^2}\Big)\Big]$$

$$= \Big[\Big(\sqrt{1-\Big((\nu_{ij}^{A})^2\cdot(\nu_{ij}^{B})^2+(\nu_{ij}^{C})^2-(\nu_{ij}^{A})^2\cdot(\nu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2\Big)},$$

$$\sqrt{(\nu_{ij}^{A})^2\cdot(\mu_{ij}^{B})^2+(\mu_{ij}^{C})^2-(\nu_{ij}^{A})^2\cdot(\nu_{ij}^{B})^2\cdot(\nu_{ij}^{C})^2}\Big)\Big] \tag{3.6}$$

from (3.5) and (3.6), (i) is true.
(*ii*) It can be proved similarly.

□

Theorem 3.3. *Let $A, B, C \in PFSM_{m\times n}$. Then*
(*i*) $\Box\,(A\oplus(B\otimes C)) = \Box A\oplus(\Box B\otimes\Box C)$.
(*ii*) $\Box\,((A\otimes B)\oplus C) = \Box A\otimes(\Box B\oplus\Box C)$.

Proof. (*i*) $(B\otimes C) = \Big[\Big(\mu_{ij}^{B}\cdot\mu_{ij}^{C}, \sqrt{(\nu_{ij}^{B})^2+(\mu_{ij}^{C})^2-(\nu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2}\Big)\Big]$
$(A\oplus(B\otimes C)) = \Big[\Big(\mu_{ij}^{A}, \mu_{ij}^{B}\Big)\oplus\Big(\mu_{ij}^{B}\cdot\mu_{ij}^{C}, \sqrt{(\mu_{ij}^{B})^2+(\mu_{ij}^{C})^2-(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2}\Big)\Big]$

$$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2+(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2-(\mu_{ij}^{A})^2\cdot(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2},$$

$$\mu_{i,j}^{B}\cdot\sqrt{(\mu_{ij}^{B})^2+(\mu_{ij}^{C})^2-(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2}\Big)\Big]$$

$\Box(A\oplus(B\otimes C))$

$$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2+(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2-(\mu_{ij}^{A})^2\cdot(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2},$$

$$\sqrt{1-((\mu_{ij}^{A})^2+(\mu_{ij}^{B})\cdot(\mu_{ij}^{C})^2-(\mu_{ij}^{A})^2\cdot(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2)}\Big)\Big] \tag{3.7}$$

$(\Box B\otimes\Box C)$
$= \Big[\Big(\mu_{ij}^{B}\cdot\mu_{ij}^{C}, \sqrt{(1-\mu_{ij}^{B})^2+(1-\mu_{ij}^{C})^2-(1-\mu_{ij}^{B})^2\cdot(1-\mu_{ij}^{C})^2}\Big)\Big]$
$\Box A\oplus(\Box B\otimes\Box C)$.
$= \Big[\Big(\mu_{ij}^{A}, \sqrt{(1-\mu_{ij}^{A})^2}\Big)\Big]\oplus\Big[\Big(\mu_{ij}^{B}\cdot\mu_{ij}^{C}, \sqrt{(1-\mu_{ij}^{B})^2+(1-\mu_{ij}^{C})^2-(1-\mu_{ij}^{B})^2\cdot(1-\mu_{ij}^{C})^2}\Big)\Big]$
$= \Big[\Big(\sqrt{(\mu_{ij}^{A})^2+(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2-(\mu_{ij}^{A})^2\cdot(\mu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2},$
$\sqrt{(1-\mu_{ij}^{A})^2}\cdot\sqrt{(1-\mu_{ij}^{B})^2+(1-\mu_{ij}^{C})^2-(1-\mu_{ij}^{B})^2\cdot(1-\mu_{ij}^{C})^2}\Big)\Big]$

$$= \Big[\Big(\sqrt{(\nu_{ij}^{A})^2+(\mu_{ij}^{B})^2\cdot(\nu_{ij}^{C})^2-(\mu_{ij}^{A})^2\cdot(\nu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2},$$

$$\sqrt{1-((\nu_{ij}^{A})^2+(\mu_{ij}^{B})^2\cdot(\nu_{ij}^{C})^2-(\mu_{ij}^{A})^2\cdot(\nu_{ij}^{B})^2\cdot(\mu_{ij}^{C})^2)}\Big)\Big] \tag{3.8}$$

from (3.7) and (3.8), (i) is true.
(*ii*) It can be proved similarly.

□

Theorem 3.4. *Let $A, B, C \in PFSM_{m\times n}$. Then*
(*i*) $\Diamond\,(A\oplus(B\otimes C)) = \Diamond A\oplus(\Diamond B\otimes\Diamond C)$.
(*ii*) $\Diamond\,((A\otimes B)\oplus C) = \Diamond A\otimes(\Diamond B\oplus\Diamond C)$.

Proof. (i) $(B \otimes C) = \left[\left(\mu_{ij}^B \cdot \mu_{ij}^C, \sqrt{(\nu_{ij}^B)^2 + (\mu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\mu_{ij}^C)^2}\right)\right]$

$$(A \oplus (B \otimes C)) = \left[\left(\mu_{ij}^A, \nu_{ij}^A\right)\right] \oplus \left[\left(\mu_{ij}^B \cdot \mu_{ij}^C, \sqrt{(\nu_{ij}^B)^2 + (\nu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\nu_{ij}^C)^2}\right)\right]$$

$$= \left[\left(\sqrt{(\mu_{ij}^A)^2 + (\mu_{ij}^B)^2 \cdot (\mu_{ij}^C)^2 - (\mu_{ij}^A)^2 \cdot (\mu_{ij}^B)^2 \cdot (\mu_{ij}^C)^2}, \nu_{i,j}^A \cdot \sqrt{(\nu_{ij}^B)^2 + (\nu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\nu_{ij}^C)^2}\right)\right]$$

$\Diamond (A \oplus (B \otimes C))$

$$= \left[\left(\sqrt{1 - ((\nu_{ij}^A)^2 + (\nu_{ij}^B)^2 \cdot (\mu_{ij}^C)^2 - (\mu_{ij}^B)^2 \cdot (\mu_{ij}^C)^2 \cdot (\mu_{ij}^A)^2}, \nu_{i,j}^A \cdot \sqrt{(\nu_{ij}^B)^2 + (\nu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\nu_{ij}^C)^2}\right)\right] \tag{3.9}$$

$$(\Diamond B \otimes \Diamond C). = \left[\left(\sqrt{1 - \nu_{ij}^{B^2}}, \nu_{ij}^B\right) \otimes \left(\sqrt{1 - \nu_{ij}^{C^2}}, \nu_{ij}^C\right)\right]$$

$$= \left[\left(\sqrt{(1 - \nu_{ij}^B)^2} \cdot \sqrt{(1 - \nu_{ij}^C)^2}, \sqrt{(\nu_{ij}^B)^2 + (\nu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\nu_{ij}^C)^2}\right)\right]$$

$\Diamond A \oplus (\Diamond B \otimes \Diamond C)$.

$$= \left[\left(\sqrt{(1 - \nu_{ij}^A)^2}, \nu_{ij}^A\right) \oplus \left(\sqrt{(1 - \nu_{ij}^B)^2} \cdot \sqrt{(1 - \nu_{ij}^C)^2}, \sqrt{(\nu_{ij}^B)^2 + (\nu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\nu_{ij}^C)^2}\right)\right]$$

$$= \left[\left(\sqrt{(1 - \nu_{ij}^A)^2 + (1 - \nu_{ij}^B)^2 \cdot (1 - \nu_{ij}^C)^2 - (1 - \nu_{ij}^A)^2 \cdot (1 - \nu_{ij}^B)^2 \cdot (1 - \nu_{ij}^C)^2}, \nu_{i,j}^A \cdot \sqrt{(\nu_{ij}^B)^2 + (\nu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\nu_{ij}^C)^2}\right)\right]$$

$$= \left[\left(\sqrt{1 - ((\nu_{ij}^A)^2 + (\nu_{ij}^B)^2 \cdot (\mu_{ij}^C)^2 - (\mu_{ij}^B)^2 \cdot (\mu_{ij}^C)^2 \cdot (\mu_{ij}^A)^2}, \nu_{i,j}^A \cdot \sqrt{(\nu_{ij}^B)^2 + (\nu_{ij}^C)^2 - (\nu_{ij}^B)^2 \cdot (\nu_{ij}^C)^2}\right)\right] \tag{3.10}$$

from (3.9) and (3.10), (i) is true.
(*ii*) It can be proved similarly.

□

Definition 3.2. *Let* $A = [(\mu_{ij}^A, \nu_{ij}^A)], B = [(\mu_{ij}^B, \nu_{ij}^B)] \in PFSM_{m\times n}$. *Then the operator* $\mapsto$ *(Implication) denoted by* $A \mapsto B$ *is defined by* $A \mapsto B = \left[(\max\{\nu_{ij}^A, \mu_{ij}^B\}, \min\{\mu_{ij}^A, \nu_{ij}^B\})\right]$ *for all* i *and* j.

In the following we have to prove some new results related to the implication operators of Pythagorean fuzzy soft matrices.

Proposition 3.1. *Let* $A = [(\mu_{ij}^A, \nu_{ij}^A)], B = [(\mu_{ij}^B, \nu_{ij}^B)],\ C = [(\mu_{ij}^C, \nu_{ij}^C)] \in PFSM_{m\times n}$. *Then*
$(i)[A \cap B] \mapsto C \supseteq [A \mapsto C] \cap [B \mapsto C]$
$(ii)[A \cup B] \mapsto C \subseteq [A \mapsto C] \cup [B \mapsto C]$
$(iii)[A \cap B] \mapsto C = [A \mapsto C] \cup [B \mapsto C]$

Proof. Let $A = [\left(\mu_{ij}^A, \nu_{ij}^A\right)]$, $B = [\left(\mu_{ij}^B, \nu_{ij}^B\right)]$, $C = [\left(\mu_{ij}^C, \nu_{ij}^C\right)] \in PFSM_{m\times n}$. Then for all i and j,
$(i)(A \cap B) \mapsto C$

$$= \left[\left(\min\{\mu_{ij}^A, \mu_{ij}^B\}, \max\{\nu_{ij}^A, \nu_{ij}^B\}\right)\right] \mapsto \left[\left(\{\mu_{ij}^C, \nu_{ij}^C\}\right)\right]$$

$$= \Big[\Big(\max\{\max\{\nu_{ij}^A, \nu_{ij}^B\}, \mu_{ij}^C\}, \min\{\min\{\mu_{ij}^A, \mu_{ij}^B\}, \nu_{i,j}^C\}\Big)\Big] \quad (3.11)$$

$[A \mapsto C] \cap [B \mapsto C]$

$$= \Big[\Big(\max\{\nu_{ij}^A, \mu_{ij}^C\}, \min\{\mu_{ij}^A, \nu_{ij}^C\}\Big)\Big] \cap \Big[\Big(\max\{\nu_{ij}^B, \mu_{ij}^C\}, \min\{\mu_{ij}^B, \nu_{ij}^C\}\Big)\Big]$$

$$= \Big[\Big(\min\{\max\{\nu_{ij}^A, \mu_{ij}^C\}, \max\{\nu_{ij}^B, \mu_{ij}^C\}\}, \max\{\min\{\mu_{ij}^A, \nu_{ij}^C\}, \min\{\mu_{ij}^B, \nu_{ij}^C\}\}\Big)\Big]$$

$$= \Big[\Big(\min\{\max\{\nu_{ij}^A, \nu_{ij}^B\}, \mu_{ij}^C\}, \max\{\min\{\mu_{ij}^A, \nu_{ij}^B\}, \nu_{ij}^C\}\Big)\Big] \quad (3.12)$$

From (3.11) and (3.12) it is clear that.
$[A \cap B] \mapsto C \supseteq [A \mapsto C] \cap [B \mapsto C]$

(ii) $(A \cup B) \mapsto C$

$$= \Big[\Big(\max\{\mu_{ij}^A, \mu_{ij}^B\}, \min\{\nu_{ij}^A, \nu_{ij}^B\}\Big)\Big] \mapsto \Big[\Big(\{\mu_{ij}^C, \nu_{ij}^C\}\Big)\Big]$$

$$= \Big[\Big(\max\{\min\{\nu_{ij}^A, \nu_{ij}^B\}, \mu_{ij}^C\}, \min\{\max\{\mu_{ij}^A, \mu_{ij}^B\}, \nu_{ij}^C\}\Big)\Big] \quad (3.13)$$

$[A \mapsto C] \cup [B \mapsto C]$

$$= \Big[\Big(\max\{\nu_{ij}^A, \mu_{ij}^C\}, \min\{\mu_{ij}^A, \nu_{ij}^C\}\Big] \cup \Big[\{\max\{\nu_{ij}^A, \mu_{ij}^C\}, \min\{\mu_{ij}^B, \nu_{ij}^C\}\Big)\Big]$$

$$= \Big[\Big(\max\{\max\{\nu_{ij}^A, \mu_{ij}^C\}, \max\{\nu_{ij}^B, \mu_{ij}^C\}\}, \min\{\min\{\mu_{ij}^A, \nu_{ij}^C\}, \min\{\mu_{ij}^B, \nu_{ij}^C\}\}\Big)\Big]$$

$$= \Big[\Big(\max\{\max\{\nu_{ij}^A, \nu_{ij}^B\}, \mu_{ij}^C\}, \min\{\min\{\mu_{ij}^A, \nu_{ij}^B\}, \nu_{ij}^C\}\}\Big)\Big] \quad (3.14)$$

From (3.13) and (3.14) it is clear that
$[A \cap B] \mapsto C \supseteq [A \mapsto C] \cup [B \mapsto C]$.

(iii) $[A \cap B] \mapsto C = [A \mapsto C] \cup [B \mapsto C]$
Proof follows from (3.11) and (3.14).

□

Proposition 3.2. *Let* $A = [(\mu_{ij}^A, \nu_{ij}^A)], B = [(\mu_{ij}^B, \nu_{ij}^B)],\ C = [(\mu_{ij}^C, \nu_{ij}^C)] \in PFSM_{m\times n}$. *Then*
(i) $[A + B] \mapsto C \supseteq [A \mapsto C] + [B \mapsto C]$
(ii) $[A.B] \mapsto C \subseteq [A \mapsto C].[B \mapsto C]$
(iii) $A \mapsto [B + C] \subseteq [A \mapsto B] + [A \mapsto C]$
(iv) $A \mapsto [B.C] \supseteq [A \mapsto B].[A \mapsto C]$
(v) $A \mapsto A^0 = A^0$

Proof. Let $A = [\Big(\mu_{ij}^A, \nu_{ij}^A\Big)]$, $B = [\Big(\mu_{ij}^B, \nu_{ij}^B\Big)]$, $C = [\Big(\mu_{ij}^C, \nu_{ij}^C\Big)] \in PFSM_{m\times n}$. Then for all i and j,
(i) $[A + B] \mapsto C$

$$= \Big[\Big(\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B, \nu_{ij}^A \cdot \nu_{ij}^B\Big)\Big] \mapsto \Big[\Big(\mu_{ij}^C, \nu_{ij}^C\Big)\Big]$$

$$= \Big[\Big(\max\{\nu_{ij}^A \cdot \nu_{ij}^B, \mu_{ij}^C\}, \min\{\mu_{ij}^A + \mu_{ij}^B - \mu_{ij}^A \cdot \mu_{ij}^B\}, \nu_{ij}^C\Big)\Big] \quad (3.15)$$

$[A \mapsto C] + [B \mapsto C]$

$= \Big[\Big(\max\{\nu_{ij}^A, \mu_{ij}^C\}, \min\{\mu_{ij}^A, \nu_{ij}^C\}\Big)\Big] + \Big[\Big(\{\max\{\nu_{ij}^B, \mu_{ij}^C\}, \min\{\mu_{ij}^B, \nu_{ij}^C\}\Big)\Big]$

$$= \Big[\Big(\max\{\nu_{ij}^A, \mu_{ij}^C\} + \max\{\mu_{ij}^B, \mu_{ij}^C\} - \max\{\nu_{ij}^A, \mu_{ij}^C\} \cdot \max\{\mu_{ij}^B, \mu_{ij}^C\}, \min\{\mu_{ij}^A, \nu_{ij}^C\} \cdot \min\{\mu_{ij}^B, \nu_{ij}^C\}\Big)\Big] \tag{3.16}$$

From (3.15) and (3.16), it follows that
$[A + B] \mapsto C \supseteq [A \mapsto C] + [B \mapsto C]$.

Similarly, we can prove (ii), (iii), (iv), (v). □

4. Conclusions

We have defined some modal operators and an implication for Pythagorean fuzzy soft matrices and discussed their algebraic properties with some existing operators of Pythagorean fuzzy soft matrices ($\oplus$ and $\otimes$).

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