TWMS J. App. and Eng. Math. V.12, N.1, 2022, pp. 107-119

MODIFIED DIFFERENTIAL TRANSFORMATION METHOD FOR SOLVING CLASSES OF NON-LINEAR DIFFERENTIAL EQUATIONS

S. ALAHMAD[1], N. R. ANAKIRA[2], M. MAMAT[1], I. SULAIMAN[1], R. ALAHMAD[3], §

Abstract. In this research article, a numerical scheme namely modified differential transformation method (MDTM) is employed successfully to obtain accurate approximate solutions for classes of nonlinear differential equations. This scheme based on differential transform method (DTM), Laplace transform and Padé approximants. Validity and efficiency of MDTM are tested upon several examples and comparisons.are made to demonstrate that. The results lead to conclude that the MDTM is effective, explicit and easy to use.

Keywords: Boundary value problems, Differential transform method, Laplace transform, Padé approximants.

AMS Subject Classification: 41A21, 34B60

1. Introduction

Nonlinear differential equations play an important role in many of fields of applied science and engineering due to it's wide applications in mechanical systems, fluid dynamics and simulation of electrical networks [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Most of these applications are modeled by differential equations especially the nonlinear one which means the equations are difficult to solve either numerically or analytically. Thus, finding exact or approximate solutions for these models are great interesting and very important. Several numerical or approximated methods has been employed to find exact or accurate approximate solutions, for instance, a class of high-order nonlinear differential equations has been solved using a collocation method based on Bessel functions of the first kind [10]. Moreover, Nonlinear differential equations have been solved using the Taylor matrix

[1] Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Terengganu, Malaysia.
e-mail: alahmad.shadi@yu.edu.jo; ORCID: https://orcid.org/0000-0001-7690-1444.
e-mail: must@unisza.edu.my; ORCID: https://orcid.org/0000-0002-4802-3733.
e-mail: kademi4u@yahoo.com; ORCID: https://orcid.org/0000-0001-5246-6636.
[2] Department of Mathematics, Faculty of Science and Technology, Irbid National University, Jordan.
e-mail: alanaghreh_nedal@yahoo.com; ORCID: https://orcid.org/0000-0003-4839-6342.
[3] Faculty of Engineering, Higher Colleges of Technology, Ras Alkhaimah, UAE.
Department of Mathematics, Faculty of Science, Yarmouk University, Irbid, Jordan 21163.
e-mail: rami_thenat@yu.edu.jo; ORCID: https://orcid.org/0000-0002-0233-072x.
§ Manuscript received: November 7, 2019; accepted: January 6, 2020.

method [11], the closed-form method [12], Modified Legendre operational matrix method [13], Optimal homotopy asymptotic method [14], the subdomain finite element method [15], predictor optimal homotopy asymptotic method [16], variational iteration method [17], the homotopy perturbation method [18], multistage optimal homotopy asymptotic method (MOHAM) [19].

The most important challenge faces the researchers is to search and find a better and more effective method which gives the exact solution or accurate approximate solution to the nonlinear models. One of the well-known techniques is the DTM which is one of the top ten techniques for solving linear and nonlinear problems proposed by Zhou [20]. The idea of DTM is based on the concept of Taylor series [21, 22, 23], and the solution is usually in a series form. Unfortunately, DTM has some drawbacks or difficulties especially in large time span or region and it gives at most a good approximation which is closed to exact one in small region [24]. Therefore, it is necessary to develop and improve some nonlinear analytical approximations valid for large parameters. So that to improve the accuracy of DTM, we construct alternative scheme which modifies the series solution for classes of boundary value problems starting the process by applying Laplace transformation to the truncated series obtained by DTM, then convert the transformed series into a meromorphic function by Padé approximants, and finally applying the inverse Laplace transform to obtain highly accurate results or exact solutions for differential equations.

The structure of this paper has been formulated in 4-Sections; In Section 2 basic definitions of differential transform method, operational properties of the differential transformation and Padé approximants are presented. Numerical examples have been presented in Section 3 to illustrate the effectiveness of the proposed scheme. While, the conclusion and discussion are included in last section.

2. Preliminaries

This section presents some basic ideas and concepts of differential transform method and Padé approximants.

2.1. **Differential Transform Method.**

Definition 2.1. [25]
If a function $f(x)$ is analytical with respect to x in the domain of interest, then

$$F(k) = \frac{f^{(k)}(x_0)}{k!}. \tag{1}$$

The inverse differential transform of $F(k)$ is defined as:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{2}$$

For more information about the basic operations of DTM, see [25]

Theorem 2.1. [5]
If $f(y) = y^m$, then

$$F(k) = \begin{cases} (Y(0))^m, & k = 0 \\ \frac{1}{Y(0)} \sum_{r=1}^{k} \left(\frac{(m+1)r-k}{k} \right) Y(r) F(k-r), & k \geq 1 \end{cases}$$

Theorem 2.2. [5]
If $f(y) = e^{ay}$*, then*

$$F(k) = \begin{cases} e^{aY(0)}, & k = 0 \\ a\sum_{r=0}^{k-1} \frac{r+1}{k} Y(r+1) F(k-1-r), & k \geq 1 \end{cases}$$

Theorem 2.3. [5]
If $f(y) = \sin(\alpha y)$ *and* $g(y) = \cos(\alpha y)$ *, then*

$$F(k) = \begin{cases} \sin(\alpha Y(0)), & k = 0 \\ \alpha\sum_{r=0}^{k-1} \frac{k-r}{k} G(r) Y(k-r), & k \geq 1 \end{cases}$$

and

$$G(k) = \begin{cases} \cos(\alpha Y(0)), & k = 0 \\ -\alpha\sum_{r=0}^{k-1} \frac{k-r}{k} F(r) Y(k-r), & k \geq 1 \end{cases}$$

Using the differential transform, a differential equation in the domain of interest can be transformed into an algebraic equation in the K-domain and $f(t)$ can be obtained by the finite-term Taylor series expansion plus a remainder,as

$$f(t) = \sum_{k=0}^{N} F(k) \frac{(t-t_0)^k}{k!} + R_{N+1}(t). \tag{3}$$

The series solution (3) converges rapidly only in a small region; in the wide region, they may have very slow convergence rates, and then their truncations yield inaccurate results. In the MDTM, we apply a Laplace transform to the series obtained by DTM, then convert the transformed series into a meromorphic function by forming its Padé approximants, and then invert the approximant to obtain an analytic solution, which may be periodic or a better approximation solution than the DTM truncated series solution. For further reference on DTM see [25, 26, 27, 28, 29].

2.2. **Padé Approximation.** Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $y(x)$.
The $[L/M]$ Padé approximants to a function $y(x)$ are given by

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i,$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = \mathrm{O}\left(x^{L+M+1}\right), \tag{4}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, then we impose the normalization condition

$$Q_M(0) = 1. \tag{5}$$

Finally, we require that $P_L(x)$ and $Q_M(x)$ have no common factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\begin{cases} P_L(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L \\ Q_M(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M \end{cases} \tag{6}$$

then, by (5) and (6), we may multiply (4) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (4) in more detail as

$$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ \cdot \\ \cdot \\ \cdot \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{cases} \tag{7}$$

$$\begin{cases} a_0 = p_0 \\ a_0 + a_0 q_1 = p_1 \\ a_2 + a_1 q_1 + a_0 q_2 = p_2 \\ \cdot \\ \cdot \\ \cdot \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \end{cases} \tag{8}$$

To solve these equations, we start with (7), which is a set of linear equations for all the unknown $q's$. Once the $q's$ are known, then (8) gives an explicit formula for the unknown $p's$, which complete the solution.
If (8) and (7) are non-singular, then we can solve them directly and obtain (9) [30], where (9) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \dots & a_{L+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_L & a_{L+1} & \dots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \dots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \dots & a_{L+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_L & a_{L+1} & \dots & a_{L+M} \\ x^M & x^{M-1} & \dots & 1 \end{bmatrix}} \tag{9}$$

To obtain diagonal Padé approximants of different order such as [2 /2], [4/4] or [6/6] we can use the symbolic calculation software, such as matlab, mathematica and maple.

3. Numerical Results

In this section, several test examples have been illustrated to demonstrate our procedure.

3.1. **Example 1.** Consider the following quadratic Riccati differential equation taken from Aminikhah [19].

$$y^{'}(t) = e^t - e^{3t} + 2e^{2t} y(t) - e^t y^2(t),\ 0 \leq t \leq 1, \tag{10}$$

$$y(0) = 1, \tag{11}$$

The exact solution of above equation is

$$y(t) = e^t.$$

To solve this problem, we apply the differential transform for both sides of (10), which gives

$$(k+1)Y(k+1)=\frac{1}{k!}-\frac{3^k}{k!}+2\sum_{i=0}^{k}\frac{2^i}{i!}Y(k-i)-\sum_{m=0}^{k}\frac{1}{m!}G(k-m). \tag{12}$$

So,

$$Y(k+1)=\frac{1}{k+1}\Big[\frac{1}{k!}-\frac{3^k}{k!}+2\sum_{i=0}^{k}\frac{2^i}{i!}Y(k-i)-\sum_{m=0}^{k}\frac{1}{m!}G(k-m)\Big]. \tag{13}$$

Where $G(k)$ is the differential transform of $g(y)=y^2$.
By Theorem 2.1, the differential transform $G(k)$ in Eq.(13)

$$G(0)=(Y(0))^2=1. \tag{14}$$

$$G(k)=\sum_{r=1}^{k}\Big(\frac{3r-k}{k}\Big)Y(r)G(k-r),\ k\geq 1 \tag{15}$$

Based on Eq.(1) of definition (2.1), the initial condition given by Eq.(11) will be $Y(0)=1$, substituting Eq.(14) in Eq.(13), we get $Y(1)=1$.
Consequently,
$Y(2)=\frac{1}{2}, Y(3)=\frac{1}{6}, Y(4)=\frac{1}{24}, Y(5)=\frac{1}{120}, Y(6)=\frac{1}{720}, Y(7)=\frac{1}{5040}$, and using the inverse transformation rule (2), the approximate solution of Eq.(10) will be

$$y(t)=\sum_{k=0}^{\infty}Y(k)t^k=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}+\dots. \tag{16}$$

And this in the limit of infinitely many terms, yields the exact solution of Eq.(10).
In order to prove the efficiency of the MDTM, and using just the first four terms from the DTM series solution (16), we implement the MDTM as follows:
Applying the Laplace transform to the first four terms from the DTM series solution (16), yields

$$\mathcal{L}\left(y\left(t\right)\right)=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}+\frac{1}{s^4}+\dots.$$

For simplicity, let $s=\frac{1}{z}$; then

$$\mathcal{L}\left(y\left(t\right)\right)=z+z^2+z^3+z^4+\dots.$$

The Padé approximants $\left[\frac{2}{2}\right]$gives

$$\left[\frac{2}{2}\right]=-\frac{z}{z-1}.$$

Recalling $z=\frac{1}{s}$, we obtain $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right]=\frac{1}{s-1}.$$

By using the inverse Laplace transform to the [2/2] Padé approximate, we obtain the modified approximate solution

$$y(t)=e^t. \tag{17}$$

3.2. **Example 2.** In this exampe, we consider the following nonlinear problem which is taken from Guler [11] and Mukherjee [34]

$$ty^{''}(t)+2y^{'}(t)+ty^{5}(t)=0,\ 0\leq t\leq 1, \tag{18}$$

with initial conditions

$$y(0)=1, y^{'}(0)=0, \tag{19}$$

where the exact solution of above equation is

$$y(t)=\sqrt{\frac{3}{3+t^{2}}}.$$

Now, taking the differential transform for both sides of (18), we obtain

$$\sum_{i=0}^{k}\delta(i-1)(k+1-i)\,(k+2-i)\,Y\,(k+2-i)+2\,(k+1)\,Y\,(k+1)$$
$$+\sum_{i=0}^{k}\delta\,(i-1)\,G(k-i)=0. \tag{20}$$

Where $G(k)$ is the differential transform of $g(y)=y^{5}$.
The initial conditions for this problem can be transformed to the following form based on Eq.(1) of definition (2.1)

$$Y(0)=1, Y(1)=0,$$

Based on Theorem 2.1, the differential transform $G(k)$ in (20) is

$$G(0)=(Y(0))^{5}=1, \tag{21}$$

$$G(k)=\sum_{r=1}^{k}\Big(\frac{6r-k}{k}\Big)Y(r)G(k-r),\ k\geq 1. \tag{22}$$

Hence

$$Y\,(2)=-\frac{1}{6},\ Y\,(3)=0,\ Y\,(4)=\frac{1}{24},\ Y\,(5)=0,\ Y\,(6)=-\frac{5}{432},\ Y\,(7)=0,\ Y\,(8)=\frac{35}{10368}\ .$$

Using the inverse transformation rule represented by Eq.(2), the approximate solution of Eq.(18) will be

$$y(t)=\sum_{k=0}^{\infty}U(k)t^{k}=1-\frac{t^{2}}{6}+\frac{t^{4}}{24}-\frac{5t^{6}}{432}+\frac{35t^{8}}{10368}+\dots. \tag{23}$$

To improve the accuracy of the differential transform solution (23), we use the MDTM by applying the Laplace transform [32] to the series solution (23), yields

$$\mathcal{L}\,(y\,(t))=\frac{1}{s}-\frac{1}{3}\frac{1}{s^{3}}+\frac{1}{s^{5}}-\frac{25}{3}\frac{1}{s^{7}}+\frac{1225}{9}\frac{1}{s^{9}}+\dots.$$

For the purpose of simplification, let $s=\frac{1}{z}$; then

$$\mathcal{L}\,(u\,(t))=z-\frac{1}{3}z^{3}+z^{5}-\frac{25}{3}z^{7}+\frac{1225}{9}z^{9}+\dots.$$

The Padé approximants $\left[\frac{5}{4}\right]$ gives

$$\left[\frac{5}{4}\right]=\frac{568z^{5}+369z^{3}+18z}{675z^{4}+375z^{2}+18}.$$

TABLE 1. Absolute error for Example 3.2

t_i	Exact Solution	MDTM Solution	Absolute Error	Absolute Error [34]	Absolute Error [11]
0.0	1.0000000	1.0000000	0	0	0
0.2	0.9933993	0.9933993	3.17×10^{-11}	1.02×10^{-10}	7.30×10^{-7}
0.4	0.9743547	0.9743547	3.00×10^{-8}	1.01×10^{-7}	4.50×10^{-5}
0.6	0.9449112	0.9449127	1.52×10^{-6}	5.52×10^{-6}	4.90×10^{-4}
0.8	0.9078413	0.9078640	2.27×10^{-5}	9.10×10^{-5}	2.56×10^{-3}
1.0	0.8660254	0.8661960	1.71×10^{-4}	7.76×10^{-4}	8.97×10^{-3}

Recalling $z = \frac{1}{s}$, we obtain [5/4] in terms of s

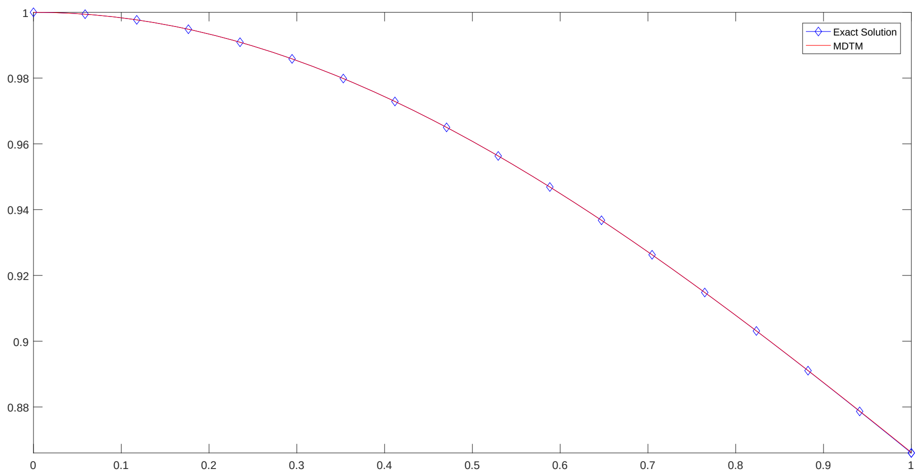
$$\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{18s^4 + 369s^2 + 568}{18s^5 + 375s^3 + 675s}.$$

By using the inverse Laplace transform to the [5/4] Padé approximant, we obtain the modified approximate solution

$$y(t) = 0.00106003\ cos\,(4.34087877\ t) + 0.15745849\ cos(1.41071075\ t) + \frac{568}{675}.$$

Table 1 shows absolute error for Example 3.2 using MDTM, DTM [34] and Taylor matrix method [11], and Fig. (1) shows the graphs of approximated and exact solution $y(t)$ for Example 3.2.

FIGURE 1. The plots of the approximate and exact solutions y(t) for Example 3.2



3.3. **Example 3.** The third example considered for the following nonlinear problem taken from Guler [11]

$$y(t)y^{'}(t) + ty(t) + y^2(t) + t^2y^3(t) = te^{-t} + t^2e^{-3t}, 0 \leq t \leq 1, \tag{24}$$

with initial condition

$$y(0) = 1. \tag{25}$$

The analytic solution of this problem is given by

$$y(t) = e^{-t}.$$

To solve the differential Eq. (24), taking differential transform for both sides, to get

$$\sum_{k_1=0}^{k} Y(k_1)(k-k_1+1)Y(k-k_1+1) + \sum_{k_2=0}^{k} \delta(k_2-1)Y(k-k_2) + \sum_{k_3=0}^{k} Y(k_3)Y(k-k_3) + \sum_{k_4=0}^{k} G(k_4)\delta(k-k_4-2) = \sum_{k_5=0}^{k} \delta(k_5-1)\frac{(-1)^{k-k_5}}{(k-k_5)!} + \sum_{k_6=0}^{k} \delta(k_6-2)\frac{(-3)^{k-k_6}}{(k-k_6)!}. \quad (26)$$

From (1), the initial condition given in (25) can be transformed as

$$Y(0) = 1, \quad (27)$$

Where $G(k)$ is the differential transform of $g(y) = y^3$.
By Theorem 2.1, the differential transform $G(k)$ in (26) is

$$G(0) = (Y(0))^3 = 1, \quad (28)$$

$$G(k) = \sum_{r=1}^{k} \Big(\frac{4r-k}{k}\Big) Y(r)G(k-r), \ k \geq 1. \quad (29)$$

Therefore, substituting (27) and (28) in (29), then in (26), yields the following:

$$Y(1) = -1, Y(2) = \frac{1}{2}, Y(3) = -\frac{1}{6}, \ and\ G(1) = -3, \ then\ Y(4) = \frac{1}{24}, \ and\ G(2) = \frac{9}{2},$$

$$then\, Y(5) = -\frac{1}{120}, \ and\, G(3) = -\frac{9}{2}, \ then\, Y(6) = \frac{1}{720}, \ and\, G(4) = \frac{27}{8}, \ then\, Y(7) = -\frac{1}{5040}.$$

Using the inverse transformation rule (2), we obtain an approximate solution of (24) in the form

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} + \frac{t^6}{720} - \frac{t^8}{5040} + \dots . \quad (30)$$

And this in the limit of infinitely many terms, yields the exact solution of (24).
In order to prove the efficiency of the MDTM, and using just the first four terms from the DTM series solution (30), we implement the MDTM as follows:
Applying the Laplace transform to the first four terms from the DTM series solution (30), yields

$$\mathcal{L}(y(t)) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \frac{1}{s^4} + \dots .$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}(y(t)) = z - z^2 + z^3 - z^4 + \dots .$$

The Padé approximants $\left[\frac{2}{2}\right]$ gives

$$\left[\frac{2}{2}\right] = \frac{z}{z+1}.$$

Recalling $z = \frac{1}{s}$, we obtain $\left[\frac{2}{2}\right]$ in terms of s

$$\left[\frac{2}{2}\right] = \frac{1}{s+1}.$$

. By using the inverse Laplace transform to the [2/2] Padé approximant, we obtain the modified approximate solution

$$y(t) = e^{-t}. \tag{31}$$

3.4. **Example 4.** The following fourth order nonlinear differential equation taken from [10] is considered

$$y^{(4)}(t) - y^2(t)y^{'''}(t) + sin(t)y^{'}(t)y^{''}(t) + cos(t)y(t) = sin(t)(1 - cos(t)). \tag{32}$$

subject to the initial conditions

$$y(0) = 0, y^{'}(0) = 1, y^{''}(0) = 0, y^{'''}(0) = -1,\ 0 \le t \le 1. \tag{33}$$

To get approximate for the above problem, we apply the differential transform on both sides, which gives

$$\begin{aligned} Y(k+4) = \frac{k!}{(k+4)!}[&\sum_{k_1=0}^{k}\sum_{k_2=0}^{k_1}(k_1+1)(k_1+2)(k_1+3)Y(k_1+3)Y(k_1-k_2)Y(k-k_1) \\ &- \sum_{k_3=0}^{k}\sum_{k_4=0}^{k_3}(k_3+1)(k_3+2)Y(k_3+2)(k_3-k_4+1)\frac{Y(k_3-k_4+1)}{(k-k_3)!}sin(\frac{\pi(k-k_3)}{2}) \\ &- \sum_{k_5=0}^{k}\frac{Y(k-k_5)}{k_5!}cos(\frac{\pi k_5}{2}) + \frac{1}{k!}sin(\frac{\pi k}{2}) + \frac{2^{(k-1)}}{k!}sin(\frac{\pi k}{2}). \end{aligned} \tag{34}$$

Based on Eq. (1), the initial conditions (33) can be transformed into the following form

$$Y(0) = 0, Y(1) = 1, Y(2) = 0, Y(3) = -\frac{1}{6}.$$

Therefore,

$$Y(4) = 0, Y(5) = \frac{1}{120}, Y(6) = 0, and\ Y(7) = -\frac{1}{5040}.$$

By using the inverse transformation rule (2), the approximate solution of Eq.(32) becomes

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \dots. \tag{35}$$

And this in the limit of infinitely many terms, yields the exact solution of (32).

In order to prove the efficiency of the MDTM, and using just the first two terms from the DTM series solution (35), we implement the MDTM as follows:
Applying the Laplace transform to the first two terms from the DTM series solution (35), yields

$$\mathcal{L}(y(t)) = \frac{1}{s^2} - \frac{1}{s^4} + \dots.$$

For simplicity, let $s = \frac{1}{z}$; then

$$\mathcal{L}(y(t)) = t^2 - t^4 + \dots. \tag{36}$$

The Padé approximants [2/2] gives

$$\left[\frac{2}{2}\right] = \frac{z^2}{z^2+1}.$$

Recalling $z = \frac{1}{s}$, we obtain [2/2] in terms of s

$$\left[\frac{2}{2}\right] = \frac{1}{s^2+1}.$$

By using the inverse Laplace transform to the [2/2] Padé approximant, we obtain the modified approximate solution

$$y(t) = sin(t).$$

4. Conclusions

In this research article, The MDTM was successfully employed and accurate approximate solution was obtained for classes of nonlinear differential equations. The MDTM is very powerful and efficient scheme and this is observed and demonstrated throughout the obtained results. The results also show that this scheme is a very promising one and can be easily applied to other differential equations.

Acknowledgement. The authors would like to express their sincere gratitude to the anonymous referees for their useful and valuable comments.

References

[1] Bulbul, B., Sezer, M., (2015), A numerical approach for solving generalized Abel-type nonlinear differential equations. Applied Mathematics and Computation, 262 pp. 169-177.

[2] Davis, H. T., (1961), Introduction to nonlinear differential and integral equations. US Government Printing Office.

[3] Al-Hawary, T., Amourah, A. A., Darus, M., (2014), Differential sandwich theorems for p-valent functions associated with two generalized differential operator and integral operator. International Information Institute (Tokyo). Information, 17(8) pp. 3559-3570.

[4] Hassan, I. A. H., (2008), Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems. Chaos, Solitons & Fractals, 36(1) pp. 53-65.

[5] Alomari, A. K., Anakira, N. R., Bataineh, A. S., Hashim, I., (2013), Approximate solution of nonlinear system of BVP arising in fluid flow problem. Mathematical Problems in Engineering, 2013 Article ID 136043, pp. 1-7.

[6] Al-Hawary, T., Amourah, A. A., Yousef, F., Darus, M., (2015), A certain fractional derivative operator and new class of analytic functions with negative coefficients. International Information Institute (Tokyo). Information, 18(11) pp. 4433-4441.

[7] Lebrun, J., (2019), On two coupled Abel-type differential equations arising in a magnetostatic problem. Il Nuovo Cimento A (1965-1970), 1990. 103(10) pp. 1369-1379.

[8] Amourah, A. A., (2019), Faber polynomial coefficient estimates for a class of analytic bi-univalent functions. In AIP Conference Proceedings, 2096(1) pp.1–9.

[9] Yazdani, A., J. Vahidi, and S. Ghasempour, (2016), Comparison Between Differential Transform Method and Taylor Series method for Solving Linear and Nonlinear Ordinary Differential Equations. International Journal of Mechatronics, Electrical and Computer Technology, 6(20) pp. 2872-2877.

[10] Yuzbasi, S., (2017), A numerical scheme for solutions of a class of nonlinear differential equations. Journal of Taibah University for Science, 11(6) pp. 1165–1181.

[11] Guler, C., 2007, A new numerical algorithm for the Abel equation of the second kind. International Journal of Computer Mathematics, 84(1) pp. 109–119.

[12] Markakis, M., (2009), Closed-form solutions of certain Abel equations of the first kind. Applied Mathematics Letters, 22(9) pp. 1401–1405.

[13] Alomari, A. K., Syam, M., Al-Jamal, M. F., Bataineh, A. S., Anakira, N. R., Jameel, A. F., (2018), Modified Legendre operational matrix of differentiation for solving strongly nonlinear dynamical systems. International Journal of Applied and Computational Mathematics, 4(5) pp. 1–13.

[14] Anakira, N. R., Alomari, A. K., Hashim, I., (2013), "Optimal homotopy asymptotic method for solving delay differential equations." Mathematical Problems in Engineering 2013 pp. 1–12.

[15] Geyikli, T., Karakoc, S. G., (2015), Subdomain finite element method with quartic B-splines for the modified equal width wave equation. Computational Mathematics and Mathematical Physics, 55(3) pp. 410-421.

[16] Alomari, A. K., Anakira, N. R., Hashim, I., (2014), Multiple solutions of problems in fluid mechanics by predictor optimal homotopy asymptotic method. Advances in Mechanical Engineering, 6 pp. 1–7.

[17] Abbasbandy, S., (2007), A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials. Journal of Computational and Applied Mathematics, 207(1): pp. 59–63.

[18] Abbasbandy, S., (2006), Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method. Applied Mathematics and Computation, 172(1) pp. 485–490.

[19] Anakira, N. R., Alomari, A. K., Jameel, A. F., Hashim, I., (2017), Multistage optimal homotopy asymptotic method for solving boundary value problems with robin boundary conditions. Far East Journal of Mathematical Sciences, 102(8) pp. 1727-1744.

[20] Zhou, J. K. (1986), Differential Transform and its Applications for Electrical Circuits; Huarjung University Press, Wuhan, China.

[21] Ayaz, F. (2004), Solutions of the system of differential equations by differential transform method. Applied Mathematics and Computation, 147(2) pp.547–567.

[22] Chen, C. L., & Sy-Hong, L., (1996), Application of Taylor transformation to nonlinear predictive control problem. Applied mathematical modelling, 20(9), pp. 699-710.

[23] Odibat, Z. M., (2008), Differential transform method for solving Volterra integral equation with separable kernels. Mathematical and Computer Modelling, 48(7-8), pp. 1144-1149.

[24] Momani, S., and Ertúrk, V. S., (2008), Solutions of non-linear oscillators by the modified differential transform method. Computers and Mathematics with Applications, 55(4) pp. 833-842.

[25] Al-Ahmad, S., Mamat, M., AlAhmad, R., Sulaiman, I. M., Ghazali, P. L., and Mohamed, M. A., (2018), On New Properties of Differential Transform via Difference Equations. International Journal of Engineering and Technology, 7(3.28) pp. 321-324.

[26] Arikoglu, A., Ozkol, I., (2006), Solution of difference equations by using differential transform method. Applied Mathematics and Computation, 174(2) pp. 1216-1228.

[27] Ali, A. H., (2017), Applications of Differential Transform Method To Initial Value Problems.American Journal of Engineering Research, 6(12) pp. 365-371.

[28] Jameel, A. F., Anakira, N. R., Rashidi, M. M., Alomari, A. K., Saaban, A., and Shakhatreh, M. A., (2018), Differential Transformation Method For Solving High Order Fuzzy Initial Value Problems. Italian Journal of Pure and Applied Mathematics, 39 pp. 194-208.

[29] Kanth, A. R., and Aruna, K., (2008), Differential transform method for solving linear and non-linear systems of partial differential equations. Physics Letters A, 372(46) pp. 6896-6898.

[30] Baker, G. A. (1975), Essentials of Padé approximants. academic press.

[31] Bender, C. M., and Orszag, S. A., (2013), Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory. Springer Science and Business Media.

[32] Biazar, J., and Eslami, M. (2010), Differential transform method for quadratic Riccati differential equation. International Journal of Nonlinear Science, 9(4) pp. 444-447.

[33] AlAhmad, R. (2020), Laplace transform of the product of two functions. Italian journal of pure and applied Mathematics. No 43 (to appear).

[34] Mukherjee, S., Roy, B., and Chaterjee, P. K., (2011), Solution of Lane-Emden equation by differential transform method. Int. J. Nonlinear Sci, 12(4) pp.478-484.

S. Al-Ahmad is currently a PhD student in the Faculty of Informatics and Computing, University of Sultan Zainal Abidin (UniSZA), Malaysia.His research interests are mainly in the field of differential equations. In particular, he is working on special topics on differential transform method and modified differential transform method to solving several classes on ordinary differential equations. Also, he is interested in studying the relations connecting differential transform and difference operators and related special functions such as incomplete gamma functions.

N. R. Anakira is the Head of Mathematics Department at Irbid National University, Jordan. He received his PhD in Applied Mathematics from the School of Mathematical Science, University Kebangsaan, Malaysia in 2015. His research interests are in the areas of applied mathematics, numerical solution of ordinary and partial differential equations, delay differential equations and mathematical modeling.

M. Mamat is a professor of computational & applied mathematics and the Dean of Graduate School, University of Sultan Zainal Abidin, K. Terengganu, Malaysia. He obtained his Ph.D from UMT in 2007 specialization in optimization field. His research interest includes unconstrained optimization such as hybrid conjugate gradient methods, three term methods, Quasi-Newton methods and chaotic systems.

I. M. Sulaiman is currently a post-doctoral researcher at the Faculty of informatics and computing, University of Sultan Zainal Abidin (UniSZA), Malaysia. He obtained his PhD from UniSZA in 2018 specializing in the field of fuzzy systems. His research interest includes Fuzzy nonlinear systems and unconstrained optimization.

R. AlAhmad is an associated professor of applied mathematics in Yarmouk University. His research interests are operator theory (differential and difference operators), inverse problems on differential and difference equations, special functions, and probability theory. Recently, he is working on special topics on differential transform, difference operators, and related topics such as incomplete gamma functions, incomplete beta functions and Stirling numbers.