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ON A FRACTIONAL INTEGRAL OPERATOR CONTAINING ψ-GENERALIZED MITTAG LEFFLER FUNCTION IN ITS KERNEL AND PROPERTIES

MEENAKSHI SINGHAL[1], EKTA MITTAL[2] §

Abstract. This paper is devoted to the study of ψ-generalized Mittag-Leffler function associated with ψ-generalized beta function. We also obtain integral representations and other useful properties of it for example, Mellin transforms, recurrence relations etc. Further we develop derivative formulas and some fractional differ-integral properties for this ψ-generalized Mittag-Leffler function.
Other than this we also establish fractional integral operator containing ψ-generalized Mittag-Leffler function as its kernel and obtain some associated properties.

Keywords: ψ-beta function,ψ-hypergeometric function, ψ-generalized Mittag-Leffler function, Mellin transform, generalized fractional integral operator.

AMS Subject Classification: 26A33,33B15,33C05,33C65

1. Introduction

The Mittag-Leffler function occurs in the solution of fractional order differential and integral equations, and also in the investigations of the fractional generalization of the kinetic equation,Levy flights, random walks,super diffusive transport and in the study of complex systems etc.

The study of the Mittag-Leffler function and its various generalizations has become popular because of its applications in the Fractional Calculus.
Mittag-Leffler has introduced a function $E_\sigma(z)$

$$E_\sigma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\sigma\, k+1)} \qquad (z \in C; R(\sigma) > 0), \tag{1}$$

in the year 1903, known as Mittag-Leffler function.This function is a generalization of the exponential series.Since for $\sigma = 1$,we have the exponential function.

[1] Department of Mathematics, IIS (Deemed to be university), Jaipur, India.
e-mail: meenakshi.singhal60@gmail.com; ORCID: https://orcid.org/0000-0003-2715-4723 .
e-mail: ekta.mittal@iisuniv.ac.in; ORCID: https://orcid.org/0000-0001-7235-1852.
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Wiman introduced a generalization of the Mittag-Leffler function(1).

$$E_{\sigma,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\sigma\ k+\eta)}, \qquad (z,\eta \in C; R(\sigma) > 0). \tag{2}$$

Afterthat Prabhakar[14] investigated the following generalization of the function $E_{\sigma,\eta}(z)$by

$$E^{\delta}_{\sigma,\eta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(\sigma\ k+\eta)} \frac{z^k}{k!}, \qquad (z,\eta,\delta,\sigma \in C; R(\sigma) > 0), \tag{3}$$

where $(\lambda)_n$ denotes the Pocchammer symbol defined (for $\lambda \in C$) by:

$$(\lambda)_n = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)...(\lambda+n-1) & (n \in N). \end{cases} \tag{4}$$

Shukla and Prajapati[16] introduced a generalized Mittag Leffler function $E^{\delta,c}_{\sigma,\eta}(z)$ by

$$E^{\delta,c}_{\sigma,\eta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_{kc}}{\Gamma(\sigma\ k+\eta)} \frac{z^k}{k!}, \tag{5}$$

$$(z,\eta,\delta,\sigma \in C; min\{R(\sigma), R(\eta), R(\delta)\} > 0; c \in (0,1) \cup N).$$

Özarslan and Yilmaz[13] introduced the following extended Mittag-Leffler function $E^{\delta,c}_{\sigma,\eta}(z;p)$ by

$$E^{\delta,c}_{\sigma,\eta}(z;p) = \sum_{k=0}^{\infty} \frac{\mathcal{B}_p(\delta+k, c-\delta)}{\mathcal{B}(\delta, c-\delta)} \frac{(c)_k}{\Gamma(\sigma\ n+\eta)} \frac{z^k}{k!}, \tag{6}$$

$$(z,\eta,\delta,\sigma \in C; R(c) > R(\delta) > 0, R(\sigma) > 0, R(\eta) > 0),$$

where $\mathcal{B}_p(x,y)$is extended beta function defined by[4]

$$\mathcal{B}_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{\frac{-p}{t(1-t)}} dt, \tag{7}$$

where $Re(x) > 0, Re(y) > 0, Re(p) > 0$.
For $p = 0$ it gives the familiar beta function.
Mittal. et al.[11] presented a new extended Mittag-Leffler function $E^{\delta,\rho,c}_{\sigma,\eta}(z;p)$ by

$$E^{\delta,\rho,c}_{\sigma,\eta}(z;p) = \sum_{k=0}^{\infty} \frac{\mathcal{B}_p(\delta+k\rho, c-\delta)}{\mathcal{B}(\delta, c-\delta)} \frac{(c)_{k\rho}}{\Gamma(\sigma\ k+\eta)} \frac{z^k}{k!}, \tag{8}$$

$$(z,\eta,\delta,\sigma \in C; R(c) > R(\delta) > 0, R(\sigma) > 0, R(\eta) > 0, \rho > 0)$$

where $\mathcal{B}_p(x,y)$is extended beta function defined by[4].
Many more generalizations of Mittag -Leffler function has been presented and investigated by many authors [2, 14, 12, 7].
In 2018 Enes Ata [3] introduced the ψ- generalized beta functions as

$${}^{\psi}\Gamma^{(\alpha,\beta)}_{p}(x) = \int_0^{\infty} u^{x-1} {}_1\Psi_1\Big(\alpha, \beta; -u - \frac{p}{u}\Big) du. \tag{9}$$

where $\alpha, \beta \in C; \mathcal{R}(x) > 0, \mathcal{R}(\alpha) > 0, \mathcal{R}(\beta) > 1, p > 0$, and ${}_1\Psi_1(\alpha,\beta,z)$ is the Wright function defined by

$${}_1\Psi_1(\alpha,\beta,z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}$$

and

$$ {}^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(x,y) = \int_0^{\infty} t^{x-1}(1-t)^{y-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{t(1-t)}\Big)dt, \tag{10} $$

$$ (Re(x) > 0, Re(y) > 0, Re(\alpha) > 0, Re(\beta) > 1, Re(p) > 0). $$

It was observe that
For $p \neq 0$, $\alpha = 0$ and β=1,then (10) gives (7)
For $p = 0$ and β=1 then (10) gives the original beta function.

Here, we define ψ-generalized Mittag-Leffler function such as

$$ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{{}^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(\eta+k, c-\eta)}{\mathcal{B}(\eta, c-\eta)} \frac{(c)_k}{\Gamma(\mu\, k+\sigma)} \frac{z^k}{k!}, \tag{11} $$

$$ (z,\eta,\mu,\sigma,c \in C; R(c) > R(\eta) > 0, R(\sigma) > 0, R(\mu) > 0, p \geq 0, Re(\alpha) > 0, Re(\beta) > 1) $$

where ${}^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(x,y)$ is ψ-generalized beta function defined by(10).
For $p \neq 0$ and $\alpha = 0$ and β=1 in (11) then it reduces to (6).
For $p = 0$ and β=1 in (11) then it reduces to (3).

2. Properties of the ψ-generalized Mittag-Leffler function

Theorem 2.1. *Let $p \geq 0$, and $c,\mu,\sigma,\eta,\alpha,\beta \in C$ with $R(c) > R(\eta) > 0$ and $R(\mu) > 0, Re(\alpha) > 0, Re(\beta) > 1$. Then*

$$ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z) = \frac{1}{\mathcal{B}(\eta,c-\eta)} \int_0^1 u^{\eta-1}(1-u)^{c-\eta-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)E_{\mu,\sigma}^{c}(uz)du. \tag{12} $$

Proof. using equation(10) in equation (11), we have

$$ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z) = \sum_{k=0}^{\infty}\Big\{\int_0^1 u^{\eta+k-1}(1-u)^{c-\eta-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)du\Big\}\frac{(c)_k}{\mathcal{B}(\eta,c-\eta)}\frac{z^k}{\Gamma(\mu k+\sigma)k!}, $$

changing order of summation and integration, we obtain

$$ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z) = \int_0^1 u^{\eta-1}(1-u)^{c-\eta-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)\sum_{k=0}^{\infty}\frac{(c)_k}{\mathcal{B}(\eta,c-\eta)}\frac{(uz)^k}{\Gamma(\mu k+\sigma)k!}du, $$

after using(3) in the above equation, we get the required result. □

Corollary 2.1. *Let $p \geq 0$, and $Re(\alpha) > 0, Re(\beta) > 1; c,\mu,\sigma,\eta \in C$ with $R(c) > R(\eta) > 0$ and $R(\mu) > 0$ Then substituting $u = \frac{w}{1+w}$ in Theorem(2.1), we get*

$$ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z) = \frac{1}{\mathcal{B}(\eta,c-\eta)} \int_0^{\infty} \frac{w^{\eta-1}}{(w+1)^c}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p(1+w)^2}{w}\Big)E_{\mu,\sigma}^{c}\Big(\frac{wz}{1+w}\Big)dw. \tag{13} $$

Corollary 2.2. *Let $p \geq 0$, and $Re(\alpha) > 0, Re(\beta) > 1; c,\mu,\sigma,\eta \in C$ with $R(c) > R(\eta) > 0$ and $R(\mu) > 0$. Then substituting $u = Sin^2\theta$ in Theorem(2.1), we get*

$$ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z) = \frac{2}{\mathcal{B}(\eta,c-\eta)} \int_0^{\pi/2} sin^{2\eta-1}\theta\, cos^{2c-2\eta-1}\theta\, {}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{sin^2\theta\, cos^2\theta}\Big)E_{\mu,\sigma}^{c}(z sin^2\theta)d\theta. \tag{14} $$

Kurulay and Bayram[8] defined the following identity

$$E^{c}_{\mu,\sigma}(uz) = \sigma\, E^{c}_{\mu,\sigma+1}(uz) + \mu\, z \frac{d}{dz} E^{c}_{\mu,\sigma+1}(uz). \tag{15}$$

With the help of this we give the recurrence relation for the newly defined Mittag-Leffler function.

Corollary 2.3. *Let $p \geq 0$, and $Re(\alpha) > 0, Re(\beta) > 1; c, \mu, \sigma, \eta \in C$ with $R(c) > R(\eta) > 0$ and $R(\mu) > 0)$. Then using (15)in (12) we get*

$$^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z) = \sigma\, {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma+1,p}(z) + \mu\, z \frac{d}{dz}\, {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma+1,p}(z). \tag{16}$$

In the next theorem, we apply the Mellin transforms on (11) and obtain the result in terms of Wright hypergeometric function which is defined as

$$\begin{aligned} {}_p\Psi_q(z) &= {}_p\Psi_q\left[\begin{matrix}(\lambda_i, \eta_i)_{1,p} \\ (\mu_i, \zeta_i)_{1,q}\end{matrix} ; z\right] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\lambda_1 + \eta_1 n)\cdots\Gamma(\lambda_p + \eta_p n)}{\Gamma(\mu_1 + \zeta_1 n)\cdots\Gamma(\mu_q + \zeta_q n)} \frac{z^n}{n!}, \end{aligned} \tag{17}$$

where the coefficients $\eta_r (r = 1, ..., p)$ and $\zeta_s (s = 1, ..., q)$ are positive real numbers such that

$$1 + \sum_{s=1}^{q} \zeta_s - \sum_{r=1}^{p} \eta_r \geq 0.$$

Theorem 2.2. *The Mellin transform of the ψ-generalized Mittag-Leffler function is given by*

$$\mathcal{M}\{{}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z); w\} = \frac{\Gamma(w)\Gamma(c + w - \eta)}{\Gamma(\beta - \alpha\, w)\Gamma(\eta)\Gamma(c - \eta)}\, {}_2\Psi_2 \left[\begin{matrix}(c, 1), (\eta + w, 1) \\ (\sigma, \mu), (c + 2w, 1)\end{matrix} ; z\right], \tag{18}$$

$(Re(\mu) > 0, Re(\sigma) > 0, Re(w) > 0, R(c) > R(\eta) > 0$ *and* $p \geq 0), Re(\alpha) > 0, Re(\beta) > 1.$

where ${}_2\psi_2(-)$ is the Wright generalized hypergeometric function.

Proof. By applying Mellin transform of the ψ-generalized Mittag- Leffler function then we get

$$\mathcal{M}\{{}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z); w\} = \int_0^{\infty} p^{w-1}\, {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z) dp, \tag{19}$$

applying (12) in (19), we obtain

$$\begin{aligned} \mathcal{M}\{{}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z); w\} = \frac{1}{\mathcal{B}(\eta, c - \eta)} \int_0^{\infty} p^{w-1} \Big\{ \int_0^1 u^{\eta-1}(1 - u)^{c-\eta-1} {}_1\Psi_1\Big(\alpha, \beta; \frac{-p}{u(1 - u)}\Big)\Big\} \\ \times E^{c}_{\mu,\sigma}(uz) du dp, \end{aligned}$$

interchanging the order of integration

$$\begin{aligned} \mathcal{M}\{{}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z); w\} = \frac{1}{\mathcal{B}(\eta, c - \eta)} \int_0^1 \Big\{ u^{\eta-1}(1 - u)^{c-\eta-1} E^{c}_{\mu,\sigma}(uz)\Big\} \\ \times \int_0^{\infty} p^{w-1} {}_1\Psi_1\Big(\alpha, \beta; \frac{-p}{u(1 - u)}\Big) dp du, \end{aligned}$$

on solving inner integral by substituting $v = \frac{p}{u(1-u)}$ in it, we get

$$\begin{aligned}\int_0^\infty p^{w-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)dp &= \int_0^\infty v^{w-1}u^w\,(1-u)^w{}_1\Psi_1(\alpha,\beta,-v)dv\\ &= u^w\,(1-u)^w\int_0^\infty v^{w-1}{}_1\Psi_1(\alpha,\beta,-v)dv\end{aligned}$$

using the integral representation of ${}_1\Psi_1(\alpha,\beta,-v)$[4]

$$\int_0^\infty p^{w-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)dp = u^w\,(1-u)^w\int_0^\infty v^{w-1}\times\Big[\frac{1}{2\pi\iota}\int_{-\infty}^0 u^{-\beta}\,e^{u-vu^{-\alpha}}du\Big]dv$$

changing the order of integration

$$= u^w\,(1-u)^w\frac{1}{2\pi\iota}\int_{-\infty}^0 u^{-\beta}e^u\,\Big[\int_0^\infty v^{w-1}e^{-vu^{-\alpha}}dv\Big]du$$

Put $vu^{-\alpha} = t$ then $u^{-\alpha}dv = dt$

$$\begin{aligned}\int_0^\infty p^{w-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)dp &= u^w\,(1-u)^w\frac{1}{2\pi\iota}\int_{-\infty}^0 u^{-(\beta-\alpha w)}e^u\,\Big[\int_0^\infty t^{w-1}e^{-t}dt\Big]du\\ &= u^w\,(1-u)^w\Gamma(w)\frac{1}{2\pi\iota}\int_{-\infty}^0 u^{-(\beta-\alpha w)}e^u du\\ &= \frac{u^w\,(1-u)^w\Gamma(w)}{\Gamma(\beta-\alpha\,w)}\end{aligned}$$

hence

$$\begin{aligned}\mathcal{M}\{{}^\psi E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z);w\} &= \frac{\Gamma(w)}{\Gamma(\beta-\alpha\,w)\mathcal{B}(\eta,c-\eta)}\sum_{k=0}^\infty\frac{(c)_k}{\Gamma(\mu\,k+\sigma)}\frac{z^k}{k!}\int_0^1 u^{\eta+k+w-1}(1-u)^{c+w-\eta-1}du\\ &= \frac{\Gamma(w)}{\Gamma(\beta-\alpha\,w)\mathcal{B}(\eta,c-\eta)}\sum_{k=0}^\infty\frac{(c)_k}{\Gamma(\mu\,k+\sigma)}\frac{z^k}{k!}\frac{\Gamma(\eta+k+w)\Gamma(c+w-\eta)}{\Gamma(c+k+2w)}\\ &= \frac{\Gamma(w)\,\Gamma(c+w-\eta)}{\Gamma(\beta-\alpha\,w)\Gamma(\eta)\,\Gamma(c-\eta)}\sum_{k=0}^\infty\frac{\Gamma(c+k)\,\Gamma(\eta+k+w)}{\Gamma(\mu\,k+\sigma)\,\Gamma(c+k+2w)}\frac{z^k}{k!}\\ &= \frac{\Gamma(w)\,\Gamma(c+w-\eta)}{\Gamma(\beta-\alpha\,w)\Gamma(\eta)\,\Gamma(c-\eta)}{}_2\Psi_2\left[\begin{matrix}(c,1),(\eta+w,1)\\(\sigma,\mu),(c+2w,1)\end{matrix};z\right]\end{aligned}$$

which is the required result. □

Corollary 2.4. *put $w=1$ in the Theorem (2.2), we get*

$$\int_0^\infty {}^\psi E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(z)dp = \frac{\Gamma(c+1-\eta)}{\Gamma(\beta-\alpha)\Gamma(\eta)\Gamma(c-\eta)}\,{}_2\Psi_2\left[\begin{matrix}(c,1),(\eta+1,1)\\(\sigma,\mu),(c+2,1)\end{matrix};z\right]. \tag{11}$$

Corollary 2.5. *After applying inverse mellin transform of the equation (18), we get the following integral representation*

$$
{}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z) = \frac{1}{2\pi\iota\,\Gamma(\eta)\Gamma(c-\eta)} \int_{\nu-\iota\infty}^{\nu+\iota\infty} \frac{\Gamma(w)\Gamma(c+w-\eta)}{\Gamma(\beta-\alpha\, w)} \times {}_2\Psi_2\left[\begin{matrix}(c,1),(\eta+w,1)\\(\sigma,\mu),(c+2w,1)\end{matrix} ;w\right] p^{-w}dw, \quad (12)
$$

where $w > 0, Re(p) > 0$.

Theorem 2.3. *The following derivative formula holds for the* ψ*-generalized Mittag-Leffler function,*

$$
\frac{d^n}{dz^n}\{{}^{\psi}E_{\mu,\eta,p}^{\eta,c,\alpha,\beta}(z)\} = (c)_n\ {}^{\psi}E_{\mu,\sigma+n\mu,p}^{\eta+n,c+n,\alpha,\beta}(z), \quad (13)
$$

where $(Re(c) > 0, Re(\eta) > 0, Re(\mu) > 0, Re(\beta) > 1, R(\alpha) > 0$ *and* $p \geq, 0\ \sigma, c, \mu, \eta \in C)$.

Proof. We take the derivative w.r.t. z of equation (12) and apply Leibnitz rule in right hand side then we have

$$
\begin{aligned}
\frac{d}{dz}\{{}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z)\} &= \frac{1}{\mathcal{B}(\eta,c-\eta)} \int_0^1 \Big\{u^{\eta-1}(1-u)^{c-\eta-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big) \\
&\quad \times \frac{\partial}{\partial z}E_{\mu,\sigma}^{c}(uz)\Big\}du \\
&= \frac{1}{\mathcal{B}(\eta,c-\eta)} \int_0^1 \Big\{u^{\eta-1}(1-u)^{c-\eta-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big) \\
&\quad \times \sum_{k=1}^{\infty} \frac{(c)_k}{\Gamma(\mu k+\sigma)} \frac{k u^k z^{k-1}}{k!}\Big\}du \\
&= \frac{1}{\mathcal{B}(\eta,c-\eta)} \int_0^1 \Big\{u^{\eta+k}(1-u)^{c-\eta-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)\Big\}du \\
&\quad \times \sum_{k=0}^{\infty} \frac{(c)_{k+1}}{\Gamma(\mu k+\mu+\sigma)} \frac{z^k}{k!} \\
&= \frac{c}{\mathcal{B}(\eta,c-\eta)} \int_0^1 \Big\{u^{(\eta+1)+k-1}(1-u)^{c-\eta-1}{}_1\Psi_1\Big(\alpha,\beta;\frac{-p}{u(1-u)}\Big)\Big\}du \\
&\quad \times \sum_{k=0}^{\infty} \frac{(c+1)_k}{\Gamma(\mu k+\mu+\sigma)} \frac{z^k}{k!}
\end{aligned}
$$

by applying(12) in right hand side of above, we get

$$
\frac{d}{dz}\{{}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z)\} = c\ {}^{\psi}E_{\mu,\sigma+\mu,p}^{\eta+1,c+1,\alpha,\beta}(z). \quad (14)
$$

again differenciating both side of equation (23) with respect to z , we have

$$
\frac{d^2}{dz^2}\{{}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(z)\} = c(c+1)\ {}^{\psi}E_{\mu,\sigma+2\mu,p}^{\eta+2,c+2,\alpha,\beta}(z). \quad (15)
$$

continuing this process n times, we obtain the required result.

□

Theorem 2.4. *The following derivative formula also holds for the ψ-generalized Mittag-Leffler function*

$$\frac{d^k}{dz^k}\{z^{\sigma-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\lambda z^{\mu})\} = z^{\sigma-k-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma-k,p}(\lambda z^{\mu}), \tag{16}$$

$(\sigma, \lambda, c, \mu, \eta \in C$ *and* $Re(c) > 0, Re(\eta) > 0, Re(\mu) > 0, p \geq 0, Re(\beta) > 1, R(\alpha) > 0)$.

Proof. Using equation (11), we have

$$\begin{aligned}\frac{d^k}{dz^k}\{z^{\sigma-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\lambda z^{\mu})\} &= \frac{d^k}{dz^k}\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k, c-\eta)}{\mathcal{B}(\eta, c-\eta)}\frac{\lambda^k\,(c)_k}{\Gamma(\mu\,k+\sigma)}\frac{z^{\mu\,k+\sigma-1}}{k!}\\ &= \sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k, c-\eta)}{\mathcal{B}(\eta, c-\eta)}\frac{\lambda^k\,(c)_k}{\Gamma(\mu\,k+\sigma)}\frac{d^k}{dz^k}\left[\frac{z^{\mu\,k+\sigma-1}}{k!}\right]\\ &= \sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k, c-\eta)}{\mathcal{B}(\eta, c-\eta)}\frac{\lambda^k\,(c)_k}{\Gamma(\mu\,k+\sigma-k)}\frac{z^{\mu\,k+\sigma-1-k}}{k!}\\ &= z^{\sigma-k-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma-k,p}(\lambda z^{\mu})\end{aligned}$$

□

3. Fractional calculus of the ψ-generalised Mittag-Leffler function

Fractional calculus is very useful in designing models of real-world problem. Many researchers investigated in this field and presented its importance in various areas of engineering, science and finance[1, 18] The fractional derivative and integrals are important aspects of fractional calculus. The left- and right-sided Riemann -Liouville fractional integral operator I^{τ}_{a+} and I^{τ}_{b-} are defined by

$$(I^{\tau}_{a+}f)(x) = \frac{1}{\Gamma(\tau)}\int_a^x (x-v)^{\tau-1}f(v)dv, \qquad (\tau \in C, Re(\tau) > 0). \tag{17}$$

and

$$(I^{\tau}_{b-}f)(x) = \frac{1}{\Gamma(\tau)}\int_x^b (x-v)^{\tau-1}f(v)dv, \qquad (\tau \in C, Re(\tau) > 0). \tag{18}$$

Also the left- and right-sided Riemann -Liouville fractional derivative operator is as follows:

$$(D^{\tau}_{a+}f)(x) = \Big(\frac{d}{dx}\Big)^k (I^{k-\tau}_{a+}f)(x). \tag{19}$$

and

$$(D^{\tau}_{a-}f)(x) = \Big(-\frac{d}{dx}\Big)^k (I^{k-\tau}_{a-}f)(x). \tag{20}$$

Hilfer[5, 6] has established the generalization of the RL fractional derivative operator $D^{\tau,\lambda}_{a+}$ with order $0 < \tau < 1$ and $0 \leq \lambda \leq 1$ by

$$(D^{\tau,\lambda}_{a+}f)(x) = (I^{\lambda(1-\tau)}_{a+}\frac{d}{dx}(I^{(1-\tau)(1-\lambda)}_{a+}f))(x). \tag{21}$$

Mathai and Haubold [9] has given the following result on right-sided fractional integral operator I^{τ}_{a+} as

$$(I^{\tau}_{a+}[(v-a)^{\sigma-1}])(x) = \frac{\Gamma(\sigma)}{\Gamma(\tau+\sigma)}(x-a)^{\tau+\sigma-1}, \qquad (\tau, \sigma \in C, Re(\tau) > 0, Re(\sigma) > 0). \tag{22}$$

Also Srivastava and Tomovski[17] has established the following result

$$(D_{a+}^{\tau,\lambda}[(v-a)^{\mu-1}](x) = \frac{\Gamma(\mu)}{\Gamma(\mu-\tau)}(x-a)^{\mu-\tau-1}, \tag{23}$$

where $\tau, \mu \in C, Re(\tau) > 0, Re(\mu) > 0$ *and* $0 < \tau < 1, 0 \leq \lambda \leq 1$.

Theorem 3.1. *If $x > a$ such that $(a \in R^+ = [0,\infty])$ and $\alpha, \beta, \tau, \eta, \sigma, \delta \in C, R(\mu) > 0, R(\sigma) > 0, R(\tau) > 0, R(\delta) > 0$ and $R(\alpha) > 0, R(\beta) > 1$. then*

$$I_{a+}^{\tau}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})](x) = (x-a)^{\tau+\sigma-1}\ {}^{\psi}E_{\mu,\sigma+\tau,p}^{\eta,c,\alpha,\beta}(\delta(x-a)^{\mu}), \tag{24}$$

$$D_{a+}^{\tau}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})](x) = (x-a)^{\sigma-\tau-1}\ {}^{\psi}E_{\mu,\sigma-\tau,p}^{\eta,c,\alpha,\beta}(\delta(x-a)^{\mu}), \tag{25}$$

and

$$D_{a+}^{\tau,\lambda}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})](x) = (x-a)^{\sigma-\tau-1}\ {}^{\psi}E_{\mu,\sigma-\tau,p}^{\eta,c,\alpha,\beta}(\delta(x-a)^{\mu}). \tag{26}$$

Proof.

$$\begin{aligned}
I_{a+}^{\tau}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})](x) &= \frac{1}{\Gamma(\tau)}\int_a^x \frac{(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})}{(x-v)^{1-\tau}}dv \\
&= \frac{1}{\Gamma(\tau)\mathcal{B}(\eta,c-\eta)}\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(\eta+k,c-\eta)(c)_k(\delta)^k}{\Gamma(\mu k+\sigma)k!} \\
&\quad \times \int_a^x (v-a)^{\sigma+\mu k-1}(x-v)^{\tau-1}dv \\
&= \frac{1}{\mathcal{B}(\eta,c-\eta)}\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(\eta+k,c-\eta)(c)_k(\delta)^k}{\Gamma(\mu k+\sigma)k!} \\
&\quad \times (I_{a+}^{\tau}[(v-a)^{\sigma+\mu k-1}])(x),
\end{aligned}$$

using result of equation (31) in right hand side of above equation we have

$$\begin{aligned}
I_{a+}^{\tau}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})] &= \frac{1}{\mathcal{B}(\eta,c-\eta)}\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(\eta+k,c-\eta)(c)_k(\delta)^k}{\Gamma(\mu k+\sigma)k!} \\
&\quad \times (x-a)^{\sigma+\tau+\mu k-1}\frac{\Gamma(\mu k+\sigma)}{\Gamma(\mu k+\sigma+\tau)} \\
&= (x-a)^{\sigma+\tau-1}\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}_p^{(\alpha,\beta)}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k}{\Gamma(\mu k+\sigma+\tau)} \\
&\quad \times \frac{[\delta(x-a)^{\mu}]^k}{k!} \\
&= (x-a)^{\sigma+\tau-1}\ {}^{\psi}E_{\mu,\sigma+\tau,p}^{\eta,c,\alpha,\beta}(\delta(x-a)^{\mu}).
\end{aligned}$$

which is the required result.
Now for next result we have

$$(D_{a+}^{\tau}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})])(x) = \left(\frac{d}{dx}\right)^k\{I_{a+}^{k-\tau}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma-\tau,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})]\},$$

by using equation (33), we have

$$(D_{a+}^{\tau}[(v-a)^{\sigma-1}\ {}^{\psi}E_{\mu,\sigma,p}^{\eta,c,\alpha,\beta}(\delta(v-a)^{\mu})])(x) = \left(\frac{d}{dx}\right)^k\{(x-a)^{\sigma-\tau+k-1}\ {}^{\psi}E_{\mu,\sigma-\tau+k,p}^{\eta,c,\alpha,\beta}(\delta(x-a)^{\mu})\}$$

by the use of (25), we reach at the desired result

$$D^{\tau}_{a+}[(v-a)^{\sigma-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(v-a)^{\mu})] = (x-a)^{\sigma-\tau-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma-\tau,p}(\delta(x-a)^{\mu})$$

Next to prove equation (35), we have

$$\begin{aligned}(D^{\tau,\lambda}_{a+}[(v-a)^{\sigma-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(v-a)^{\mu})])(x) &= \Big(D^{\tau,\lambda}_{a+}\Big[\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k}{\Gamma(\mu k+\sigma)}\frac{(\delta)^k}{k!}\\ &\quad\times(v-a)^{\mu k+\sigma-1}\Big]\Big)(x)\\ &=\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k}{\Gamma(\mu k+\sigma)}\frac{(\delta)^k}{k!}\\ &\quad\times\Big(D^{\tau,\lambda}_{a+}[(v-a)^{\mu k+\sigma-1}]\Big)(x),\end{aligned}$$

using equation (32)in the right hand side in above equation , we get the desired result

$$\begin{aligned}D^{\tau,\lambda}_{a+}[(v-a)^{\sigma-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(v-a)^{\mu})](x) &= \sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k(\delta)^k}{\Gamma(\mu k+\sigma)k!}\frac{\Gamma(\mu k+\sigma)}{\Gamma(\mu k+\sigma-\tau)}\\ &\quad\times(v-a)^{\mu k+\sigma-\tau-1}\\ &=(v-a)^{\sigma-\tau-1}\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k}{\Gamma(\mu k+\sigma-\tau)}\\ &\quad\times\frac{[\delta(x-a)^{\mu}]^k}{k!}\\ &=(x-a)^{\sigma-\tau-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma-\tau,p}(\delta(x-a)^{\mu})\end{aligned}$$

□

4. An integral operator containing the ψ-generalised Mittag Leffler function

In this section we introduce fractional integral operator of the ψ-generalised Mittag-Leffler function.

Definition 1: Let $\delta,\eta,\mu,c,\sigma,\alpha,\beta\in C, R(\mu)>0, R(\sigma)>0, R(\delta)>0, R(c)>0, R(\eta)>0, R(\alpha)>0, R(\beta)>1$ then

$$(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}f)(x)=\int_a^x (x-v)^{\sigma-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(x-v)^{\mu})f(v)dv. \tag{27}$$

For $\alpha=0,\beta=1$ we obtain an integral operator given by Rahman et al.[15] as

$$(\varepsilon^{\delta,\eta,c}_{a+,\mu,\sigma,p}f)(x)=\int_a^x (x-v)^{\sigma-1}\ E^{\eta,c}_{\mu,\sigma,p}(\delta(x-v)^{\mu})f(v)dv. \tag{28}$$

For $\alpha=0,\beta=1,p=0$ we obtain an integral operator given by Srivastava and Tomovski[17] as

$$(\varepsilon^{\delta,\eta}_{a+,\mu,\sigma}f)(x)=\int_a^x (x-v)^{\sigma-1}\ E^{\eta}_{\mu,\sigma}(\delta(x-v)^{\mu})f(v)dv. \tag{29}$$

when $\delta=0$ then the integral operator reduces to the classical R-L fractional integral operator.

Theorem 4.1. *Let* $\delta, \eta, \mu, c, \sigma, \alpha, \beta \in C, R(\mu) > 0, R(\sigma) > 0, R(\lambda) > 0, R(\delta) > 0, R(c) > 0, R(\eta) > 0, R(\alpha) > 0, R(\beta) > 1,$ *then*

$$(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}[(v-a)^{\lambda-1}])(x) = (x-a)^{\lambda+\sigma-1}\Gamma(\lambda)^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma+\lambda,p}(\delta(x-a)^{\mu}). \tag{30}$$

Proof. by the use of equation (36), we have

$$\begin{aligned}(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}[(v-a)^{\lambda-1}])(x) &= \int_a^x (x-v)^{\sigma-1}(v-a)^{\lambda-1}\,{}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma+\lambda,p}(\delta(x-v)^{\mu})dv \\ &= \sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k}{\Gamma(\mu k+\sigma)}\frac{(\delta)^k}{k!} \\ &\quad\times\Big(\int_a^x (v-a)^{\lambda-1}(x-v)^{\sigma+\mu k-1}dv\Big) \\ &= \sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k}{\Gamma(\mu k+\sigma)}\frac{(\delta)^k}{k!}I^{\mu k+\sigma}_{a+}[(v-a)^{\lambda-1}] \\ &= (x-a)^{\sigma+\lambda-1}\Big\{\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{(c)_k}{\Gamma(\mu k+\sigma)}\frac{[\delta(x-a)^{\mu}]^k}{k!} \\ &\quad\times\frac{\Gamma(\lambda)\Gamma(\mu k+\sigma)}{\Gamma(\mu k+\sigma+\lambda)}\Big\} \\ &= (x-a)^{\sigma+\lambda-1}\Gamma(\lambda)^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma+\lambda,p}(\delta(x-a)^{\mu})\end{aligned}$$

which completes the proof. □

Theorem 4.2. *Let* ϕ *be the function in the space of Lebesgue measurable functions* $L(a,d)$ *on a finite interval* $[a,d](d>a)$ *of the real line* R *given by*

$$L(a,d) = \Big\{f : ||f||_1 = \int_a^d |f(x)|dx < \infty\Big\}. \tag{31}$$

If $\eta, \mu, \sigma, \delta, c \in C, R(\mu) > 0, R(\sigma) > 0, R(\delta) > 0, R(\eta) > 0, R(c) > 0, R(\alpha) > 0, R(\beta) > 1$ *then*

$$||(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}\phi||_1 \leq B||\phi||, \tag{32}$$

where

$$B = (d-a)^{R(\sigma)}\sum_{k=0}^{\infty}\frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_{p}(\eta+k,c-\eta)}{\mathcal{B}(\eta,c-\eta)}\frac{|(c)_k|}{\Gamma(\mu k+\sigma)(R(\sigma)+R(\mu)k)}\frac{|\delta(d-a)^{\mu}|^k}{k!}. \tag{33}$$

Proof.

$$||(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}\phi||_1 = \int_a^d |\int_a^x (x-v)^{\sigma-1\psi} E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(x-v)^{\mu})\phi(v)dv|dx$$
$$\leq \int_a^d \left[\int_v^d (x-v)^{R(\sigma-1)}]|^{\psi} E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(x-v)^{\mu})|dx\right]|\phi(v)|dv$$

Put $(x-v) = w,$ we have

$$= \int_a^d \left[\int_0^{d-v} (w)^{R(\sigma-1)} \; |^{\psi} E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(w)^{\mu})|dw\right]|\phi(v)|dv$$
$$\leq \int_a^d \left[\sum_{k=0}^{\infty} \frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_p(\eta+k, c-\eta)}{\mathcal{B}(\eta, c-\eta)} \frac{(c)_k}{\Gamma(\mu k+\sigma)} \frac{|\delta|^k}{k!} \left(\frac{(w)^{R(\sigma)+R(\mu)k}}{R(\sigma)+R(\mu)k}\right)_0^{d-a}\right]$$
$$\times \int_a^d |\phi(v)|dv$$
$$\leq \left\{(d-a)^{R(\sigma)} \sum_{k=0}^{\infty} \frac{{}^{\psi}\mathcal{B}^{(\alpha,\beta)}_p(\eta+k, c-\eta)}{\mathcal{B}(\eta, c-\eta)\Gamma(\mu k+\sigma)} \frac{|c|_k}{(R(\sigma)+R(\mu)k)} \frac{|\delta(d-a)^{\mu}|^k}{k!}\right\}$$
$$\times \int_a^d |\phi(v)|dv$$
$$= B||\phi||_1$$

Hence

$$||(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}\phi||_1 \leq B||\phi||_1$$

□

Theorem 4.3. *If $\tau, \eta, \mu, \sigma, \delta, c \in C, R(\mu) > 0, R(\sigma) > 0, R(\eta) > 0, R(\delta) > 0, R(c) > 0, R(\alpha) > 0, R(\beta) > 1$ and $x > a$ then for any function $f \in L(\mu, \sigma)$*

$$(I^{\tau}_{a+}[\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma;p}f])(x) = (\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma+\tau,p}f)(x) = (\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}[I^{\tau}_{a+}f])(x). \quad (34)$$

Proof. From (26) and (36) we have

$$(I^{\tau}_{a+}[\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}f])(x) = \frac{1}{\Gamma(\tau)} \int_a^x (x-v)^{\tau-1}[(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}f)(v)]dv$$
$$= \frac{1}{\Gamma(\tau)} \int_a^x (x-v)^{\tau-1} \times \left[\int_a^v (v-w)^{\sigma-1} \; {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(v-w)^{\mu})f(w)dw\right]dv,$$

change the order of integration and use the Dirichlet's formula, we get

$$(I^{\tau}_{a+}[\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}f])(x) = \int_a^x \left[\frac{1}{\Gamma(\tau)} \int_w^x (x-v)^{\tau-1}(v-w)^{\sigma-1} \; {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(v-w)^{\mu})dv\right]f(w)dw,$$

Using (26) we get

$$(I^{\tau}_{a+}[\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}f])(x) = \int_a^x \Big(I^{\tau}_{w+}[(v-w)^{\sigma-1} \; {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(v-w)^{\mu})dvf(w)dw]\Big)(x)$$

By (33) we obtain

$$(I^{\tau}_{a+}[\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}f])(x) = \int_a^x (x-w)^{\tau+\sigma-1} \; {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma+\tau,p}(\delta(x-w)^{\mu})f(w)dw$$

Hence

$$(I^{\tau}_{a+}[\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}f])(x) = (\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma+\tau,p}f)(x).$$

Now for the second part of the equation (43) using equations (26) and (36), we have

$$(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}[I^{\tau}_{a+}f])(x) = \int_a^x (x-v)^{\sigma-1\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(x-v)^{\mu})\Big[I^{\tau}_{a+}f\Big](v)dv$$

$$= \int_a^x (x-v)^{\sigma-1\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(x-v)^{\mu})\left(\frac{1}{\Gamma(\tau)}\int_a^v \frac{f(w)}{(v-w)^{1-\tau}}dw\right)dv,$$

change the order of integration and use the Dirichlet's formula[10] we get

$$(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}[I^{\tau}_{a+}f])(x) = \int_a^x \frac{1}{\Gamma(\tau)}\Big[\int_w^x (x-v)^{\sigma-1}(v-w)^{\tau-1}\ {}^{\psi}E^{\eta,c,\alpha,\beta}_{\mu,\sigma,p}(\delta(x-v)^{\mu})dv\Big]f(w)dw,$$

Now we follow the above similar process by using equations (26) and (33), we get

$$(\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma,p}[I^{\tau}_{a+}f])(x) = (\varepsilon^{\delta,\eta,c,\alpha,\beta}_{a+,\mu,\sigma+\lambda,p}f)(x).$$

□

5. Conclusions

In this investigation,we developed some results like integral transforms, derivative formulas and fractional differ-integral properties of ψ-generalized mittag-leffler function by using ψ-generalized beta function.

Other than this we also evaluated some properties for fractional integral operator containing ψ-generalized mittag-leffler function in its kernel.The result shown in this paper are general in nature but can be extended to establish other properties of special functions.

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Meenakshi Singhal is working as an assistant professor in the Department of Mathematics, IIS (deemed to be University) Jaipur. She completed her U. G. from Kota University, kota, M.Sc.Tech. Mathematics in 2012 from the Central University of Rajasthan, Ajmer. She is working on her Ph.D. at the same university. Her research interest focus on Special functions, Integral Transforms and Fractional Calculus.

Ekta Mittal is working as an assistant professor in the Department of Mathematics, IIS deemed to be University Jaipur. She completed her U. G., P. G., M.Phil (2004) at Rajasthan University Jaipur and Ph.D. from Manipal Rajasthan University Jaipur. Her research interest area are Special functions, Integral Transforms and Fractional Calculus.