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L^2- OPTIMAL ORDER ERROR FOR TWO-DIMENSIONAL COUPLED BURGERS' EQUATIONS BY WEAK GALERKIN FINITE ELEMENT METHOD

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Abstract. In this paper, we present a continuous time and discrete time weak Galerkin finite element schemes for solving non linear two Dimensional coupled Burgers' equations with a stabilization term. We use special weak form (trilinear form) for nonlinear term. The optimal order error in L^2- norm is obtained based on dual argument technique for both continuous time and discrete time weak Galerkin finite element schemes. The Numerical examples are in good agreement with the theoretical analysis and polynomial mixture $\{P_k(K), P_{k-1}(\partial K), [P_{k-1}(K)]^2\}$.

Keywords: Weak Galerkin Finite Element Method (WG-FEM), Burgers' Equations, Optimal order error.

AMS Subject Classification: 65N15, 65N30.

1. Introduction

Two dimensional coupled Burgers' Equations serves as a useful model for many interesting problems in applied mathematics. It models effectively certain problems of a fluid flow nature, in which either shocks or viscous dissipation is a significant factor as shock flows, traffic flow, acoustic transmission in fog, air flow over an air, oil, gas dynamics etc. Besides its importance in understanding convection diffusion phenomena, Burgers' equation can be used, especially for computational purposes, as a precursor of the Naiver-Stokes equations for fluid flow problems (see [11] [12], [13]). In fact, it can be used as a model for any nonlinear wave propagation problem subject to dissipation. Depending on the problem being modeled, this dissipation may result from viscosity, heat conduction, mass diffusion, thermal radiation, chemical reaction, or other source.

In this paper, we consider nonlinear two dimensional coupled Burgers' problem [1].

$$\boldsymbol{u}_t - \epsilon\nabla^2\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{f}(x,y,t), \qquad (x,y,t) \in \Omega\times(0,T], \tag{1.1}$$

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with Dirchlet boundary conditions

$$\boldsymbol{u}(x,y,t) = \boldsymbol{\eta}(x,y,t), \qquad (x,y,t) \in \partial\Omega \times (0,T], \tag{1.2}$$

and initial conditions

$$\boldsymbol{u}(x,y,0) = \boldsymbol{u}^0(x,y), \qquad (x,y) \in \Omega, \tag{1.3}$$

where $\Omega = \{(x,y), a_1 \leq x \leq a_2, b_1 \leq y \leq b_2\} \subset \mathbb{R}^2$ is the computational domian, $\partial\Omega$ it's boundary, $\boldsymbol{u} = (u,v)$, u and v are the velocity components, $\boldsymbol{u^0} = (u^0, v^0)$, $\boldsymbol{\eta} = (\eta_u, \eta_v)$ are known functions, $\boldsymbol{u}_t$ is unsteady term, $\epsilon\nabla^2\boldsymbol{u}$ is the diffusion term, $\epsilon = \frac{1}{Re}$ is diffusion constant, Re is the Reynolds number and $\boldsymbol{f} = (f_1, f_2)$ is the source term (Often equal to zero).

Rest of the paper is organized as follows. In Section 2, we introduce the definition of discrete weak derivative, discrete weak gradient, weak finite element spaces and some lemmas which are necessary in error estimate. Section 3 is devoted to variational form and weak variational form for continuous and discrete time WG-FEM. In section 4 we derive the optimal order error for both continuous and discrete time WG-FEM in L^2-norm. Finally, in section 5 numerical experiments are presented to show the efficacy of the WG-FEM and confirm our theoretical analysis.

2. The Weak Galerkin Method

In this section, we introduce some important weak function spaces, weak differential operators, which are useful in the error analysis of WG-FEM. Let $K \subset \Omega$ be any polygonal region with boundary ∂K.

For any triangle $K \in T_h$ and ∂K. A weak function $w = \{w_0, w_b\}$ on K has two pieces, $w_0 \in L^2(K)$ and $w_b \in L^2(\partial K)$, the first pieces represents the values of w in the interior K and the second pieces on triangle boundary ∂K. The space of weak functions and corresponding vector space defined on K are given by

$$W(K) = \{w = \{w_0, w_b\} | w_0 \in L^2(K), w_b \in L^2(\partial K)\}. \tag{2.1}$$

Define a space

$$H(div, K) = \{\boldsymbol{w}, \boldsymbol{w} \in (L^2(K))^2, \nabla \cdot \boldsymbol{w} \in L^2(K)\}. \tag{2.2}$$

Definition 2.1. *Let $w \in W(K)$, the weak derivative operator of w in the direction x_j is defined as a linear functional $\frac{\partial_d w}{\partial x_j}$ on $H^1(K)$ such that,*

$$\int_K \frac{\partial_d w}{\partial x_j} q dx = -\int_K w_0 \frac{\partial q}{\partial x_j} dx + \int_{\partial K} w_b q n_{x_j} ds, \quad \forall q \in H^1(K). \tag{2.3}$$

Definition 2.2. *Let $w \in W(K)$, the weak gradient operator of w is defined as a linear functional $\nabla_d w \in H(div, K)$ on each element K, by the following equation:*

$$\int_K \nabla_d w \cdot \boldsymbol{q} dK = -\int_K w_0 (\nabla \cdot \boldsymbol{q}) dK + \int_{\partial K} w_b (\boldsymbol{q} \cdot \boldsymbol{n}) ds, \quad \forall \boldsymbol{q} \in H(div, K), \tag{2.4}$$

where $\mathbf{n}$ is the outward normal direction of ∂K.

Definition 2.3. *Let $w \in W(K)$, the discrete weak derivative operator of w in the direction x_j is defined as unique polynomial $\frac{\partial_{d,r} w}{\partial x_j}$ on $P_{k-1}(K)$ such that*

$$\int_K \frac{\partial_{d,r} w}{\partial x_j} q dx = -\int_K w_0 \frac{\partial q}{\partial x_j} dx + \int_{\partial K} w_b q n_{x_j} ds, \quad \forall q \in P_{k-1}(K). \tag{2.5}$$

Definition 2.4. *Let $w \in W(K)$, the discrete weak gradient operator of w is defined as unique polynomial $\nabla_{d,r} w \in [P_{k-1}(K)]^2$ on each element K,by the following equation:*

$$\int_K \nabla_{d,r} w \cdot \boldsymbol{q} dK = -\int_K w_0 (\nabla \cdot \boldsymbol{q}) dK + \int_{\partial K} w_b (\boldsymbol{q} \cdot \boldsymbol{n}) ds, \quad \forall \boldsymbol{q} \in [P_{k-1}(K)]^2. \tag{2.6}$$

By applying the usual integration by part to the first term on the right hand side of (2.6), we can rewrite the equation (2.6) as follows

$$\int_K \nabla_{d,r} w \cdot \boldsymbol{q} \, dK = \int_K \nabla w_0 \, \boldsymbol{q} \, dK + \int_{\partial K} (w_0 - w_b) \, (\boldsymbol{q} \cdot \boldsymbol{n}) ds, \quad \forall \boldsymbol{q} \in [P_{k-1}(K)]^2. \tag{2.7}$$

Let T_h be a partition of the domain Ω with mesh size $h = \max h_K, \forall K \in T_h$, where h_K is longest side of K. In this paper we assume that T_h is shape regular, namely, satisfying the shape regularity assumptions ***A1-A4*** in [4].

A discrete weak function $w = \{w_0, w_b\}$ refers to a polynomial with two components in which the first component w_0 is associated with the interior K and w_b is defined on each edge $e, e \in \partial K$. Note that w_b may or may not equal to the trace of w_0 on ∂K. Now we introduce two trial finite element spaces as follows:

$$W_h = \{w = \{w_0, w_b\} : \{w_0, w_b\}|_K \in P_k(K) \times P_{k-1}(\partial K)\}, \tag{2.8}$$

$$\boldsymbol{W}_h = \{\boldsymbol{w} = \{u, v\} : u \in W_h, v \in W_h\}, \tag{2.9}$$

with test space,

$$\boldsymbol{W}_h^0 = \{\boldsymbol{w} \in \boldsymbol{W}_h : \boldsymbol{w}_b \mid_{\partial K \cap \partial \Omega} = 0\}. \tag{2.10}$$

Let $V_{k-1}(K) = \{[P_{k-1}(K)]^2 \equiv$ set of vector-valued polynomial of degree no more than $k-1$ on $K\}$.

To derive the error estimates for the WG-FEM, we define two projection operators, the first $Q_h \boldsymbol{w} = \{Q_0 \boldsymbol{w}, Q_b \boldsymbol{w}\}$ is $\boldsymbol{L}^2-$ projection of $\boldsymbol{H}^1(\Omega)$ on to $\boldsymbol{P}_k(K) \times \boldsymbol{P}_{k-1}(\partial K)$ with $\boldsymbol{w}_0 \mid_K = Q_0 \boldsymbol{w}, \quad \boldsymbol{w}_b \mid_e = Q_b \boldsymbol{w}, \quad \forall K \in T_h, e \in \partial K$ and the other projection is R_h, the $\boldsymbol{L}^2-$ projection of $[L^2(K)]^2$ onto $V_{k-1}(K)$ (i.e. R_h is the L^2-projection to the space of piecewise polynomials of degree $k-1$).

Lemma 2.1. *[6] Let T_h be the finite element partition of Ω satisfying the shape regularity assumption **A1-A4**. Let $Q_h \boldsymbol{w} = \{Q_0 \boldsymbol{w}, Q_b \boldsymbol{w}\}$ is $\boldsymbol{L}^2-$ projection operator. Then, we have*

$$\nabla_d (Q_h \boldsymbol{w}) = R_h (\nabla \boldsymbol{w}), \quad \forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega). \tag{2.11}$$

Lemma 2.2. *[4] Let T_h be the finite element partition of Ω satisfying the shape regularity assumption **A1-A4**. Then, for any $\boldsymbol{w} \in \boldsymbol{H}^{k+1}(\Omega)$, we have*

$$\sum_{K \in T_h} \| \boldsymbol{w} - Q_0 \boldsymbol{w} \|_K^2 + \sum_{K \in T_h} h_K^2 \| \nabla (\boldsymbol{w} - Q_0 \boldsymbol{w}) \|_K^2 \leq C h^{2(k+1)} \|\boldsymbol{w}\|_{k+1}^2, \tag{2.12}$$

$$\sum_{K \in T_h} \| (\nabla \boldsymbol{w} - R_h (\nabla \boldsymbol{w})) \|_K^2 \leq C h^{2k} \|\boldsymbol{w}\|_{k+1}^2. \tag{2.13}$$

In addition, for any function $\boldsymbol{w} \in \boldsymbol{H}^1(K)$, the following trace inequality holds.

$$\| \boldsymbol{w} \|_{\partial K}^2 \leq C (h_K^{-1} \| \boldsymbol{w} \|_K^2 + h_K \| \nabla \boldsymbol{w} \|_K^2), \quad \forall K \in T_h. \tag{2.14}$$

Lemma 2.3. *Let $\boldsymbol{w} \in \boldsymbol{H}^{k+1}(K)$, there exists constant $C > 0$ such that*

$$\|Q_0\boldsymbol{w} - Q_b\boldsymbol{w}\|_{\partial K} \leq Ch^{k+\frac{1}{2}}\|\boldsymbol{w}\|_{k+1}. \tag{2.15}$$

Proof. From the definition of the $\boldsymbol{L}^2$-projection, Cauchy-Schwarz inequality, trace inequality and Lemma (2.2) that

$$\begin{aligned}
\|Q_0\boldsymbol{w} - Q_b\boldsymbol{w}\|^2_{\partial K} &= (Q_0\boldsymbol{w} - Q_b\boldsymbol{w}, Q_0\boldsymbol{w} - Q_b\boldsymbol{w})_{\partial K} = (Q_0\boldsymbol{w} - \boldsymbol{w}, Q_0\boldsymbol{w} - Q_b\boldsymbol{w})_{\partial K} \\
&\leq \|Q_0\boldsymbol{w} - \boldsymbol{w}\|_{\partial K}\|Q_0\boldsymbol{w} - Q_b\boldsymbol{w}\|_{\partial K} \\
&\leq (h_K^{-1}\|Q_0\boldsymbol{w} - \boldsymbol{w}\|^2 + h_K\|\nabla(Q_0\boldsymbol{w} - \boldsymbol{w})\|^2)^{\frac{1}{2}}\|Q_0\boldsymbol{w} - Q_b\boldsymbol{w}\|_{\partial K} \\
&= Ch^{-\frac{1}{2}}(h^{2(k+1)}\|\boldsymbol{w}\|^2_{k+1})^{\frac{1}{2}}\|Q_0\boldsymbol{w} - Q_b\boldsymbol{w}\|_{\partial K} \\
&= Ch^{k+\frac{1}{2}}\|\boldsymbol{w}\|_{k+1}\|Q_0\boldsymbol{w} - Q_b\boldsymbol{w}\|_{\partial K}.
\end{aligned}$$

□

3. Variational form and Weak Variational form

Multiplying equations (1.1) by $\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega)$ and integrating both side on Ω. We get.

$$(\boldsymbol{u}_t, \boldsymbol{w}) + \epsilon(\nabla\boldsymbol{u}, \nabla\boldsymbol{w}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}), \tag{3.1}$$

$$(\boldsymbol{u}(x, y, 0), \boldsymbol{w}) = (\boldsymbol{u}^0, \boldsymbol{w}).$$

The third term in (3.1) can be written as (see [9])

$$((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{w}) = \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}, \boldsymbol{u}). \tag{3.2}$$

Substituting (3.2) in to equation (3.1), the Variational form is find $\boldsymbol{u} \in \boldsymbol{H}^1(0, T, \boldsymbol{H}^1_0(\Omega))$ such that

$$\begin{cases} (\boldsymbol{u}_t, \boldsymbol{w}) + \epsilon(\nabla\boldsymbol{u}, \nabla\boldsymbol{w}) + \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{w}), \\ \boldsymbol{u}(x, y, 0) = \boldsymbol{u}^0(x, y) \qquad \forall(x, y) \in \Omega \qquad \forall\boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega). \end{cases} \tag{3.3}$$

Define two bilinear form $a_0(.,.)$, $s(.,.)$ and trilinear form $a_1(.;.,.)$ on $\boldsymbol{W}_h$, for any $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{W}_h$

$$a_0(\boldsymbol{u}, \boldsymbol{w}) = \sum_{K\in T_h}(\epsilon\nabla_d\boldsymbol{u}, \nabla_d\boldsymbol{w}), \tag{3.4}$$

$$a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w}) = \sum_{K\in T_h}\frac{1}{2}\{(\boldsymbol{u}_0\cdot\nabla_d\boldsymbol{u}, \boldsymbol{w}_0) - (\boldsymbol{u}_0\cdot\nabla_d\boldsymbol{w}, \boldsymbol{u}_0)\}, \tag{3.5}$$

$$s(\boldsymbol{u}, \boldsymbol{w}) = \sum_{K\in T_h} h_k^{-1}(Q_b\boldsymbol{u}_0 - \boldsymbol{u}_b, Q_b\boldsymbol{w}_0 - \boldsymbol{w}_b)_{\partial K}, \tag{3.6}$$

where $s(\boldsymbol{u}, \boldsymbol{w})$ is also called a stabilizer, the stabilizer term is used to control the gap between $\boldsymbol{u}_0$ and $\boldsymbol{u}_b$ and thus the gap of $\boldsymbol{u}_0$ over the boundary of K.
We defined the trip-bar norm as follows, for any $\boldsymbol{u} \in \boldsymbol{W_h}$, we have

$$|||\boldsymbol{u}||| = \left(\sum_{K\in T_h}((\nabla_d\boldsymbol{u}, \nabla_d\boldsymbol{u})_K + (\boldsymbol{u}_0, \boldsymbol{u}_0)_K + h_k^{-1}(Q_b\boldsymbol{u}_0 - \boldsymbol{u}_b, Q_b\boldsymbol{u}_0 - \boldsymbol{u}_b)_{\partial K})\right)^{\frac{1}{2}}, \tag{3.7}$$

and H^1- equivalent norm

$$\|\boldsymbol{u}\|_{w,1} = \left(\sum_{K\in T_h} (\|\nabla_d \boldsymbol{u}\|_K^2 + h_K^{-1}\|Q_b\boldsymbol{u}_0 - \boldsymbol{u}_b\|_{\partial K}^2) \right)^{\frac{1}{2}}. \tag{3.8}$$

In the finite element space $\boldsymbol{W}_h$, we introduce a discrete H^1- semi norm as follows

$$\|\boldsymbol{u}\|_{h,1} = \left(\sum_{K\in T_h} (\|\nabla \boldsymbol{u}_0\|_K^2 + h_K^{-1}\|Q_b\boldsymbol{u}_0 - \boldsymbol{u}_b\|_{\partial K}^2) \right)^{\frac{1}{2}}, \tag{3.9}$$

where

$$\|\nabla_d \boldsymbol{u})\|^2 = \sum_{K\in T_h} (\nabla_d \boldsymbol{u}, \nabla_d \boldsymbol{u})_K, \quad \|\boldsymbol{u}_0\|^2 = \sum_{K\in T_h} (\boldsymbol{u}_0, \boldsymbol{u}_0)_K.$$

Lemma 3.1. *[6] There exists two constant D_1 and $D_2 > 0$ such that*

$$D_1\|\boldsymbol{w}\|_{h,1} \le \|\boldsymbol{w}\|_{w,1} \le D_2\|\boldsymbol{w}\|_{h,1}, \quad \forall \boldsymbol{w} \in \boldsymbol{W_h}. \tag{3.10}$$

Lemma 3.2. *There exists two constant M_1 and $M_2 > 0$ such that*

$$M_1\|\boldsymbol{w}\|_{h,1} \le |||\boldsymbol{w}||| \le M_2\|\boldsymbol{w}\|_{h,1}, \quad \forall \boldsymbol{w} \in \boldsymbol{W_h}. \tag{3.11}$$

Lemma 3.3. *[10] With T_h is shape regular, we have*

$$|||\boldsymbol{w} - Q_h\boldsymbol{w}|||^2 \le Ch^{2k}\|\boldsymbol{w}\|_{k+1}^2, \quad \forall \boldsymbol{w} \in \boldsymbol{H}^{k+1}(\Omega). \tag{3.12}$$

Now we can describe the WG-FEM for coupled Burgers' equations based on variational formulation (3.3), the continuous time WG-FEM is find $\boldsymbol{u}_h(t) = (\boldsymbol{u}_0(.,t), \boldsymbol{u}_b(.,t)) \in \boldsymbol{W}_h^0$ satisfying $\boldsymbol{u}_b = Q_b\boldsymbol{\eta}$ and $\boldsymbol{u}_h(0) = Q_h\boldsymbol{u}^0$, such that

$$(\boldsymbol{u}_{h,t}(t), \boldsymbol{w}_0) + a(\boldsymbol{u}_h(t); \boldsymbol{u}_h(t), \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}_0), \forall \boldsymbol{w} \in \boldsymbol{W}_h^0, \tag{3.13}$$

where

$$a(\boldsymbol{u}_h(t); \boldsymbol{u}_h(t), \boldsymbol{w}) = a_0(\boldsymbol{u}_h(t), \boldsymbol{w}) + a_1(\boldsymbol{u}_h(t); \boldsymbol{u}_h(t), \boldsymbol{w}) + s(\boldsymbol{u}_h(t), \boldsymbol{w}). \tag{3.14}$$

Let $0 = t_0 < t_1 < .... < t_N = T$ be a partition for time interval $[0, T]$ and $\tau > 0$ be a time step size satisfying $N\tau = T$ with N is positive integer, denote by $\boldsymbol{U}_n \in \boldsymbol{W}_h(k, k-1)$ the approximate solution of $\boldsymbol{u}(t_n)$. the backward Euler WG-FEM is defined by replacing the time derivative in equation (3.3) by a backward difference quotient $\widetilde{\partial}_t\boldsymbol{U}_n = (\boldsymbol{U}_n - \boldsymbol{U}_{n-1})/\tau$

$$(\widetilde{\partial}_t\boldsymbol{U}_n, \boldsymbol{w}_0) + a(\boldsymbol{U}_n; \boldsymbol{U}_n, \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}_0), \forall \boldsymbol{w} \in \boldsymbol{W}_h^0. \tag{3.15}$$

There are some properties of the trilinear form $a(.;.,.)$, which is easy to prove.

Lemma 3.4. *Let $\boldsymbol{W}_h(k, k-1)$ be the WG-FEM defined in (2.8) and $a(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{w})$ be the trilinear form given in (3.14), there exists a positive constants δ, λ, such that*

$$a(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{u}_h) \ge \delta|||\boldsymbol{u}_h|||^2, \tag{3.16}$$

$$|a(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{w})| \le \lambda|||\boldsymbol{u}_h||||||\boldsymbol{w}|||. \tag{3.17}$$

4. Error Analysis

In this section we estimate the optimal order error for both continues and discrete time WG-FEM, the error estimate will be measured in L^2 norm. Throughout this work the constant C have different values in each occurence (i.e. general constant).

4.1. **Error Equation.** Let $\boldsymbol{u} \in \boldsymbol{H}^1(K)$ and $\boldsymbol{w} \in \boldsymbol{W}_h$ be any finite element function, from Lemma (2.1), definition (2.1) and the integration by part, we get

$$a_0(Q_h\boldsymbol{u}, \boldsymbol{w}) = (\epsilon\nabla\boldsymbol{u}, \nabla\boldsymbol{w}_0)_K - (\epsilon(R_h\nabla\boldsymbol{u})\cdot\mathbf{n}, \boldsymbol{w}_0 - \boldsymbol{w}_b)_{\partial K}.$$

This implies that

$$(\epsilon\nabla\boldsymbol{u}, \nabla\boldsymbol{w}_0)_K = (\epsilon\nabla_d Q_h\boldsymbol{u}, \nabla_d\boldsymbol{w})_K + (\epsilon(R_h\nabla\boldsymbol{u})\cdot\mathbf{n}, \boldsymbol{w}_0 - \boldsymbol{w}_b)_{\partial K}. \tag{4.1}$$

From definition of trilinear form (3.5), we have

$$a_1(Q_h\boldsymbol{u}; Q_h\boldsymbol{u}, \boldsymbol{w}) = \sum_{K\in T_h} \frac{1}{2}\{(Q_0\boldsymbol{u}\cdot\nabla_d Q_h\boldsymbol{u}, \boldsymbol{w}_0)_K - (Q_0\boldsymbol{u}\cdot\nabla_d\boldsymbol{w}, Q_0\boldsymbol{u})_K\}. \tag{4.2}$$

Since

$$(Q_0\boldsymbol{u}\cdot\nabla_d\boldsymbol{w}, Q_0\boldsymbol{u})_K = (Q_0\boldsymbol{u}\cdot\nabla\boldsymbol{w}_0, Q_0\boldsymbol{u})_K - (\boldsymbol{w}_0 - \boldsymbol{w}_b, (Q_0\boldsymbol{u}\cdot\boldsymbol{n})Q_0\boldsymbol{u})_{\partial K}. \tag{4.3}$$

Substitution (4.3) in (4.2), we have

$$\begin{aligned} a_1(Q_h\boldsymbol{u}; Q_h\boldsymbol{u}, \boldsymbol{w}) &= \sum_{K\in T_h} \frac{1}{2}(Q_0\boldsymbol{u}\cdot\nabla_d Q_h\boldsymbol{u}, \boldsymbol{w}_0)_K - \sum_{K\in T_h} \frac{1}{2}(Q_0\boldsymbol{u}\cdot\nabla\boldsymbol{w}_0, Q_0\boldsymbol{u})_K \\ &+ \sum_{K\in T_h} \frac{1}{2}(\boldsymbol{w}_0 - \boldsymbol{w}_b, (Q_0\boldsymbol{u}\cdot\boldsymbol{n})Q_0\boldsymbol{u})_{\partial K}. \end{aligned} \tag{4.4}$$

In the same manner, we have

$$\begin{aligned} a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w}_0) &= \sum_{K\in T_h} \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{w}_0)_K - \sum_{K\in T_h} \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}_0, \boldsymbol{u})_K \\ &+ \sum_{K\in T_h} \frac{1}{2}(\boldsymbol{w}_0 - \boldsymbol{w}_b, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u})_{\partial K}. \end{aligned} \tag{4.5}$$

Then

$$\begin{aligned} a_1(Q_h\boldsymbol{u}; Q_h\boldsymbol{u}, \boldsymbol{w}) - a_1(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w}_0) &= \left(\sum_{K\in T_h} \frac{1}{2}(Q_0\boldsymbol{u}\cdot\nabla_d Q_h\boldsymbol{u}, \boldsymbol{w}_0)_K - \sum_{K\in T_h} \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{w}_0)_K\right) \\ &- \left(\sum_{K\in T_h} \frac{1}{2}(Q_0\boldsymbol{u}\cdot\nabla\boldsymbol{w}_0, Q_0\boldsymbol{u})_K - \sum_{K\in T_h} \frac{1}{2}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}_0, \boldsymbol{u})_K\right) \\ &+ \left(\sum_{K\in T_h} \frac{1}{2}(\boldsymbol{w}_0 - \boldsymbol{w}_b, (Q_0\boldsymbol{u}\cdot\boldsymbol{n})Q_0\boldsymbol{u})_{\partial K} - \sum_{K\in T_h} \frac{1}{2}(\boldsymbol{w}_0 - \boldsymbol{w}_b, (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u})_{\partial K}\right). \end{aligned} \tag{4.6}$$

From Lemma (2.1), add and subtract the terms $(\boldsymbol{u}\cdot R_h(\nabla\boldsymbol{u}), \boldsymbol{w}_0)$, $(\boldsymbol{u}\cdot\nabla\boldsymbol{w}_0, Q_0\boldsymbol{u})$, $(\boldsymbol{w}_0 - \boldsymbol{w}_b, (\boldsymbol{u}\cdot\boldsymbol{n})Q_0\boldsymbol{u})$, to the Eq.(4.6), we get

$$a_1\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w}_0) = a_1(Q_h\boldsymbol{u}; Q_h\boldsymbol{u}, \boldsymbol{w}) - \sum_{i=1}^{6} I_i(\boldsymbol{u}, \boldsymbol{w}), \tag{4.7}$$

where

$$\sum_{i=1}^{6} I_i(\boldsymbol{u}, \boldsymbol{w}) = \frac{1}{2}\sum_{K\in T_h} ((Q_0\boldsymbol{u} - \boldsymbol{u})\cdot R_h(\nabla\boldsymbol{u}), \boldsymbol{w}_0)_K + \frac{1}{2}\sum_{K\in T_h} (\boldsymbol{u}\cdot(R_h(\nabla\boldsymbol{u}) - \nabla\boldsymbol{u}), \boldsymbol{w}_0)_K$$

$$
-\frac{1}{2}\sum_{K\in T_h}((Q_0\boldsymbol{u}-\boldsymbol{u})\cdot\nabla\boldsymbol{w}_0, Q_0\boldsymbol{u})_K-\frac{1}{2}\sum_{K\in T_h}(\boldsymbol{u}\cdot\nabla\boldsymbol{w}_0, Q_0\boldsymbol{u}-\boldsymbol{u})_K
$$

$$
+\frac{1}{2}\sum_{K\in T_h}(\boldsymbol{w}_0-\boldsymbol{w}_b,(Q_0\boldsymbol{u}-\boldsymbol{u})\cdot\boldsymbol{n}\,Q_0\boldsymbol{u})_{\partial K}+\frac{1}{2}\sum_{K\in T_h}(\boldsymbol{w}_0-\boldsymbol{w}_b,(\boldsymbol{u}\cdot\boldsymbol{n})\,(Q_0\boldsymbol{u}-\boldsymbol{u}))_{\partial K}. \tag{4.8}
$$

Let $\boldsymbol{w}\in\boldsymbol{W}_h^0$ be a test function, testing equation (1.1) by $\boldsymbol{w}_0$, we have

$$
(\boldsymbol{u}_t,\boldsymbol{w}_0)+(-\epsilon\nabla^2\boldsymbol{u},\boldsymbol{w}_0)+((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{w}_0)=(\boldsymbol{f},\boldsymbol{w}_0). \tag{4.9}
$$

To estimate the error, we need to reformulate equation (4.9) as following:
Integration by part for the second term, we get

$$
\sum_{K\in T_h}(-\epsilon\nabla\cdot(\nabla\boldsymbol{u}),\boldsymbol{w}_0)_K=\sum_{K\in T_h}(\epsilon\nabla\boldsymbol{u},\nabla\boldsymbol{w}_0)_K-\sum_{K\in T_h}(\boldsymbol{w}_0,\epsilon\nabla\boldsymbol{u}\cdot\boldsymbol{n})_{\partial K}. \tag{4.10}
$$

Substitution Eq.(4.1) in (4.10), we get

$$
\begin{aligned}
\sum_{K\in T_h}(-\epsilon\nabla\cdot(\nabla\boldsymbol{u}),\boldsymbol{w}_0)_K \;&=\; \sum_{K\in T_h}(\epsilon\nabla_d Q_h\boldsymbol{u},\nabla_d\boldsymbol{w})_K+\sum_{K\in T_h}(\epsilon(R_h\nabla\boldsymbol{u})\cdot\boldsymbol{n},\boldsymbol{w}_0-\boldsymbol{w}_b)_{\partial K}\\
&-\;\sum_{K\in T_h}(\boldsymbol{w}_0,\epsilon\nabla\boldsymbol{u}\cdot\boldsymbol{n})_{\partial K}.
\end{aligned} \tag{4.11}
$$

using the fact that $\sum_{K\in T_h}(\epsilon\nabla\boldsymbol{u}\cdot\boldsymbol{n},\boldsymbol{w}_b)_{\partial K}=0$, after adding it, we obtain

$$
\sum_{K\in T_h}(-\epsilon\nabla\cdot(\nabla\boldsymbol{u}),\boldsymbol{w}_0)_K=\sum_{K\in T_h}(\epsilon\nabla_d Q_h\boldsymbol{u},\nabla_d\boldsymbol{w})_K+\sum_{K\in T_h}(\epsilon(R_h\nabla\boldsymbol{u}-\nabla\boldsymbol{u})\cdot\boldsymbol{n},\boldsymbol{w}_0-\boldsymbol{w}_b)_{\partial K}. \tag{4.12}
$$

In other words it's really

$$
\sum_{K\in T_h}(-\epsilon\nabla\cdot(\nabla\boldsymbol{u}),\boldsymbol{w}_0)_K=a_0(Q_h\boldsymbol{u},\boldsymbol{w})-I_7(\boldsymbol{u},\boldsymbol{w}), \tag{4.13}
$$

where $\quad I_7(\boldsymbol{u},\boldsymbol{w})=\sum_{K\in T_h}(\epsilon(\nabla\boldsymbol{u}-R_h\nabla\boldsymbol{u})\cdot\boldsymbol{n},\boldsymbol{w}_0-\boldsymbol{w}_b)_{\partial K}$.
Substitution (4.7) and (4.13) in (4.9), with fact $(\boldsymbol{u}_t,\boldsymbol{w}_0)=(Q_h\boldsymbol{u}_t,\boldsymbol{w}_0)$ and adding the term $s(Q_h\boldsymbol{u},\boldsymbol{w})$. to both side, gives

$$
(\boldsymbol{f},\boldsymbol{w}_0)+s(Q_h\boldsymbol{u},\boldsymbol{w})=(Q_h\boldsymbol{u}_t,\boldsymbol{w}_0)+a(Q_h\boldsymbol{u};Q_h\boldsymbol{u},\boldsymbol{w})-\sum_{i=1}^{7}I_i(\boldsymbol{u},\boldsymbol{w}). \tag{4.14}
$$

Subtract (3.13) from (4.14), we have an error equation

$$
((\boldsymbol{u}_h-Q_h\boldsymbol{u})_t,\boldsymbol{w}_0)+a(\boldsymbol{u}_h-Q_h\boldsymbol{u};\boldsymbol{u}_h-Q_h\boldsymbol{u},\boldsymbol{w})=\sum_{i=1}^{8}I_i(\boldsymbol{u},\boldsymbol{w}), \tag{4.15}
$$

where $I_8(\boldsymbol{w})=s(Q_h\boldsymbol{u},\boldsymbol{w})$.
To estimate $I_i(\boldsymbol{w}), i=1\sim 8$, We need the following inequality [3] with using Lemma (3.2)

$$
\sum_{K\in T_h}h_K^{-1}\|\boldsymbol{w}_0-\boldsymbol{w}_b\|_{\partial K}^2\le C\|\boldsymbol{w}\|_{h,1}^2\le C|||\boldsymbol{w}|||^2. \tag{4.16}
$$

We can estimate I_i terms in the error equation (4.15), by using Cauchy-Schwarz inequality, Lemma (2.2) and from definition of $|||\boldsymbol{w}|||$, as following

$$\begin{aligned} |I_1(\boldsymbol{u},\boldsymbol{w})| &= \left|\frac{1}{2}\sum_{K\in T_h}((Q_0\boldsymbol{u}-\boldsymbol{u})\cdot R_h(\nabla\boldsymbol{u}),\boldsymbol{w}_0)_K\right| \\ &\leq \frac{1}{2}\|\nabla\boldsymbol{u}\|_\infty\left(\sum_{K\in T_h}\|Q_0\boldsymbol{u}-\boldsymbol{u}\|_K\|\boldsymbol{w}_0\|_K\right) \\ &\leq Ch^{(k+1)}\|\boldsymbol{u}\|_{k+1}\|\boldsymbol{w}_0\| \leq Ch^k\|\boldsymbol{u}\|_{k+1}|||\boldsymbol{w}|||. \end{aligned}$$

In the same manner for $I_i, i = 2 \sim 8$, therefore, we have

$$|\sum_{i=1}^{8} I_i(\boldsymbol{u},\boldsymbol{w})| \leq Ch^k\|\boldsymbol{u}\|_{k+1}|||\boldsymbol{w}|||. \tag{4.17}$$

5. Optimal order error estimates

In this section we derived the optimal order error estimate in L^2-norm for continues and discrete time WG-FEM. Let $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)\bigcap\boldsymbol{H}^2(\Omega)$ and $P_h\boldsymbol{u}$ denote the elliptic projection of u onto finite element space $\boldsymbol{W}_h^0$, which satisfies the following inequality

$$a(P_h\boldsymbol{u};P_h\boldsymbol{u},\boldsymbol{w}) = (-\nabla\cdot(\epsilon\nabla\boldsymbol{u}),\boldsymbol{w}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{w}), \qquad \forall\boldsymbol{w}\in\boldsymbol{W}_h^0. \tag{5.1}$$

Lemma 5.1. *Suppose that the exact solution of the problem (1.1) is so regular that* $\boldsymbol{u}\in\boldsymbol{H}^{k+1}(\Omega)$ *then there exists a constant* C *such that*

$$(\boldsymbol{a})\ |||Q_h\boldsymbol{u}-P_h\boldsymbol{u}||| \leq Ch^k\|\boldsymbol{u}\|_{k+1}, \qquad (\boldsymbol{b})\ \|Q_h\boldsymbol{u}-P_h\boldsymbol{u}\| \leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}.$$

Proof. $(\boldsymbol{a})$, From Equation (5.1), we have

$$a(P_h\boldsymbol{u};P_h\boldsymbol{u},\boldsymbol{w}) = (\boldsymbol{f},\boldsymbol{w}) - (\boldsymbol{u}_t,\boldsymbol{w}), \qquad \forall\boldsymbol{w}\in\boldsymbol{W}_h^0. \tag{5.2}$$

Let $\boldsymbol{\theta} = Q_h\boldsymbol{u} - P_h\boldsymbol{u}$, testing (1.1) by $\boldsymbol{w}\in\boldsymbol{W}_h^0$, similarity for equation (4.14), we have

$$a(Q_h\boldsymbol{u};Q_h\boldsymbol{u},\boldsymbol{w}) = (\boldsymbol{f},\boldsymbol{w}) - (\boldsymbol{u}_t,\boldsymbol{w}) + \sum_{i=1}^{8} I_i(\boldsymbol{u},\boldsymbol{w}), \qquad \forall\boldsymbol{w}\in\boldsymbol{W}_h^0. \tag{5.3}$$

Subtract (5.2) from (5.3), we get

$$a(\boldsymbol{\theta};\boldsymbol{\theta},\boldsymbol{w}) = \sum_{i=1}^{8} I_i(\boldsymbol{u},\boldsymbol{w}), \qquad \forall\boldsymbol{w}\in\boldsymbol{W}_h^0. \tag{5.4}$$

Setting $\boldsymbol{w} = \boldsymbol{\theta}$ and coercivity of the trilinear form $a(\cdot;\cdot,\cdot)$, we obtain

$$|||\boldsymbol{\theta}|||^2 \leq C|\sum_{i=1}^{8} I_i(\boldsymbol{u},\boldsymbol{\theta})|. \tag{5.5}$$

From Eq.(4.17), we have

$$|\sum_{i=1}^{8} I_i(\boldsymbol{u},\boldsymbol{\theta})| \leq Ch^k\|\boldsymbol{u}\|_{k+1}|||\boldsymbol{\theta}|||. \tag{5.6}$$

Subsituation (5.6) in (5.5), we complete the prove.
To prove part ***(b)***, we use the dual problem, find $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega) \bigcap \boldsymbol{H}^2(\Omega)$, satisfy

$$-\nabla \cdot (\epsilon \nabla \boldsymbol{\phi}) + (\boldsymbol{\phi} \cdot \nabla)\boldsymbol{\phi} = \boldsymbol{\theta}, \qquad in \qquad \Omega. \tag{5.7}$$

and $\boldsymbol{\phi}$ is $\boldsymbol{H}^2$-regularity i.e. there exists a positive constant C such that $\|\boldsymbol{\phi}\|_2 \leq C\|\boldsymbol{\theta}\|$
Testing Eq.(5.7) by $\boldsymbol{\theta}$

$$\begin{aligned} \|\boldsymbol{\theta}\|^2 &= (-\nabla \cdot (\epsilon \nabla \boldsymbol{\phi}), \boldsymbol{\theta}) + ((\boldsymbol{\phi} \cdot \nabla)\boldsymbol{\phi}, \boldsymbol{\theta}) \\ &= a_0(Q_h\boldsymbol{\phi}, \boldsymbol{\theta}) - I_7(\boldsymbol{\phi}, \boldsymbol{\theta}) + a_1(Q_h\boldsymbol{\phi}, Q_h\boldsymbol{\phi}, \boldsymbol{\theta}) - \sum_{i=1}^{6} I_i(\boldsymbol{\phi}, \boldsymbol{\theta}) \\ &= a(Q_h\boldsymbol{\phi}, Q_h\boldsymbol{\phi}, \boldsymbol{\theta}) - s(Q_h\boldsymbol{\phi}, \boldsymbol{\theta}) - \sum_{i=1}^{7} I_i(\boldsymbol{\phi}, \boldsymbol{\theta}). \end{aligned} \tag{5.8}$$

From Eq.(5.4) with $\boldsymbol{w} = Q_h\boldsymbol{\phi}$, we get

$$\|\boldsymbol{\theta}\|^2 = \sum_{i=1}^{7} I_i(\boldsymbol{\phi}, \boldsymbol{\theta}) + \sum_{i=1}^{8} I_i(\boldsymbol{u}, Q_h\boldsymbol{\phi}) - s(Q_h\boldsymbol{\phi}, \boldsymbol{\theta}). \tag{5.9}$$

For $I_i(\boldsymbol{\phi}, \boldsymbol{\theta})$, we use Cauchy-Schwarz inequality, Lemma (2.2), trace inequality (2.14), Ponicaré inequality and embedding theorem, we obtain

$$\begin{aligned} |I_1(\boldsymbol{\phi}, \boldsymbol{\theta})| &= \left| \frac{1}{2} \sum_{K \in T_h} ((Q_0\boldsymbol{\phi} - \boldsymbol{\phi}) \cdot R_h(\nabla \boldsymbol{\phi}), \boldsymbol{\theta}_0)_K \right| \\ &\leq \frac{1}{2} \sum_{K \in T_h} \|Q_0\boldsymbol{\phi} - \boldsymbol{\phi}\| \|\nabla \boldsymbol{\phi}\| \|\boldsymbol{\theta}_0\| \\ &\leq Ch^2 \|\boldsymbol{\phi}\|_2 \|\|\boldsymbol{\phi}\|_1 \|\|\boldsymbol{\theta}\|\| \leq Ch^2 \|\boldsymbol{\phi}\|_2 \|\|\boldsymbol{\phi}\|_2 \|\|\boldsymbol{\theta}\|\| \\ &\leq Ch^2 \|\boldsymbol{\phi}\|_2^2 \|Ch^k \|\boldsymbol{u}\|_{k+1} \leq Ch^{k+2} \|\boldsymbol{u}\|_{k+1} \|\|\boldsymbol{\theta}\|, \end{aligned}$$

$$\begin{aligned} |I_2(\boldsymbol{\phi}, \boldsymbol{\theta})| &= \left| \frac{1}{2} \sum_{K \in T_h} (\boldsymbol{\phi} \cdot (R_h(\nabla \boldsymbol{\phi}) - \nabla \boldsymbol{\phi}), \boldsymbol{\theta}_0)_K \right| \\ &\leq \frac{1}{2} \sum_{K \in T_h} \|\boldsymbol{\phi}\| \|R_h(\nabla \boldsymbol{\phi}) - \nabla \boldsymbol{\phi}\| \|\boldsymbol{\theta}_0\| \\ &\leq \|\boldsymbol{\phi}\|_2 (Ch\|\boldsymbol{\phi}\|_2) \|\|\boldsymbol{\theta}\|\| \leq (Ch\|\boldsymbol{\phi}\|_2^2) Ch^k \|\boldsymbol{u}\|_{k+1} \\ &\leq Ch^{k+1} \|\boldsymbol{u}\|_{k+1} \|\|\boldsymbol{\theta}\|, \end{aligned}$$

$$\begin{aligned} |I_3(\boldsymbol{\phi}, \boldsymbol{\theta})| &= \left| -\frac{1}{2} \sum_{K \in T_h} ((Q_0\boldsymbol{\phi} - \boldsymbol{\phi}) \cdot \nabla \boldsymbol{\theta}_0), Q_0\boldsymbol{\phi})_K \right| \\ &\leq \frac{1}{2} \sum_{K \in T_h} \|Q_0\boldsymbol{\phi} - \boldsymbol{\phi}\| \|\nabla \boldsymbol{\theta}_0\| \|Q_0\boldsymbol{\phi}\| \\ &\leq Ch^2 \|\boldsymbol{\phi}\|_2 \|\|\boldsymbol{\theta}\|\| (\|\boldsymbol{\phi}\| + \|\boldsymbol{\phi} - Q_0\boldsymbol{\phi}\|) \\ &\leq Ch^2 \|\boldsymbol{\phi}\|_2 Ch^k \|\boldsymbol{u}\|_{k+1} \| (C\|\boldsymbol{\phi}\|_2 + Ch^2 \|\boldsymbol{\phi}\|_2) \\ &\leq Ch^{k+2} \|\boldsymbol{u}\|_{k+1} \|\|\boldsymbol{\theta}\| + Ch^{k+4} \|\boldsymbol{u}\|_{k+1} \|\|\boldsymbol{\theta}\| \\ &\leq Ch^{k+2} \|\boldsymbol{u}\|_{k+1} \|\|\boldsymbol{\theta}\|, \end{aligned}$$

$$
\begin{aligned}
|I_4(\boldsymbol{\phi},\boldsymbol{\theta})| &= \left| -\frac{1}{2}\sum_{K\in T_h} (\boldsymbol{\phi}\cdot\nabla\boldsymbol{\theta}_0, Q_0\boldsymbol{\phi}-\boldsymbol{\phi})_K \right| \\
&\leq \frac{1}{2}\sum_{K\in T_h} \|\boldsymbol{\phi}\|\|\nabla\boldsymbol{\theta}_0\|\|Q_0\boldsymbol{\phi}-\boldsymbol{\phi}\| \\
&\leq \|\boldsymbol{\phi}\|_2 |||\boldsymbol{\theta}|||(Ch^2\|\boldsymbol{\phi}\|_2) \leq \|\boldsymbol{\phi}\|_2 (Ch^k\|\boldsymbol{u}\|_{k+1})(Ch^2\|\boldsymbol{\phi}\|_2) \\
&\leq Ch^{k+2}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|,
\end{aligned}
$$

$$
\begin{aligned}
|I_5(\boldsymbol{\phi},\boldsymbol{\theta})| &= \left| \frac{1}{2}\sum_{K\in T_h} (\boldsymbol{\theta}_0-\boldsymbol{\theta}_b, (Q_0\boldsymbol{\phi}-\boldsymbol{\phi})\cdot\boldsymbol{n}\, Q_0\boldsymbol{\phi})_{\partial K} \right| \\
&\leq \frac{1}{2}\sum_{K\in T_h} (h^{-\frac{1}{2}}\|\boldsymbol{\theta}_0-\boldsymbol{\theta}_b\|_{\partial K})(h^{\frac{1}{2}}\|Q_0\boldsymbol{\phi}-\boldsymbol{\phi}\|_{\partial K}\|Q_0\boldsymbol{\phi}\|_{\partial K}) \\
&\leq |||\boldsymbol{\theta}|||h^{\frac{1}{2}}(h^{-\frac{1}{2}}\|Q_0\boldsymbol{\phi}-\boldsymbol{\phi}\| + h^{\frac{1}{2}}\|\nabla(Q_0\boldsymbol{\phi}-\boldsymbol{\phi})\|)(h^{-\frac{1}{2}}\|Q_0\boldsymbol{\phi}\| + h^{\frac{1}{2}}\|\nabla Q_0\boldsymbol{\phi}\|) \\
&\leq |||\boldsymbol{\theta}|||(h^2\|\boldsymbol{\phi}\|_2)h^{-\frac{1}{2}}(\|\boldsymbol{\phi}\| + \|\boldsymbol{\phi}-Q_0\boldsymbol{\phi}\|) + h^{\frac{1}{2}}(\|\boldsymbol{\phi}\|_1 + \|\boldsymbol{\phi}-Q_0\boldsymbol{\phi}\|_1) \\
&\leq |||\boldsymbol{\theta}|||(h^2\|\boldsymbol{\phi}\|_2)(h^{-\frac{1}{2}}(C\|\boldsymbol{\phi}\|_2 + Ch^2\|\boldsymbol{\phi}\|_2) + h^{\frac{1}{2}}(C\|\boldsymbol{\phi}\|_2 + Ch\|\boldsymbol{\phi}\|_2)) \\
&\leq Ch^{\frac{3}{2}}|||\boldsymbol{\theta}|||\|\boldsymbol{\phi}\|_2^2 + Ch^{\frac{7}{2}}|||\boldsymbol{\theta}|||\|\boldsymbol{\phi}\|_2^2 + Ch^{\frac{5}{2}}|||\boldsymbol{\theta}|||\|\boldsymbol{\phi}\|_2^2 + Ch^{\frac{7}{2}}|||\boldsymbol{\theta}|||\|\boldsymbol{\phi}\|_2^2 \\
&\leq (Ch^{k+\frac{3}{2}} + Ch^{k+\frac{7}{2}} + Ch^{k+\frac{5}{2}} + Ch^{k+\frac{7}{2}})\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\| \leq Ch^{k+\frac{3}{2}}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|.
\end{aligned}
$$

Similarity for $|I_6(\boldsymbol{\phi},\boldsymbol{\theta})| \leq Ch^{k+\frac{3}{2}}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|$,

$$
\begin{aligned}
|I_7(\boldsymbol{\phi},\boldsymbol{\theta})| &= \left| \sum_{K\in T_h} (\epsilon(\nabla\boldsymbol{\phi}-R_h\nabla\boldsymbol{\phi})\cdot\boldsymbol{n}, \boldsymbol{\theta}_0-\boldsymbol{\theta}_0)_{\partial K} \right| \\
&\leq \Big(\sum_{K\in T_h} h_k\|\epsilon(\nabla\boldsymbol{\phi}-R_h\nabla\boldsymbol{\phi})\|_{\partial K}^2\Big)^{\frac{1}{2}}\Big(\sum_{K\in T_h} h_k^{-1}\|\boldsymbol{\theta}_0-\boldsymbol{\theta}_b\|_{\partial K}^2\Big)^{\frac{1}{2}} \\
&\leq \Big(\sum_{K\in T_h} \|\epsilon(\nabla\boldsymbol{\phi}-R_h\nabla\boldsymbol{\phi})\|_K^2 + h_k^2\|\nabla(\epsilon(\nabla\boldsymbol{\phi}-R_h\nabla\boldsymbol{\phi}))\|_K^2\Big)^{\frac{1}{2}}|||\boldsymbol{\theta}||| \\
&\leq Ch^2\|\boldsymbol{\phi}\|_2|||\boldsymbol{\theta}||| \\
&\leq Ch^{k+2}\|\boldsymbol{u}\|_{k+1}\|\boldsymbol{\theta}\|.
\end{aligned}
$$

For $I_i(\boldsymbol{u}, Q_h\boldsymbol{\phi})$, we use Cauchy-Schwarz inequality, Lemma (2.2), trace inequality (2.14), embedding theorem, we obtain

$$
\begin{aligned}
|I_1(u,Q_h\phi)| &= \left| \frac{1}{2}\sum_{K\in T_h} ((Q_0\boldsymbol{u}-\boldsymbol{u})\cdot R_h(\nabla\boldsymbol{u}), Q_0\boldsymbol{\phi})_K \right| \\
&\leq \frac{1}{2}\sum_{K\in T_h} \|Q_0\boldsymbol{u}-\boldsymbol{u}\|\cdot\|\nabla\boldsymbol{u}\|_\infty\|Q_0\boldsymbol{\phi}\| \\
&\leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|(\|\boldsymbol{\phi}\| + \|\boldsymbol{\phi}-Q_0\boldsymbol{\phi}\|) \\
&\leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|(C\|\boldsymbol{\phi}\|_2 + Ch^2\|\boldsymbol{\phi}\|_2) \leq Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|.
\end{aligned}
$$

Similarity for $I_2(\boldsymbol{u},Q_h\boldsymbol{\phi})\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|$,

$$\begin{aligned}
|I_3(\boldsymbol{u},Q_h\boldsymbol{\phi})| &= \left|-\frac{1}{2}\sum_{K\in T_h}((Q_0\boldsymbol{u}-\boldsymbol{u})\cdot\nabla Q_0\boldsymbol{\phi}),Q_0\boldsymbol{u})_K\right|\\
&\le \frac{1}{2}\sum_{K\in T_h}\|(Q_0\boldsymbol{u}-\boldsymbol{u}\|\|\nabla Q_0\boldsymbol{\phi})\|\|Q_0\boldsymbol{u}\|\\
&\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\|Q_0\boldsymbol{\phi}\|_1\\
&\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|(\|\boldsymbol{\phi}\|_1+\|\boldsymbol{\phi}-Q_0\boldsymbol{\phi}\|_1)\\
&\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|(C\|\boldsymbol{\phi}\|_2+Ch\|\boldsymbol{\phi}\|_2)\\
&\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|(C\|\boldsymbol{\theta}\|+Ch\|\boldsymbol{\theta}\|)\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|.
\end{aligned}$$

Similarity for $|I_4(\boldsymbol{u},Q_h\boldsymbol{\phi})|\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|$,

$$\begin{aligned}
|I_5(\boldsymbol{u},Q_h\boldsymbol{\phi})| &= \left|\frac{1}{2}\sum_{K\in T_h}(Q_0\boldsymbol{\phi}-Q_b\boldsymbol{\phi},(Q_0\boldsymbol{u}-\boldsymbol{u})\cdot\boldsymbol{n}\,Q_0\boldsymbol{u}\boldsymbol{a})_{\partial K}\right|\\
&\le \frac{1}{2}\sum_{K\in T_h}\|Q_0\boldsymbol{\phi}-Q_b\boldsymbol{\phi}\|_{\partial K}\|Q_0\boldsymbol{u}-\boldsymbol{u}\|_{\partial K}\|Q_0\boldsymbol{u}\|_{\partial K}\\
&\le Ch^{\frac{3}{2}}\|\boldsymbol{\phi}\|_2(h^{-1}\|Q_0\boldsymbol{u}-\boldsymbol{u}\|_K^2+h\|\nabla(Q_0\boldsymbol{u}-\boldsymbol{u})\|_K^2)^{\frac{1}{2}}\\
&\le Ch^{\frac{3}{2}}\|\boldsymbol{\theta}\|h^{-\frac{1}{2}}Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\le Ch^{k+2}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|.
\end{aligned}$$

Similarity for $|I_6(\boldsymbol{u},Q_h\boldsymbol{\phi})|\le Ch^{k+2}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|$,

$$\begin{aligned}
|I_7(\boldsymbol{u},Q_h\boldsymbol{\phi})| &= \left|\sum_{K\in T_h}(\epsilon(\nabla\boldsymbol{u}-R_h\nabla\boldsymbol{u})\cdot\boldsymbol{n},Q_0\boldsymbol{\phi}-Q_b\boldsymbol{\phi})_{\partial K}\right|\\
&\le \sum_{K\in T_h}\|\epsilon(\nabla\boldsymbol{u}-R_h\nabla\boldsymbol{u}\|_{\partial K}\|Q_0\boldsymbol{\phi}-Q_b\boldsymbol{\phi}\|_{\partial K}\\
&\le (h^{-1}\|\nabla\boldsymbol{u}-R_h\nabla\boldsymbol{u}\|_K^2+h\|\nabla(\nabla\boldsymbol{u}-R_h\nabla\boldsymbol{u})\|_K^2)^{\frac{1}{2}}Ch^{\frac{3}{2}}\|\boldsymbol{\phi}\|_2\\
&\le (h^{-1}Ch^{2k}\|\boldsymbol{u}\|_{k+1}^2+hCh^{2k-1}\|\boldsymbol{u}\|_{k+1}^2)^{\frac{1}{2}}Ch^{\frac{3}{2}}\|\boldsymbol{\phi}\|_2\\
&\le (Ch^{2k-1}\|\boldsymbol{u}\|_{k+1}^2)^{\frac{1}{2}}Ch^{\frac{3}{2}}\|\boldsymbol{\phi}\|_2\\
&\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|,
\end{aligned}$$

$$\begin{aligned}
|I_8(\boldsymbol{u},Q_h\boldsymbol{\phi})| &= \left|\sum_{K\in T_h}h_k^{-1}(Q_b(Q_0\boldsymbol{u})-Q_b\boldsymbol{u},Q_0\boldsymbol{\phi}-Q_b\boldsymbol{\phi})_{\partial K}\right|\\
&\le (\sum_{K\in T_h}h_k^{-1}\|Q_b(Q_0\boldsymbol{u})-Q_b\boldsymbol{u}\|_{\partial K}^2)^{\frac{1}{2}}(\sum_{K\in T_h}h_K^{-1}\|Q_0\boldsymbol{\phi}-Q_b\boldsymbol{\phi}\|_{\partial K}^2)^{\frac{1}{2}}\\
&\le (Ch^{2k}\|\boldsymbol{u}\|_{k+1}^2)^{\frac{1}{2}}h^{-\frac{1}{2}}\|Q_0\boldsymbol{\phi}-Q_b\boldsymbol{\phi}\|_{\partial K}\\
&\le Ch^k\|\boldsymbol{u}\|_{k+1}^2h^{-\frac{1}{2}}Ch^{\frac{3}{2}}\|\boldsymbol{\phi}\|_2\le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\|\boldsymbol{\theta}\|,
\end{aligned}$$

$$\begin{aligned}
|s(\boldsymbol{\theta}, Q_h\boldsymbol{\phi})| &= \left|\sum_{K\in T_h} h_k^{-1}(Q_b(Q_0\boldsymbol{\phi}) - Q_b\boldsymbol{\phi}, Q_b\boldsymbol{\theta}_0 - \boldsymbol{\theta}_b)_{\partial K}\right| \\
&= \left|\sum_{K\in T_h} h_k^{-1}(Q_0\boldsymbol{\phi} - \boldsymbol{\phi}, Q_b\boldsymbol{\theta}_0 - \boldsymbol{\theta}_b)_{\partial K}\right| \\
&\le (\sum_{K\in T_h} h_k^{-1}\|Q_0\boldsymbol{\phi} - \boldsymbol{\phi}\|^2_{\partial K})^{\frac{1}{2}} (\sum_{K\in T_h} h_k^{-1}\|Q_b\boldsymbol{\theta}_0 - \boldsymbol{\theta}_b\|_{\partial K})^{\frac{1}{2}} \\
&\le Ch\|\boldsymbol{\phi}\|_2 |||\boldsymbol{\theta}||| \le Ch\|\boldsymbol{\theta}\| Ch^k\|\boldsymbol{u}\|_{k+1} \le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}\|\boldsymbol{\theta}\|.
\end{aligned}$$

Substituting $\sum_{i=1}^{8} I_i(\boldsymbol{u}, Q_h\boldsymbol{\phi})$, $\sum_{i=1}^{7} I_i(\boldsymbol{\phi}, \boldsymbol{\theta})$ and $s(Q_h\boldsymbol{\phi}, \boldsymbol{\theta})$ into (5.9), we complete proof part ***(b)*** ☐

5.1. **Error estimate for the continuous time WG scheme.**

Theorem 5.1. *Suppose that $\boldsymbol{u}(x, y, t), \boldsymbol{u}_h(x, y, t)$ be the solutions to the Burgers' equation (1.1) and the continuous time WG scheme (3.13), respectively, assume that the exact solution is so regular that $\boldsymbol{u}, \boldsymbol{u}_t \in \boldsymbol{H}^{k+1}(\Omega)$. Then there exists a constant C such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2 \le C\left(\|\boldsymbol{u}^0 - \boldsymbol{u}_h^0\|^2 + h^{2(k+1)}(\|\boldsymbol{u}^0\|^2_{k+1} + \int_0^t \|\boldsymbol{u}_t\|^2_{k+1}dt)\right). \tag{5.10}$$

Proof. Suppose that $\rho^{\boldsymbol{u}} = \boldsymbol{u} - Q_h\boldsymbol{u}$, $\mu^{\boldsymbol{u}} = Q_h\boldsymbol{u} - P_h\boldsymbol{u}$, $e^{\boldsymbol{u}} = P_h\boldsymbol{u} - \boldsymbol{u}_h$, we can write

$$\boldsymbol{u} - \boldsymbol{u}_h = \rho^{\boldsymbol{u}} + \mu^{\boldsymbol{u}} + e^{\boldsymbol{u}}. \tag{5.11}$$

From Lemma(2.2), we have

$$\|\rho^{\boldsymbol{u}}\| \le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}, \qquad \|\rho_t^{\boldsymbol{u}}\| \le Ch^{k+1}\|\boldsymbol{u}_t\|_{k+1}, \tag{5.12}$$

$$\|\mu^{\boldsymbol{u}}\| \le Ch^{k+1}\|\boldsymbol{u}\|_{k+1}, \qquad \|\mu_t^{\boldsymbol{u}}\| \le Ch^{k+1}\|\boldsymbol{u}_t\|_{k+1}. \tag{5.13}$$

We must estimate $e^{\boldsymbol{u}}$, we can write

$$\begin{aligned}
(e_t^{\boldsymbol{u}}, \boldsymbol{w}) + a(e^{\boldsymbol{u}}; e^{\boldsymbol{u}}, \boldsymbol{w}) &= (P_h\boldsymbol{u}_t, \boldsymbol{w}) + a(P_h\boldsymbol{u}, P_h\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{u}_{h,t}, \boldsymbol{w}) - a(\boldsymbol{u}_h; u_h, \boldsymbol{w}) \\
&= (P_h\boldsymbol{u}_t, \boldsymbol{w}) + a(P_h\boldsymbol{u}, P_h\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{f}, \boldsymbol{w}) \\
&= (P_h\boldsymbol{u}_t, \boldsymbol{w}) - (\nabla\cdot(\epsilon\nabla\boldsymbol{u}), \boldsymbol{w}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{w}) - (\boldsymbol{f}, \boldsymbol{w}) \\
&= (P_h\boldsymbol{u}_t, \boldsymbol{w}) - (Q_h\boldsymbol{u}_t, \boldsymbol{w}) + (Q_h\boldsymbol{u}_t, \boldsymbol{w}) - (\boldsymbol{u}_t, \boldsymbol{w}) \\
&= -(\mu_t^{\boldsymbol{u}}, \boldsymbol{w}) - (\rho_t^{\boldsymbol{u}}, \boldsymbol{w}), \ \forall \boldsymbol{w} \in \boldsymbol{W}_h^0.
\end{aligned} \tag{5.14}$$

Setting $\boldsymbol{w} = e^{\boldsymbol{u}}$, we have
$(e_t^{\boldsymbol{u}}, e^{\boldsymbol{u}}) + a(e^{\boldsymbol{u}}; e^{\boldsymbol{u}}, e^{\boldsymbol{u}}) = -(\mu_t^{\boldsymbol{u}}, e^{\boldsymbol{u}}) - (\rho_t^{\boldsymbol{u}}, e^{\boldsymbol{u}})$.
By coercivity of the trilinear form $a(.;.,.)$, Cauchy-Schwarz inequality, Young's inequality, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|e^{\boldsymbol{u}}\|^2 + \delta|||e^{\boldsymbol{u}}|||^2 &\le -(\mu_t^{\boldsymbol{u}}, e^{\boldsymbol{u}}) - (\rho_t^{\boldsymbol{u}}, e^{\boldsymbol{u}}) \\
\frac{d}{dt}\|e^{\boldsymbol{u}}\|^2 &\le C(\|\mu_t^{\boldsymbol{u}}\|^2 + \|\rho_t^{\boldsymbol{u}}\|^2 + \|e^{\boldsymbol{u}}\|^2).
\end{aligned}$$

Integration with respect to t, we get the following inequality

$$\|e^{\boldsymbol{u}}\|^2 \le \|e(.,0)\|^2 + C(\int_0^t \|\mu_\tau^{\boldsymbol{u}}\|^2 d\tau + \int_0^t \|\rho_\tau^{\boldsymbol{u}}\|^2 d\tau + \int_0^t \|e^{\boldsymbol{u}}\|^2 d\tau). \tag{5.15}$$

For $\|e(.,0)\|$, we use Lemma (2.2) and Lemma(5.1)

$$\begin{aligned}\|e(.,0)\| &= \|P_h\boldsymbol{u}^0 - \boldsymbol{u}_h^0\| = \|P_h\boldsymbol{u}^0 - Q_0\boldsymbol{u}^0 + Q_0\boldsymbol{u}^0 - \boldsymbol{u}^0 + \boldsymbol{u}^0 - \boldsymbol{u}_h^0\| \\ &\leq \|P_h\boldsymbol{u}^0 - Q_0\boldsymbol{u}^0\| + \|Q_0\boldsymbol{u}^0 - \boldsymbol{u}^0\| + \|\boldsymbol{u}^0 - \boldsymbol{u}_h^0\| \\ &\leq Ch^{k+1}\|\boldsymbol{u}^0\|_{k+1} + \|\boldsymbol{u}^0 - u_h^0\|. \qquad (5.16)\end{aligned}$$

Substitution (5.12),(5.13) and (5.16) in (5.15), with Gronwall lemma, equation (5.10) holds.

□

5.2. **Error estimate for the discrete time WG scheme.**

Theorem 5.2. *Suppose that $\boldsymbol{u} \in \boldsymbol{H}^{k+1}(\Omega), \boldsymbol{U}_n \in \boldsymbol{W}_h(k, k-1)$ be the solutions to the Burgers' equation (1.1) and the discrete time WG scheme (3.15), respectively, let $\boldsymbol{u}_0, \boldsymbol{u}_t \in \boldsymbol{H}^{k+1}(\Omega)$, then there exists a constant C such that*

$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}_n\| \leq C\left(\|\boldsymbol{u}_0 - \boldsymbol{U}_0\| + \tau\int_0^{t_n}\|\boldsymbol{u}_{tt}\|dt + h^{k+1}(\|\boldsymbol{u}_0\|_{k+1} + \int_0^{t_n}\|\boldsymbol{u}_t\|_{k+1}dt)\right). \qquad (5.17)$$

Proof. In the same manner in Theorem(5.1), we can write

$$\boldsymbol{u}_n - \boldsymbol{U}_n = \rho_n^{\boldsymbol{u}} + \mu_n^{\boldsymbol{u}} + e_n^{\boldsymbol{u}}, \qquad (5.18)$$

where $\rho_n^{\boldsymbol{u}} = \boldsymbol{u}_n - Q_h\boldsymbol{u}_n,\ \ \mu_n^{\boldsymbol{u}} = Q_h\boldsymbol{u}_n - P_h\boldsymbol{u}_n,\ \ e_n^{\boldsymbol{u}} = P_h\boldsymbol{u}_n - \boldsymbol{U}_n$ and $\boldsymbol{u}_n = \boldsymbol{u}(t_n)$, for convenience.
From Lemma(2.2) and Lemma(5.1), we have

$$\|\rho_n^{\boldsymbol{u}}\| \leq Ch^{k+1}\|\boldsymbol{u}_n\|_{k+1} \leq Ch^{k+1}(\|\boldsymbol{u}_0\|_{k+1} + \int_0^{t_n}\|\boldsymbol{u}_\tau\|_{k+1}d\tau). \qquad (5.19)$$

$$\|\mu_n^{\boldsymbol{u}}\| \leq Ch^{k+1}\|\boldsymbol{u}_n\|_{k+1} \leq Ch^{k+1}(\|\boldsymbol{u}_0\|_{k+1} + \int_0^{t_n}\|\boldsymbol{u}_\tau\|_{k+1}d\tau). \qquad (5.20)$$

We must estimate $e^{\boldsymbol{u}}$, we can write

$$\begin{aligned}(\widetilde{\partial}_t e_n^{\boldsymbol{u}}, \boldsymbol{w}) + a(e_n^{\boldsymbol{u}}; e_n^{\boldsymbol{u}}, \boldsymbol{w}) &= (\widetilde{\partial}_t P_h\boldsymbol{u}_n, \boldsymbol{w}) + a(P_h\boldsymbol{u}_n, P_h\boldsymbol{u}_n, \boldsymbol{w}) - (\widetilde{\partial}_t\boldsymbol{U}_n, \boldsymbol{w}) - a(\boldsymbol{U}_n; \boldsymbol{U}_n, \boldsymbol{w}) \\ &= (\widetilde{\partial}_t P_h\boldsymbol{u}_n, \boldsymbol{w}) + a(P_h\boldsymbol{u}_n, P_h\boldsymbol{u}_n, \boldsymbol{w}) - (\boldsymbol{f}_n, \boldsymbol{w}) \\ &= (\widetilde{\partial}_t P_h\boldsymbol{u}_n, \boldsymbol{w}) - (\nabla\cdot(\epsilon\nabla\boldsymbol{u}_n), \boldsymbol{w}) + (\boldsymbol{u}_n\cdot\nabla\boldsymbol{u}_n, \boldsymbol{w}) - (\boldsymbol{f}_n, \boldsymbol{w}) \\ &= (\widetilde{\partial}_t P_h\boldsymbol{u}_n, \boldsymbol{w}) - (\boldsymbol{u}_t, \boldsymbol{w}) \\ &= (\widetilde{\partial}_t P_h\boldsymbol{u}_n, \boldsymbol{w}) - (\widetilde{\partial}_t Q_h\boldsymbol{u}_n, \boldsymbol{w}) \\ &+ (\widetilde{\partial}_t Q_h\boldsymbol{u}_n, \boldsymbol{w}) - (\widetilde{\partial}_t\boldsymbol{u}_n, \boldsymbol{w}) + (\widetilde{\partial}_t\boldsymbol{u}_n, \boldsymbol{w}) - (\boldsymbol{u}_t, \boldsymbol{w}) \\ &= -(\widetilde{\partial}_t\mu_n^{\boldsymbol{u}}, \boldsymbol{w}) - (\widetilde{\partial}_t\rho_n^{\boldsymbol{u}}, \boldsymbol{w}) - (\boldsymbol{u}_t - \widetilde{\partial}_t\boldsymbol{u}_n, \boldsymbol{w}),\ \ \forall\boldsymbol{w}\in\boldsymbol{W}_h^0. \qquad (5.21)\end{aligned}$$

Setting $\boldsymbol{w} = e_n^{\boldsymbol{u}}$, coercivity of the trilinear form $a(.;.,.)$, Cauchy-Schwarz inequality, we get

$$\begin{aligned}(\frac{e_n^{\boldsymbol{u}} - e_{n-1}^{\boldsymbol{u}}}{\tau}, e_n^{\boldsymbol{u}}) + \delta|||e_n^{\boldsymbol{u}}|||^2 &\leq (\|\widetilde{\partial}_t\mu_n^{\boldsymbol{u}}\| + \|\widetilde{\partial}_t\rho_n^{\boldsymbol{u}}\| + \|\boldsymbol{u}_t - \widetilde{\partial}_t\boldsymbol{u}_n\|)\|e_n^{\boldsymbol{u}}\| \\ (\|e_n^{\boldsymbol{u}}\|^2 - \|e_{n-1}^{\boldsymbol{u}}\|\|e_n^{\boldsymbol{u}}\|) &\leq \tau(\|\widetilde{\partial}_t\mu_n^{\boldsymbol{u}}\| + \|\widetilde{\partial}_t\rho_n^{\boldsymbol{u}}\| + \|\boldsymbol{u}_t - \widetilde{\partial}_t\boldsymbol{u}_n\|)\|e_n^{\boldsymbol{u}}\| \\ \|e_n^{\boldsymbol{u}}\| &\leq \|e_{n-1}^{\boldsymbol{u}}\| + \tau(\|\widetilde{\partial}_t\mu_n^{\boldsymbol{u}}\| + \|\widetilde{\partial}_t\rho_n^{\boldsymbol{u}}\| + \|\boldsymbol{u}_t - \widetilde{\partial}_t\boldsymbol{u}_n\|). \qquad (5.22)\end{aligned}$$

By induction

$$\|e_n^{\boldsymbol{u}}\| \leq \|e_0^{\boldsymbol{u}}\| + \tau(\sum_{j=1}^{n} \Lambda_j^1 + \sum_{j=1}^{n} \Lambda_j^2 + \sum_{j=1}^{n} \Lambda_j^3). \tag{5.23}$$

We have

$$\|e_0^{\boldsymbol{u}}\| \leq Ch^{k+1}\|\boldsymbol{u}_0\|_{k+1} + \|\boldsymbol{u}_0 - \boldsymbol{U}_0\|, \tag{5.24}$$

and

$$\widetilde{\partial}_t \rho_j^{\boldsymbol{u}} = -(\widetilde{\partial}_t Q_h \boldsymbol{u}(t_j) - \widetilde{\partial}_t \boldsymbol{u}(t_j)) = -\tau^{-1} \int_{t_{j-1}}^{t_j} (Q_h - I)\boldsymbol{u}_t dt, \tag{5.25}$$

$$\widetilde{\partial}_t \mu_j^{\boldsymbol{u}} = -(\widetilde{\partial}_t Q_h \boldsymbol{u}(t_j) - \widetilde{\partial}_t P_h \boldsymbol{u}(t_j)) = -\tau^{-1} \int_{t_{j-1}}^{t_j} (Q_h - P_h)\boldsymbol{u}_t dt. \tag{5.26}$$

Integration by part, we obtain

$$\widetilde{\partial}_t \boldsymbol{u}(t_j) - \boldsymbol{u}_t(t_j) = -\tau^{-1} \int_{t_{j-1}}^{t_j} (t - t_{j-1})\boldsymbol{u}_{tt} dt. \tag{5.27}$$

It follows from (2.2) and Lemma(5.1) that

$$\sum_{j=1}^{n} \Lambda_j^1 \leq \tau^{-1} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} Ch^{k+1}\|\boldsymbol{u}_t\|_{k+1} dt \leq \frac{C}{\tau} h^{k+1} \int_0^{t_n} \|\boldsymbol{u}_t\|_{k+1} dt, \tag{5.28}$$

$$\sum_{j=1}^{n} \Lambda_j^2 \leq \tau^{-1} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} Ch^{k+1}\|\boldsymbol{u}_t\|_{k+1} dt \leq \frac{C}{\tau} h^{k+1} \int_0^{t_n} \|\boldsymbol{u}_t\|_{k+1} dt, \tag{5.29}$$

$$\sum_{j=1}^{n} \Lambda_j^3 \leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\boldsymbol{u}_{tt}\|_{k+1} dt \leq \int_0^{t_n} \|\boldsymbol{u}_{tt}\|_{k+1} dt. \tag{5.30}$$

Substitution (5.24), (5.28), (5.29) and (5.30) in (5.23), with disecrete Gronwall lemma, we compelet the proof. □

6. Numerical Experiments

In this section, we use the combination of polynomial spaces $\{P_1(K), P_0(\partial K), [P_0(K)]^2\}$ of the numerical approximation i.e., space consisting of piecewise linear polynomial on the triangles and piecewise constants on the edges, we also adopt the L^2-norm and L^∞-norm to present the optimal order error between the exact solution and the numerical solution u_h, we consider two example over square domain $\Omega : [0,1] \times [0,1]$ that divided into $n \times n$ square element uniformly and into 2^{n+1} triangles by the diagonal line for two triangle. The initial and Dirichlet boundary conditions are taken from the analytical solution.

6.1. **Test problem 1.** In this subsection, we consider the system of two dimension Burgers' equations (1.1) over time interval $[0, T] = [0, 1]$. The exact solutions of two dimension Burgers' equation [2] are:

$$u(x, y, t) = -2\epsilon \frac{2\pi e^{-5\pi^2 \epsilon t} cos(2\pi x) sin(\pi y)}{2 + e^{-5\pi^2 \epsilon t} sin(2\pi x) sin(\pi y)},$$

$$v(x, y, t) = -2\epsilon \frac{\pi e^{-5\pi^2 \epsilon t} sin(2\pi x) cos(\pi y)}{2 + e^{-5\pi^2 \epsilon t} sin(2\pi x) sin(\pi y)}.$$

In the test $\epsilon = 10^{-5}, \tau = 10^{-2}$ are used to check the convergence with respect to time step size τ and mesh size $h = \frac{1}{n}, (n = 2, 4, 8, 16, 32, 64)$. Table 1 and 2 show that the L^2 and $L^\infty -$ error with respect to the velocity u and v, Figure 1 show the weak Galerkin solution and exact solution for u and v in case $(T = 1, \tau = 0.01, \epsilon = 10^{-5})$.

h	L^2 error	Order	L^∞ error	Order
1/2	5.3123e-07	-	1.3013e-06	-
1/4	9.7624e-08	2.4440	4.4168e-07	1.5589
1/8	1.7426e-08	2.4860	1.1812e-07	1.9027
1/16	3.0931e-09	2.4941	2.9753e-08	1.9892
1/32	5.7738e-10	2.4215	7.1563e-09	2.0558
1/64	1.1150e-10	2.3725	1.4624e-09	2.2909

TABLE 1. L^2 and L^∞ error for u in case $T = 1$, $\epsilon = 10^{-5}$ and $\tau = 10^{-2}$.

h	L^2 error	Order	L^∞ error	Order
1/2	1.2825e-06	-	3.1415e-06	-
1/4	1.8782e-07	2.7716	6.5072e-07	2.2713
1/8	3.4515e-08	2.4440	2.2107e-07	1.5576
1/16	6.1613e-09	2.4859	5.9353e-08	1.8971
1/32	1.0956e-09	2.4915	1.5191e-08	1.9661
1/64	2.1417e-10	2.3549	3.9046e-09	1.9600

TABLE 2. L^2 and L^∞ error for v in case $T = 1$, $\epsilon = 10^{-5}$ and $\tau = 10^{-2}$.

6.2. **Test problem 2.** In this subsection, we present the test problem to illustrate the backward Euler WG finite elements method for the time dependent coupled Burgers' equations (1.1) over time interval $[0, T] = [0, 1]$. The exact solutions of coupled Burgers equation [2] are:

$$u(x, y, t) = \frac{(x + y - 2xt)}{(1 - 2t^2)}, \qquad v(x, y, t) = \frac{(x - y - 2xt)}{(1 - 2t^2)}.$$

In the test $\tau = 0.01$ and $\epsilon = 100$ are used to check the order of convergence with respect to time step size τ and mesh size $h = \frac{1}{n}, (n = 2, 4, 8, 16, 32, 64)$ the results are shown in Table3, Table4 and Figure 2.

h	L^2 error	Order	L^∞ error	Order
1/2	1.2768e-01	-	1.4221e-01	-
1/4	1.9239e-02	2.7304	2.3120e-02	2.6209
1/8	4.4667e-03	2.1068	5.4657e-03	2.0806
1/16	1.0739e-03	2.0564	1.3498e-03	2.0176
1/32	2.4359e-04	2.1403	3.3661e-04	2.0036
1/64	4.7035e-05	2.3727	8.4140e-05	2.0002

TABLE 3. Numerical results for a test problem2 .

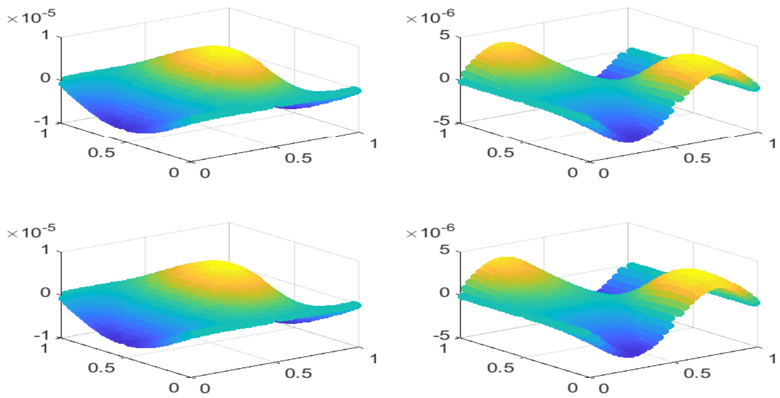

FIGURE 1. Numerical and Exact solution for u and v in case ($T = 1, \tau = 0.01, \epsilon = 10^{-5}$).

h	L^2 error	Order	L^∞ error	Order
1/2	2.4477e-01	-	4.4970e-01	-
1/4	3.8086e-02	2.6841	7.0237e-02	2.6787
1/8	8.8613e-03	2.1037	1.6499e-02	2.0899
1/16	2.1069e-03	2.0724	4.0683e-03	2.0199
1/32	4.5265e-04	2.2186	1.0141e-03	2.0042
1/64	7.9618e-05	2.5072	2.5347e-04	2.0004

TABLE 4. Numerical results for a test problem2 .

7. CONCLUSIONS

The goal of this paper is to obtain the optimal order error by applying the WG-FEM with configuration $(P_k(K), P_{k-1}(\partial K), [P_{k-1}(K)]^2)$ and stabilization term for solving two dimensional coupled Burgers' equations. The optimal order error in L^2- norm is obtained based on dual argument technique, numerically, the WG-FEM in this work gives accurate results and conforms well the theoretical analysis.

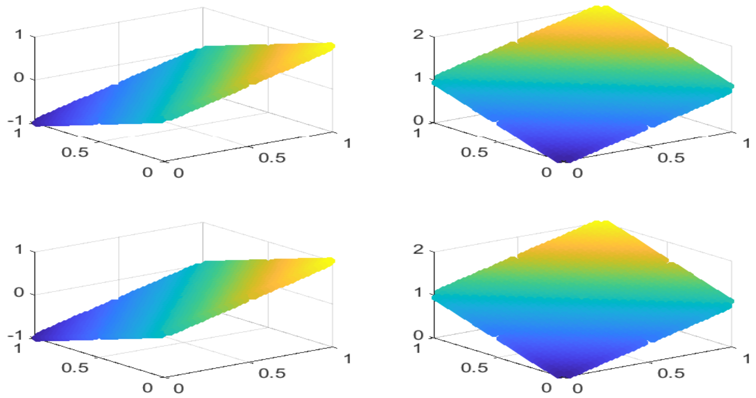

FIGURE 2. Numerical and Exact solution for u and v in case ($T = 1, \tau = 0.01, \epsilon = 100$).

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