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L(t, 1)-COLOURING OF GRAPHS

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Abstract. One of the most famous applications of Graph Theory is in the field of Channel Assignment Problems. There are varieties of graph colouring concepts that are used for different requirements of frequency assignments in communication channels. We introduce here $L(t, 1)$-colouring of graphs. This has its foundation in T-colouring and $L(p, q)$-colouring. For a given finite set T including zero, an $L(t, 1)$-colouring of a graph G is an assignment of non-negative integers to the vertices of G such that the difference between the colours of adjacent vertices must not belong to the set T and the colours of vertices that are at distance two must be distinct. The variable t in $L(t, 1)$ denotes the elements of the set T. For a graph G, the $L(t, 1)$-span of G is the minimum of the highest colour used to colour the vertices of a graph out of all the possible $L(t, 1)$-colourings. It is denoted by $\lambda_{t,1}(G)$. We study some properties of $L(t, 1)$-colouring. We also find upper bounds of $\lambda_{t,1}(G)$ of stars and multipartite graphs.

Keywords: $L(t, 1)$-colouring, Communication networks, Radio frequency, Colour span

AMS Subject Classification: 05C15, 94C15

1. Introduction

Graph colouring problems emerged as a requirement of the partitioning of the set of vertices into various classes for specific needs. The *Channel Assignment Problems* are associated with allocating the radio frequencies to various radio transmitters (channels). The concept of graph colouring is useful in the optimal assignment of radio frequencies. One of the first types of channel assignment problems was introduced by Metzger [4].

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge, two vertices called its endpoints [7]. When the endpoints of an edge coincides, the edge is called a loop. When there is more than one edge between a pair of vertices the graph is said to have multiple edges. A simple graph is a graph that neither has loops or multiple edges. If no direction is assigned to any of the edges in a graph then the graph is an undirected graph. The endpoints of an edge are also referred to as adjacent vertices. A graph is said to be connected if there exist a sequence of adjacent vertices between any pair of vertices in that graph. For all standard definitions and notations related to graphs we suggest the readers to read [7].

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In this work we study only about undirected simple connected graphs. The most important constituents of a graph are its vertices and edges. To distinguish the vertices and edges in a graph with multiple vertices and edges, labels are given to them. Colours are certain types of labels assigned to vertices and edges. If the colours are assigned to the vertices alone, then we call it as vertex colouring of graph. Similarly, we can have edge colouring. When colours are assigned simultaneously to vertices and edges, we call it as total colouring.

Colours are assigned for various practical purposes. Channel assignment problem is one of the types of colouring problems in graph theory. This is done by representing vertices as the radio transmitters and colours as the radio frequencies. To avoid interference of the radio transmissions, the frequencies of radio transmitters within certain distances must be distinct. In this scenario T-colouring is a useful mathematical tool for optimal frequency allocation.

In Section 2 we present some of the preliminary concepts in the development of $L(t, 1)$-colouring. In Section 3, we define $L(t, 1)$-colouring and present some basic results and remarks. We discuss the $L(t, 1)$-colouring of star graph in 4 and that of complete k-partite graph in 5.

2. Preliminaries

The concept of T-colouring of vertices of a graph was introduced by Hale [3]. It was further studied by Roberts in 1991 [6]. T-colouring is defined in [6] as follows.

Definition 2.1. *For a graph G and a given finite set T of non-negative integers containing 0, a T-colouring of G is an assignment c of colours (positive integers) to the vertices of G such that if $uw \in E(G)$, then $|c(u) - c(w)| \notin T$.*

T-chromatic number and T-span are the two parameters that are associated with T-colouring of graphs. The least number of colours needed in the T-colouring of a graph is the T-chromatic number of a graph. For any simple connected undirected graph both the T-chromatic number and chromatic number of a graph are the same. Hence, there was no much enthusiasm in pursuing with the studies on T-chromatic number. For a T-colouring c, the c-span is the difference between the largest value of c and the smallest value of c. T-span of a graph is the smallest among all the c-span of a graph. Although several bounds are found for the T-span, there are several families of graphs whose T-span is not found out precisely. Not much work is done on T-colouring in the last couple of years.

Although T-colouring was an answer to frequency assignment requirement, there emerged several other colouring schemes to meet many new industrial demands. In T-colouring, the condition of colouring was for adjacent transmitters whereas transmitters which are neighbours to the neighbours have also the possibility of transmission interference. Yeh in 1990 and then Griggs and Yeh in 1992 addressed this issue [1].

Labelling graphs with a condition at distance 2 was studied extensively by Griggs and Yeh [8]. The related colouring problem is known as $L(p, q)$-colouring which is presented in [1] as follows.

Definition 2.2. *For $p \geq q \geq 1$, an L(p, q)-colouring of a graph $G = (V, E)$ is an assignment c from $V(G)$ to the set of non-negative integers $\{0, 1, \dots, \lambda\}$ such that*

$$|c(u) - c(w)| \geq \begin{cases} p & \text{if } d(u, w) = 1 \\ q & \text{if } d(u, w) = 2 \end{cases}$$

where $d(u, w)$ is the distance between the vertices u and w.

When $p = 2$ and $q = 1$, we get $L(2,1)$-colouring. Among all the studies under the banner of $L(p,q)$-colouring, $L(2,1)$-colouring is a front-runner. For an $L(p,q)$-colouring c, the c-span is the difference between the largest value of c and the smallest value of c. L-span of a graph is the smallest among all the c-spans of a graph. L-span is also denoted as $\lambda_{p,q}$. Several variations of $L(p,q)$-colouring are there in the literature such as [2] and [5].

3. $L(t,1)$-colouring

We introduce a type of colouring taking the ideas from T-colouring and $L(2,1)$-colouring.

Definition 3.1. *Let $G = (V,E)$ be a graph and let $d(u,v)$ be the distance between the vertices u and v of G. Let T be a finite set of non-negative integers containing 0. An $L(t,1)$-colouring of a graph G is an assignment c of non-negative integers to the vertices of G such that $|c(u) - c(v)| \notin T$ if $d(u,v) = 1$ and $c(u) \neq c(v)$ if $d(u,v) = 2$.*

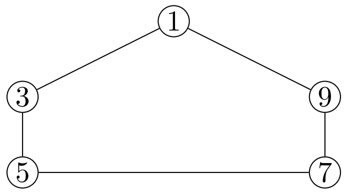

Figure 1. $L(t,1)$-colouring of C_5

Example 3.1. *For T= {0, 1, 3}, FIGURE 1 shows an $L(t,1)$-colouring of the cycle C_5.*

It is not reasonable to assume that there are infinitely many frequencies available for allotment. Therefore, it is of great importance to find the optimal assignment of the available frequencies to radio transmitters. Thus we introduce the concept of span. The span of frequencies have been studied in relation to various types of channel assignment problems. Almost in a similar fashion, we define the $L(t,1)$-span of graph G.

Definition 3.2. *For any colouring c, the c-span of the colouring is defined as the difference between the highest and the smallest colour assigned to any vertex in graph G.*

Definition 3.3. *For a graph G with a given set T, L(t, 1)-span of G denoted by the symbol $\lambda_{t,1}(G)$ is,*

$$\lambda_{t,1}(G) = min\ \{\max_{u,v \in V(G)} \{|c(u) - c(v)| : c \text{ is an } L(t,1)\text{-colouring of } G\}\}. \tag{1}$$

That is, for a fixed T, $\lambda_{t,1}(G)$ is the smallest of all the c-spans of G. In the Example 3.1, the $L(t,1)$-span is 8.

3.1. Complementary Colouring.

If c is an $L(t,1)$-colouring then it induces other $L(t,1)$-colouring as well. Complementary colouring is one among them. Let s be the largest colour assigned in an $L(t,1)$-colouring c. Then for $j \geq 0$ the colouring c' of the vertices of G defined by

$$c'(v) = s + j - c(v) \tag{2}$$

for each vertex v in $V(G)$ is called the complementary $L(t,1)$-colouring associated with the $L(t,1)$-colouring c.

Theorem 3.1. *If c is an $L(t,1)$-colouring of a graph G, then the complementary colouring c' is also an $L(t,1)$-colouring of the same graph G.*

Proof. Let c be an $L(t,1)$-colouring of a graph G with s as the largest colour assigned. Then for any two adjacent vertices u and v, we know that $|c(u) - c(v)| \notin T$. By the definition, $c'(v) = s + j - c(v)$.

Now, $|c'(u) - c'(v)| = |s + j - c(u) - (s + j - c(v))| = |c(u) - c(v)| \notin T$.

Similarly, consider two vertices u and v such that $d(u,v) = 2$. But, $c(u) \neq c(v)$. By our definition, let the complementary colours of u and v be $c'(u)$ and $c'(v)$, respectively. Then $|c'(u) - c'(v)| = |s + j - c(u) - (s + j - c(v))| = |c(u) - c(v)| \neq 0$.

Thus, the colouring c' is an $L(t,1)$-colouring. □

Consider an $L(t,1)$-colouring c. For $j = 0$, the complementary colouring c' of the graph G, the c-span and c'-span are identical. Moreover, if c is an optimal colouring of G so is c' and both will give the same $L(t,1)$-span for any graph G.

For $j \neq 0$, if the c-span of the colouring c is, say l, then the c-span for the complementary colouring c' will be $l + j$.

The following remarks are also easily verifiable.

Remark 3.1. *$L(t,1)$-colouring is analogous to T-colouring when all the vertices of the graph are at a distance one. That is, for complete graphs K_n. Here $L(t,1)$-span of G is same as T-span of G.*

Remark 3.2. *For the $L(p,q)$-colouring, if $q = 1$ and if we consider set $T = \{0, 1, \ldots, (p-1)\}$, then $L(t,1)$-colouring is analogous to $L(p,q)$-colouring. For such a set T, $L(t,1)$-span of G is same as $L(p,q)$-span of G.*

For the set T of non-negative integers including zero, let t_{max} and t_{min} denote the maximum and minimum, respectively, of all the elements in T. Obviously, $t_{min} = 0$.

Definition 3.4. *Let the set of integers between t_{min} and t_{max} that are not available in T be denoted by σ. Then, the cardinality of this set σ , $|\sigma| = t_{max} - |T| + 1$.*

4. $L(t,1)$-colouring of star graph

Star graphs are very important in the communication networks. We study the $L(t,1)$-colouring of star graphs. Also, we see the effect on $L(t,1)$-span of $K_{1,n}$ on changing the set T.

Theorem 4.1. *The $L(t,1)$-span of the star $K_{1,n}$ decreases with the increase in the value of $|\sigma|$ for $|\sigma| < n$, if the maximum value of set T remains same.*

Proof. For a graph $K_{1,n}$, let us consider a set T which consists of non-negative integers including zero. Let $t_{max} = r$.

Let $c_1 < c_2 < .. < c_{|\sigma|}$ be the positive integers that are not in T. For $|\sigma| \geq n$, all the pendant vertices of the stars can be coloured from the set of missing colours. i. e., $\{c_1, c_2, \ldots, c_{|\sigma|}\}$. The increase in the cardinality of σ does not affect the colouring unless the new term missing from set T say $c_{|\sigma+1|}$ is less than $c_{|\sigma|}$.

If $c_{|\sigma+1|}$ is less than $c_{|\sigma|}$, then the pendant vertices can be coloured using the same set but by replacing the colour of vertex coloured $c_{|\sigma|}$ by $c_{|\sigma+1|}$ which decreases the span of the graph.

If $c_{|\sigma+1|}$ is greater than $c_{|\sigma|}$, the colouring remains same, hence the span remains same.

Suppose $|\sigma| < n$.

Let $\lambda_{t,1}(K_{1,n}) = s$.

Therefore, colours that are utilized to colour this graph will belong to $\{0, 1, \ldots, s\}$. In order to colour the star using $L(t,1)$-colouring we start colouring the central vertex as zero and then we use all missing colours less than r for pendant vertices, since all pendant vertices are at a distance one from the central vertex. For the remaining vertices we give colours $r+1, r+2, \ldots, s$. Since the diameter of a star is two, all pendant vertices should get different colours. Thus the above colouring is an $L(t,1)$-colouring for a $K_{1,n}$ with span s.

Consider a set T' with cardinality $(|T|-1)$.

Therefore, the colours missing in set T' is given by set σ' whose cardinality is $|\sigma|+1$.

Case 1: $|\sigma'| < n$ Here, we can colour the graph $K_{1,n}$ in the same way as described above using colours $c_1 < c_2 < \ldots < c_{|\sigma|} < c_{|\sigma+1|}$ and continuing with colours $r+1, r+2, \ldots,$ since the number of vertices remains the same we already coloured one more vertex using missing colour from set T'. Hence we require colours till $(s-1)$ for colouring this graph. Thus, we obtain an $L(t,1)$-colouring for $K_{1,n}$ with span less than s.

Case 2: $|\sigma'| = n$ Here, we can colour the graph $K_{1,n}$ in the same way as described above using colours $c_1 < c_2 < \ldots < c_\sigma < c_{|\sigma+1|}$. Therefore, all the missing colours saturate the vertices of $K_{1,n}$ and the largest colour missing from set T' becomes the span of the graph which is always less than r. Since s is always greater than r, in this case also the theorem remains true. □

The following result is easily obtained by using above arguments.

Corollary 4.1. *The $L(t,1)$-span of the graph $K_{1,n}$ will remain same for all those sets T whose cardinality and maximum value are same. i. e., it does not vary with the set T for $n > |\sigma|$.*

Since T is finite, we denote the complement this set by T' and its element as $t'_1, t'_2, t'_3, \ldots$ such that $t'_1 \leq t'_2 \leq t'_3 \leq \ldots$.

Theorem 4.2. *For a graph $G = K_{1,n}$ and set T of non-negative integers including zero, $\lambda_{t,1}(G) \leq t'_n$, where elements of complement of set T is denoted by infinite set $T' = \{t'_1, t'_2, t'_3, \ldots\}$ such that $t'_1 \leq t'_2 \leq t'_3 \leq \ldots$.*

Proof. For a given graph $G = K_{1,n}$ with n+1 vertices, we can easily obtain the above bound by colouring the central vertex as 0 and the pendant vertices as $t'_1, t'_2, t'_3, \ldots, t'_n$. □

5. $L(t,1)$-colouring of complete k-partite graph

Wee find the bound for $\lambda_{t,1}(G)$ for a complete k-partite graph G.

Theorem 5.1. *The $L(t,1)$-span of complete k-partite graph for any finite set T, $\lambda_{t,1}(K_{m_1,m_2,m_3,\ldots,m_k}) \leq rk + \sum_{i=1}^{k} |m_i| - 1$, where r is the t_{max} of T.*

Proof. Let G be a complete k-partite graph with partite sets having cardinalities $|m_1|$, $|m_2|$, $|m_3|$, $\ldots$, $|m_k|$. Without loss of generality, choose the partite set m_1 and vertex $v_1 \in m_1$. Colour v_1 using 0. Let $T = \{0, 1, 2, \ldots, r\}$. Since all the other vertices of the other partite set apart from m_1 are at a distance one from v_1, none of them can have their colour from the set T.

Hence, to colour the vertices in the partite sets $m_2, m_3, \ldots, m_k$, we start colouring the vertices of set m_2 using colour $r+1, r+2, \ldots, r+|m_2|$. We colour the vertices of set m_3 using the colours $r+|m_2|+r+1, r+|m_2|+r+2, \ldots, r+|m_2|+r+|m_3|$, as each of vertices of m_3 is at a distance one from all the vertices of m_1. Continuing in similar way, we can colour the vertices of the partite set m_k by using colours $r+|m_2|+r+|m_3|+\ldots+r+|m_{k-1}|+r+1, r+|m_2|+r+|m_3|+\ldots+r+|m_{k-1}|+r+2, \ldots, r+|m_2|+r+|m_3|+\ldots+r+|m_{k-1}|+r+|m_k|$.

Since vertices $v_2, v_3, \ldots, v_{|m_1|}$ in the partite set m_1 need to be coloured, which are at distance two from v_1 and at a distance one from all the vertices that are already coloured we use the colour $r+|m_2|+r+|m_3|+\ldots+r+|m_{k-1}|+r+|m_k|+r+1$ for v_2. The remaining vertices can be coloured by increasing colour of v_2 by one for each vertex. Hence the colours required are $r + |m_2| + r + |m_3| + \ldots + r + |m_{k-1}| + r + |m_k| + r + 1, \ldots, r + |m_2| + r + |m_3| + \ldots + r + |m_{k-1}| + r + |m_k| + r + |m_1| - 1 = rk + \sum_{i=1}^{k} |m_i| - 1$.

Therefore, the $L(t, 1)$-span of the complete k-partite graph becomes $rk + \sum_{i=1}^{k} |m_i| - 1$. We achieve this bound only when T has the elements $\{0, 1, \ldots, r\}$. In all other cases the $\lambda_{t,1}(K_{m_1,m_2,m_3,\ldots m_k})$ is less than $rk + \sum_{i=1}^{k} |m_i| - 1$. Hence, $\lambda_{t,1}(K_{m_1,m_2,m_3,\ldots,m_k}) \leq rk + \sum_{i=1}^{k} |m_i| - 1$. □

6. Conclusion

In this paper, we have introduced $L(t, 1)$-colouring of graphs. This is a new addition to family of channel assignment problems. The significance of the set T is that certain frequencies t can be reserved or classified for the requirements of the frequency allotment agency. Those frequencies can be used for emergency services or priced differentially. $L(t, 1)$-colouring also assures that two radio channels have the same frequency only if they are at a distance of at least three. In this way the disturbances due to frequency interference can be minimized.

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