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APPROXIMATION BY LUPAŞ-STANCU OPERATORS BASED ON q-INTEGER

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Abstract. The purpose of this paper is to introduce a Stancu type generalization of the Lupaş operators based on the q-integers. We investigate the rate of convergence of operators by mean of modulus of continuity and functions belonging to the Lipschitz class as well as Peetre's K-functional.

Keywords: Lupaş-Stancu operators; q analogue; Peetre's K-functional; Korovkin's type theorem; Convergence theorems.

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1. Introduction

Approximation theory basically deals with approximation of functions by simpler functions or more easily calculated functions. Broadly it is divided into theoretical and constructive approximation. In 1912 S.N. Bernstein [4] was the first to construct sequence of positive linear operators as follows:

$$B_m(f;u) = \sum_{l=0}^{m} \binom{m}{l} u^l (1-u)^{m-l} f\left(\frac{l}{m}\right), \tag{1}$$

to provide a constructive proof of well known Weierstrass approximation theorem [31] using probabilistic approach. Here $C[0,1]$ denotes the set of all continuous functions on $[0,1]$ which is equipped with sup-norm $\|.\|_{C[0,1]}$. He showed that if $f \in C[0,1]$, then $B_m(f;u)$ converges to $f(u)$ uniformly on $[0,1]$. For some recent work on related operators, we refer to [1, 2, 18, 19, 20, 22, 23, 24, 26, 29].

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Before proceeding further, let us recall some basic definitions and notations of quantum calculus [9]. For any fixed real number $q > 0$ satisfying the conditions $0 < q < 1$, the q-integer $[l]_q$, for $l \in \mathbb{N}$ is defined as,

$$[l]_q := \begin{cases} \frac{(1-q^l)}{(1-q)}, & q \neq 1 \\ l, & q = 1, \end{cases}$$

and the q-factorial by

$$[l]_q! := \begin{cases} [l]_q[l-1]_q...[1]_q, & l \geq 1 \\ l, & l = 0. \end{cases}$$

The q-Binomial expansion is

$$(u+y)_q^m := (u+y)(u+qy)(u+q^2y)\cdots(u+q^{m-1}y),$$

and the q-binomial coefficients are as follows:

$$\left[\begin{array}{c} m \\ l \end{array}\right]_q := \frac{[m]_q!}{[l]_q![m-l]_q!}.$$

After development of q-calculus, Lupaş [15] in 1987 introduced the q-Lupaş operator (rational) as follows:

$$L_{m,q}(f;u) = \sum_{l=0}^{m} \frac{f\left(\frac{[l]_q}{[m]_q}\right) \left[\begin{array}{c} m \\ l \end{array}\right]_q q^{\frac{l(l-1)}{2}} \; u^l \; (1-u)^{m-l}}{\prod_{j=1}^{m} \{(1-u) + q^{j-1}u\}}, \tag{2}$$

and studied its approximation properties.
Similarly, Phillips [25] in 1996 constructed another q-analogue of Bernstein operators (polynomials) as follows:

$$B_{m,q}(f;u) = \sum_{l=0}^{m} \left[\begin{array}{c} m \\ l \end{array}\right]_q u^l \prod_{s=0}^{m-l-1} (1-q^s u) \; f\left(\frac{[l]_q}{[m]_q}\right), \; u \in [0,1] \tag{3}$$

where $B_{m,q} : C[0,1] \to C[0,1]$ defined for any $m \in \mathbb{N}$ and any function $f \in C[0,1]$.

Basis of these operators have been used in computer aided geometric design(CAGD) to study curves and surfaces. Then onward it became an active area of research in approximation theory as well as CAGD[10, 11, 12].
In 1968 Stancu [30] showed that the Bernstein-Stancu polynomials

$$\big(P_m^{(\gamma,\delta)} f\big)(u) = \sum_{l=0}^{m} \binom{m}{l} u^l (1-u)^{m-l} f\left(\frac{l+\gamma}{m+\delta}\right), \tag{4}$$

converge to continuous function $f(u)$ uniformly in [0,1] for each real γ, δ such that $0 \leq \gamma \leq \delta$. For more literature on Stancu operator one can refers [3, 5, 17, 21, 27]

A. Lupaş [16] introduced the linear positive operators at the International Dortmund Meeting held in Witten (Germany, March, 1995) as follows:

$$L_m(f;u) = (1-a)^{-mu} \sum_{l=0}^{\infty} \frac{(mu)_l a^l}{l!} f\left(\frac{l}{m}\right), \; u \geq 0, \tag{5}$$

with $f:[0,\infty)\to\mathbb{R}$. If we impose $L_m(u)=u$, we get $a=\frac{1}{2}$. Thus operators (5) becomes

$$L_m(f;u)=2^{-mu}\sum_{l=0}^{\infty}\frac{(mu)_l}{l!2^l}f\left(\frac{l}{m}\right),\ u\geq 0, \tag{6}$$

where $(mu)_l$ is the rising factorial defined as:

$$(mu)_0=1,\ (mu)_l=mu(mu+1)(mu+2)\cdots(mu+l-1),\ l\geq 0.$$

Recently the q-analogue of Lupaş operators (6) is defined in [28] as:

$$\mathcal{L}_m^q(f;u)=2^{-[m]_q u}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]!2^l}f\left(\frac{[l]_q}{[m]_q}\right),\ u\geq 0. \tag{7}$$

Motivated by the above mentioned work, we introduce the Stancu type generalization of Lupaş Operators based on q-integer are as follows:

Definition 1.1. *Let $0<q<1$, $0\leq\gamma\leq\delta$ and $m\in\mathbb{N}$. For $f:[0,\infty)\to\mathbb{R}$ we define q-Lupaş-Stancu operators as*

$$\mathcal{L}_{m,q}^{\gamma,\delta}(f;u)=2^{-[m]_q u}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}f\left(\frac{[l]_q+\gamma}{[m]_q+\delta}\right),\ u\geq 0. \tag{8}$$

The operators (8) are linear and positive. For $\gamma=\delta=0$, the operators (8) turn out to be q-Lupaş operators defined in (7). Next, we prove some auxiliary results for (8).

Lemma 1.1. *Let $t^m(u)=u^m$, $(m=0,1,2)$. The following equalities are true:*

(i) $\mathcal{L}_{m,q}^{\gamma,\delta}(1;u)=1$,
(ii) $\mathcal{L}_{m,q}^{\gamma,\delta}(t;u)=\frac{[m]_q}{[m]_q+\delta}u+\frac{\gamma}{[m]_q+\delta}$,
(iii) $\mathcal{L}_{m,q}^{\gamma,\delta}(t^2;u)=\frac{q[m]_q^2}{([m]_q+\delta)^2}u^2+\frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u+\frac{\gamma}{([m]_q+\delta)^2}$.

Proof. By using Lemma 1 of [28] and some basic calculations We have

(i)

$$\mathcal{L}_{m,q}^{\gamma,\delta}(1;u)=2^{-[m]_q u}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}=1.$$

(ii)

$$\begin{aligned}\mathcal{L}_{m,q}^{\gamma,\delta}(t;u) &= 2^{-[m]_q u}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}\frac{[l]_q+\gamma}{[m]_q+\delta}\\ &= \frac{2^{-[m]_q u}[m]_q}{[m]_q+\delta}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}\frac{[l]_q}{[m]_q}\\ &= \frac{[m]_q}{[m]_q+\delta}u+\frac{\gamma}{[m]_q+\delta}.\end{aligned}$$

(iii)

$$\begin{aligned}
\mathcal{L}_{m,q}^{\gamma,\delta}(t^2;u) &= 2^{-[m]_q u}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}\frac{([l]_q+\gamma)^2}{([m]_q+\delta)^2}\\
&= \frac{2^{-[m]_q u}}{([m]_q+\delta)^2}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}([l]_q+\gamma)^2\\
&= \frac{2^{-[m]_q u}}{([m]_q+\delta)^2}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}([l]_q^2+2\gamma[l]_q+\gamma^2)\\
&= \frac{2^{-[m]_q u}[m]_q^2}{([m]_q+\delta)^2}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}\frac{[l]_q^2}{[m]_q^2}\\
&+ \frac{2^{-[m]_q u}[m]_q}{([m]_q+\delta)^2}2\gamma\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l}\frac{[l]_q}{[m]_q}\\
&+ \frac{2^{-[m]_q u}[m]_q}{([m]_q+\delta)^2}\gamma^2\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]_q!2^l},
\end{aligned}$$

after solving we get,

$$\mathcal{L}_{m,q}^{\gamma,\delta}(t^2;u) = \frac{q[m]_q^2}{([m]_q+\delta)^2}u^2 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u + \frac{\gamma^2}{([m]_q+\delta)^2}.$$

□

Corollary 1.1. *Using Lemma 1.1, we get the following central moments.*
$\mathcal{L}_{m,q}^{\gamma,\delta}(t-u;u) \ = 0$
$\mathcal{L}_{m,q}^{\gamma,\delta}((t-u)^2;u) = \frac{q[m]_q^2}{([m]_q+\delta)^2}u^2 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u + \frac{\gamma^2}{([m]_q+\delta)^2} - \frac{2[m]_q}{[m]_q+\delta}u^2 - \frac{2\gamma}{[m]_q+\delta}u + u^2$
$= \rho_m(u)$ *(say).*

Theorem 1.1. *Let $f \in C_B[0,\infty)$ and $q_m \in (0,1)$, such that $q_m \to 1$, as $m \to \infty$. Then for each $u \in [0,\infty)$ we have*

$$\lim_{m\to\infty}\mathcal{L}_{m,q_m}^{\gamma,\delta}(f;u) = f(u).$$

Proof. By Korovkin's theorem it is enough to show that

$$\lim_{m\to\infty}\mathcal{L}_{m,q_m}^{\gamma,\delta}(t^m;u) = u^m,\ m = 0,1,2.$$

By Lemma 1.1, it is clear that

$$\lim_{m\to\infty}\mathcal{L}_{m,q_m}^{\gamma,\delta}(1;u) = 1$$

$$\lim_{m\to\infty}\mathcal{L}_{m,q_m}^{\gamma,\delta}(t;u) = u$$

and

$$\lim_{m\to\infty}\mathcal{L}_{m,q_m}^{\gamma,\delta}(t^2;u) = \lim_{m\to\infty}\left[\frac{q_m[m]_{q_m}^2}{([m]_{q_m}+\delta)^2}u^2 + \frac{[m]_{q_m}([2]_{q_m}+2\gamma)}{([m]_{q_m}+\delta)^2}u + \frac{\gamma^2}{([m]_{q_m}+\delta)^2}\right]$$

$$\lim_{m\to\infty} \mathcal{L}^{\gamma,\delta}_{m,q_m}(t^2;u) = u^2.$$

This completes the proof. □

2. Direct results

Let $C_B[0,\infty)$ be the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\le u<\infty} | f(u) | .$$

Let us consider the K-functional as:

$$K_2(f,\rho) = \inf_{s\in W^2}\{\| f-s \| + \rho \| s^{''} \|\},$$

where $\rho>0$ and $W^2=\{s\in C_B[0,\infty): s^{'}, s^{''}\in C_B[0,\infty)\}$.
Then as in ([6], p. 177, Theorem 2.4), there euists an absolute constant $C>0$ such that

$$K_2(f,\rho)\le C\omega_2(f,\sqrt{\rho}). \tag{9}$$

Second order modulus of smoothness of $f\in C_B[0,\infty)$ is as follows

$$\omega_2(f,\sqrt{\rho}) = \sup_{0<h\le\sqrt{\rho}} \sup_{u\in[0,\infty)} | f(u+2h)-2f(u+h)+f(u) | .$$

The usual modulus of continuity of $f\in C_B[0,\infty)$ is defined by

$$\omega(f,\rho) = \sup_{0<h\le\rho} \sup_{u\in[0,\infty)} | f(u+h)-f(u) | .$$

Theorem 2.1. *Let $f\in C_B[0,\infty)$ and $q\in(0,1)$. Then for every $u\in[0,\infty)$ we have*

$$| \mathcal{L}^{\gamma,\delta}_{m,q}(f;u) - f(u) | \le C\omega_2(f;\rho_m(u)),$$

where

$$\rho^2_m(u) = \frac{q[m]^2_q}{([m]_q+\delta)^2}u^2 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u + \frac{\gamma^2}{([m]_q+\delta)^2} - \frac{2[m]_q}{[m]_q+\delta}u^2 - \frac{2\gamma}{[m]_q+\delta}u + u^2.$$

Proof. Let $s\in\mathcal{W}^2$. Then from Taylor's expansion, we get

$$s(t) = s(u) + s'(u)(t-u) + \int_u^t (t-u)s''(u)\mathrm{d}u,\ t\in[0,\mathcal{A}],\ \mathcal{A}>0.$$

Now by Corollary 1.1, we have

$$\mathcal{L}^{\gamma,\delta}_{m,q}(s;u) = s(u) + \mathcal{L}^{\gamma,\delta}_{m,q}\left(\int_u^t (t-u)s''(u)\mathrm{d}u;u\right).$$

$$\begin{aligned} |\mathcal{L}^{\gamma,\delta}_{m,q}(s;u)(s;u) - s(u)| &\le \mathcal{L}^{\gamma,\delta}_{m,q}\left(\left|\int_u^t |(t-u)| \, | s''(u) | \, \mathrm{d}u;u\right|\right) \\ &\le \mathcal{L}^{\gamma,\delta}_{m,q}\left((t-u)^2;u\right) \| s'' \|, \end{aligned}$$

hence we get

$$\begin{aligned} &\mathcal{L}^{\gamma,\delta}_{m,q}(s;u(s;u) - s(u)| \\ \le\ & \| s'' \| \left(\frac{q[m]^2_q}{([m]_q+\delta)^2}u^2 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u + \frac{\gamma^2}{([m]_q+\delta)^2} - \frac{2[m]_q}{[m]_q+\delta}u^2 - \frac{2\gamma}{[m]_q+\delta}u + u^2\right). \end{aligned}$$

By Lemma 1.1, we have

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u)| \leq 2^{-[m]_q u} \sum_{l=0}^{\infty} \frac{([m]_q u)_l}{[l]_q! 2^l} \left| f\left(\frac{[l]_q+\gamma}{[m]_q+\delta}\right) \right| \leq \| f \| .$$

Thus, we have

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u)| \leq | \mathcal{L}_{m,q}^{\gamma,\delta}((f-s);u) - (f-s)(u) | + |\mathcal{L}_{m,q}^{\gamma,\delta}(s;u) - s(u)|.$$

After substituting all values, we get

$$\begin{aligned} |\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| &\leq \| f-s \| \\ + \quad \| s'' \| &\left(\frac{q[m]_q^2}{([m]_q+\delta)^2}u^2 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u + \frac{\gamma^2}{([m]_q+\delta)^2} - \frac{2[m]_q}{[m]_q+\delta}u^2 - \frac{2\gamma}{[m]_q+\delta}u + u^2 \right). \end{aligned}$$

By taking the infimum on the right hand side over all $s \in \mathcal{W}^2$, we get

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{C} K_2\left(f, \rho_m^2(u)\right).$$

By using the property of K-functional, we have

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{C}\omega_2\left(f, \rho_m(u)\right).$$

This completes the proof. □

3. Pointwise estimates

Theorem 3.1. *Let $0 < \alpha \leq 1$ and $\c{E}$ be any bounded subset of the interval $[0,\infty)$. If $f \in C_B[0,\infty)$, is locally Lip(α), i.e., the condition*

$$|f(v) - f(u)| \leq Ł|v-u|^{\alpha}, \ v \in \c{E} \text{ and } u \in [0,\infty) \tag{10}$$

holds, then, for each $u \in [0,\infty)$, we have

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| \leq Ł\left\{\rho_m(u)^{\frac{\alpha}{2}} + 2(d(u,\c{E}))^{\alpha}\right\}, \ u \in [0,\infty)$$

where Ł is a constant depending on α and f and $d(u;\c{E})$ is the distance between u and $\c{E}$ defined by

$$d(u,\c{E}) = inf\{|t-u|; t \in E\} \text{ and } \delta_m(u) = \mathcal{L}_{m,q}^{\gamma,\delta}((t-u)^2;u).$$

Proof. Let $\overline{\c{E}}$ be the closure of $\c{E}$ in $[0,1)$. Then, there exists a point $t_0 \in \overline{\c{E}}$ such that $d(u,\c{E}) = |u-t_0|$.
Using the triangle inequality, we have

$$|f(t) - f(u)| \leq |f(t) - f(t_0)| + |f(t_0) - f(u)|.$$

By using (10) we get,

$$\begin{aligned} |\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| &\leq \mathcal{L}_{m,q}^{\gamma,\delta}(|f(t)-f(t_0)|;u) + \mathcal{L}_{m,q}^{\gamma,\delta}(|f(u)-f(t_0)|;u) \\ &\leq Ł\left\{\mathcal{L}_{m,q}^{\gamma,\delta}(|t-t_0|^{\alpha};u) + (|u-t_0|^{\alpha};u) + |u-t_0|^{\alpha}\right\} \\ &\leq Ł\left\{\mathcal{L}_{m,q}^{\gamma,\delta}(|t-u|^{\alpha};u) + 2|u-t_0|^{\alpha}\right\}. \end{aligned}$$

By choosing $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get $\frac{1}{p} + \frac{1}{q} = 1$. Then by using Hölder's inequality we get

$$\begin{aligned} |\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| &\leq Ł\left\{\mathcal{L}_{m,q}^{\gamma,\delta}\left(|t-u|^{\alpha p};u\right)^{\frac{1}{p}}\left[\mathcal{L}_{m,q}^{\gamma,\delta}(1^q;u)\right]^{\frac{1}{q}} + 2(d(u,Ȩ))^{\alpha}\right\} \\ &\leq Ł\left\{\mathcal{L}_{m,q}^{\gamma,\delta}\left(((t-u)^2;u)\right)^{\frac{\alpha}{2}} + 2(d(u,Ȩ))^{\alpha}\right\} \\ &\leq Ł\left\{\rho_m(u)^{\frac{\alpha}{2}} + 2(d(u,Ȩ))^{\alpha}\right\}. \end{aligned}$$

Hence the proof is completed. □

Now, we recall local approximation in terms of α order Lipschitz-type maximal function given by Lenze [14] as

$$\widetilde{\omega}_{\alpha}(f;u) = \sup_{t\neq u, t\in(0,\infty)} \frac{|f(t)-f(u)|}{|t-u|^{\alpha}},\ u \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{11}$$

Then we get the next result

Theorem 3.2. *Let $f \in C_B[0,\infty)$ and $\alpha \in (0,1]$. Then, for all $u \in [0,\infty)$, we have*

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| \leq \widetilde{\omega}_{\alpha}(f;u)\Big(\rho_m(u)\Big)^{\frac{\alpha}{2}},$$

where $\rho_m(u)$ is defined in Corollary 1.1.

Proof. We know that

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| \leq \mathcal{L}_{m,q}^{\gamma,\delta}(|f(t)-f(u)|;u).$$

From equation (11), we have

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| \leq \widetilde{\omega}_{\alpha}(f;u)\mathcal{L}_{m,q}^{\gamma,\delta}(|t-u|^{\alpha};u).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$|\mathcal{L}_{m,q}^{\gamma,\delta}(f;u) - f(u)| \leq \widetilde{\omega}_{\alpha}(f;u)\big(\mathcal{L}_{m,q}^{\gamma,\delta}(|t-u|^2;u)\big)^{\frac{\alpha}{2}},$$

which proves the desired result. □

4. Weighted approximation by $\mathcal{L}_{m,q}^{\gamma,\delta}$

In this section we shall discuss weighted approximation theorems for the operators $\mathcal{L}_{m,q}^{\gamma,\delta}$ on the interval $[0,\infty)$.

Theorem 4.1. [13] *Let (T_m) be the sequence of linear positive operators from $C_{u^2}[0,\infty)$ to $B_{u^2}[0,\infty)$ satisfy*

$$\lim_{m\to\infty} \| T_m\kappa_i - \kappa_i \|_{u^2} = 0, \qquad i = 0,1,2.$$

Then for any function $f \in C_{u^2}^*[0,\infty)$

$$\lim_{m\to\infty} \| T_m f - f \|_{u^2} = 0.$$

Theorem 4.2. *Let q_m be a sequence in $(0,1)$, such that $q_m \to 1$, as $m \to \infty$.. Then for each function $f \in C_{u^2}^*[0,\infty)$, we get*

$$\lim_{m\to\infty} \| \mathcal{L}_{m,q_m}^{\gamma,\delta} f - f \|_{u^2} = 0.$$

Proof. By Theorem 4.1, it is enough to show

$$\lim_{m\to\infty} \| \mathcal{L}^{\gamma,\delta}_{m,q_m}\kappa_i - \kappa_i \|_{u^2} \;=\; 0, \qquad i = 0,1,2. \tag{12}$$

By Lemma 1.1 (i) and (ii), it is clear that

$$\begin{aligned} \| \mathcal{L}^{\gamma,\delta}_{m,q_m}(1;u) - 1 \|_{u^2} &= 0 \\ \| \mathcal{L}^{\gamma,\delta}_{m,q_m}(t;u) - u \|_{u^2} &= \sup_{u\in[0,\infty)} \frac{\left(\frac{[m]_q}{([m]_q+\delta)} - 1\right)u + \frac{\gamma}{[m]_q+\delta}}{1+u^2} \\ &\to 0, as\ m \to \infty. \end{aligned} \tag{13}$$

and by Lemma 1.1 (iii), we have

$$\begin{aligned} \| \mathcal{L}^{\gamma,\delta}_{m,q_m}(t^2;u) - u^2 \|_2 &= \sup_{u\in[0,\infty)} \frac{\left(\frac{q[m]_q^2}{([m]_q+\delta)^2} - 1\right)u^2 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u + \frac{\gamma^2}{([m]_q+\delta)^2}}{1+u^2} \\ &\le \left(\frac{q[m]_q^2}{([m]_q+\delta)^2} - 1 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2} + \frac{\gamma^2}{([m]_q+\delta)^2}\right). \end{aligned}$$

Last inequality means that (12) holds for $i = 2$. By Theorem 4.1, the proof is completed. □

Theorem 4.3. *Let q_m be a sequence in $(0,1)$, such that $q_m \to 1$, as $m \to \infty$. Let $f \in C^*_{u^2}[0,\infty)$, and its modulus of continuity $\omega_{d+1}(f;\rho)$ be defined on $[0, d+1] \subset [0,\infty)$. Then, we have*

$$|\mathcal{L}^{\gamma,\delta}_{m,q}(f;u) - f(u)\|_{C[0,d]} \le 6M_f(1+d^2)\rho_m(d) + 2\omega_{d+1}(f;\sqrt{\rho_m(d)}),$$

where $\rho_m(d) = \mathcal{L}^{\gamma,\delta}_{m,q}((t-u)^2;u) = \frac{q[m]_q^2}{([m]_q+\delta)^2}u^2 + \frac{[m]_q([2]_q+2\gamma)}{([m]_q+\delta)^2}u + \frac{\gamma^2}{([m]_q+\delta)^2} - \frac{2[m]_q}{[m]_q+\delta}u^2 - \frac{2\gamma}{[m]_q+\delta}u + u^2$.

Proof. From ([8] p. 378), for $u \in [0,d]$ and $t \in [0,\infty)$, we have

$$|f(t) - f(u)| \le 6M_f(1+d^2)(t-u)^2 + \left(1 + \frac{|t-u|}{\rho}\right)\omega_{d+1}(f;\rho).$$

Applying $\mathcal{L}^{\gamma,\delta}_{m,q}$ both the sides, we have

$$|\mathcal{L}^{\gamma,\delta}_{m,q}(f;u) - f(u)| \le 6M_f(1+d^2)\mathcal{L}^{\gamma,\delta}_{m,q}((t-u)^2;u) + \left(1 + \frac{\mathcal{L}^{\gamma,\delta}_{m,q}(|t-u|;u)}{\rho}\right)\omega_{d+1}(f;\rho).$$

Applying Cauchy-Schwarz inequality,for $u \in [0,d]$ and $t \ge 0$, we get

$$\begin{aligned} |\mathcal{L}^{\gamma,\delta}_{m,q}(f;u) - f(u)| &\le \mathcal{L}^{\gamma,\delta}_{m,q}\left(|(f;u) - f(u)|;u\right) \\ &\le 6M_f(1+d^2)\mathcal{L}^{\gamma,\delta}_{m,q}((t-u)^2;u) \\ &+ \omega_{d+1}(f;\rho)\left(1 + \frac{1}{\rho}\mathcal{L}^{\gamma,\delta}_{m,q}\left((t-u)^2;u\right)^{\frac{1}{2}}\right). \end{aligned}$$

Thus, from Lemma 1.1, for $u \in [0,d]$, we get

$$|\mathcal{L}^{\gamma,\delta}_{m,q}(f;u) - f(u)| \le 6M_f(1+d^2)\rho_m(d) + \omega_{d+1}(f;\rho)\left(1 + \frac{\sqrt{\rho_m(d)}}{\rho}\right).$$

By Choosing $\rho = \sqrt{\rho_m(d)}$, we get the required result. □

Now, we prove a theorem to approximate all functions in $C^*_{u^2}$ Such type of results are given in [7] for locally integrable functions.

Theorem 4.4. *Let q_m be a sequence in $(0,1)$, such that $q_m \to 1$, as $m \to \infty$. Then for each function $f \in C^*_{u^2}[0,\infty)$, and $\alpha > 1$*

$$\lim_{m\to\infty} \sup_{u\in[0,\infty)} \frac{| \mathcal{L}^{\gamma,\delta}_{m,q_m}(f;u) - f(u) |}{(1+u^2)^\alpha} = 0.$$

Proof. Let for any fixed $u_0 > 0$,

$$\begin{aligned} \sup_{u\in[0,\infty)} \frac{| \mathcal{L}^{\gamma,\delta}_{m,q_m}(f;u) - f(u) |}{(1+u^2)^\alpha} &\le \sup_{u\le u_0} \frac{| \mathcal{L}^{\gamma,\delta}_{m,q_m}(f;u) - f(u) |}{(1+u^2)^\alpha} + \sup_{u\ge u_0} \frac{| \mathcal{L}^{\gamma,\delta}_{m,q_m}(f;u) - f(u) |}{(1+u^2)^\alpha} \\ &\le \| \mathcal{L}^{\gamma,\delta}_{m,q_m}(f) - f \|_{[c_0,u_0]} + \| f \|_{u^2} \sup_{u\le u_0} \frac{| \mathcal{L}^{\gamma,\delta}_{m,q_m}(1+t^2;u) |}{(1+u^2)^\alpha} \\ &+ \sup_{u\ge u_0} \frac{| f(u) |}{(1+u^2)^\alpha}. \end{aligned} \tag{14}$$

Since, $| f(u) | \le M_f(1+u^2)$ we have,

$$\sup_{u\ge u_0} \frac{| f(u) |}{(1+u^2)^\alpha} \le \sup_{u\ge u_0} \frac{M_f}{(1+u^2)^{\alpha-1}} \le \frac{M_f}{(1+u^2)^{\alpha-1}}.$$

Let $\epsilon > 0$, and let us choose u_0 large then we have

$$\frac{M_f}{(1+{u_0}^2)^{\alpha-1}} < \frac{\epsilon}{3} \tag{15}$$

and in view of (1.1), we get,

$$\begin{aligned} \| f \|_{u^2} \lim_{m\to\infty} \frac{| \mathcal{L}^{\gamma,\delta}_{m,q_m}(1+t^2;u) |}{(1+u^2)^\alpha} &= \| f \|_{u^2} \frac{1+u^2}{(1+u^2)^\alpha} \\ &\le \frac{\| f \|_{u^2}}{(1+u^2)^{\alpha-1}} \\ &\le \frac{\| f \|_{u^2}}{(1+{u_0}^2)^{\alpha-1}} \\ &\le \frac{\epsilon}{3}. \end{aligned} \tag{16}$$

By using Theorem 4.3, the first term of inequality (14) becomes

$$\| \mathcal{L}^{\gamma,\delta}_{m,q_m}(f) - f \|_{[c_0,u_0]} < \frac{\epsilon}{3}, \text{as } m \to \infty. \tag{17}$$

Hence we get the required proof by combining (15)-(17)

$$\sup_{u\in[0,\infty)} \frac{| \mathcal{L}^{\gamma,\delta}_{m,q_m}(f;u) - f(u) |}{(1+u^2)^\alpha} < \epsilon.$$

□

5. Conclusion

Thus it can be concluded that the parameters γ, δ, and q, will provide more modeling flexibility for approximation of functions and basis of these operators can be used to draw curves and surfaces in CAGD.

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