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Y-CONE METRIC SPACES AND COUPLED COMMON FIXED POINT RESULTS WITH APPLICATION TO INTEGRAL EQUATION

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Abstract. This paper acquaints with a concept of Y-cone metric space and to study some topological properties of Y-cone metric space. We prove the coupled common fixed point results for mixed weakly monotone map in ordered Y-cone metric spaces. We give an example, which constitutes the main theorem.

Keywords: Coupled common fixed point, mixed weakly monotone maps, Y-cone metric space.

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1. Introduction

Fixed point theory is a phenomenal subject, with an immense number of applications in various fields of mathematics. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle, which provided the existence and uniqueness of a solution of an operator equation $Fs = s$, is the most used fixed point theorem in the analysis. The Banach contraction principle is uncomplicated and most adaptable elementary result in fixed point theory. A number of authors have defined contractive mapping on a complete metric space X, a generalization of the Banach's contraction principle. For more results, we, refer to [3, 4, 6, 7, 9, 10, 13, 15, 17, 22, 23].

Metric spaces play a significant role in the study of functional analysis and topology. The term 'metric' is derived from the word 'metor' (measure). A metric space is a set in which we can express the distance between any two of its elements. To discover a proper concept of a metric space, diverse concepts exist in this sphere. Then, new notions of distance lead to new notions of convergence and continuity. Several generalizations of the metric space have then developed in many papers see [2, 11, 14, 16, 20]. Recently, cone metric spaces

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introduced by Huang and Zhang [14]. There they described convergence in cone metric spaces and introduced the completeness. Then they proved some fixed point theorems of contractive mappings on cone metric spaces. To use this concept in topology, the theory of cone metric spaces have developed by many authors see [1, 5, 8, 12, 18, 19, 21, 24].

The organization of this paper is as follows. In section 2, the preliminary on cone metric space, A-metric space and b-metric space has been discussed. In section 3, we generalize the concepts of cone metric space, A-metric space and b-metric space named it as Y-cone metric space followed by topological properties of Y-cone metric spaces and provide definitions and lemmas. Section 4, provides the existence and uniqueness of certain coupled common fixed point results in the framework of Y-cone metric spaces. In the end, we gave an example to validate the result in section 4.

2. Preliminaries

The perception of cone metric spaces propound by Huang [14]. For more delineate the succeeding definitions, we recommend the reader to [14].

Definition 2.1. [14] *Let X be a nonempty set and let E be a Banach space equipped with the partial ordering $\leq$ regarding the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:*

(1) $\theta \leq d(x,y)\ \forall\ x,y \in X$ and $d(x,y) = \theta \iff x = y$,

(2) $d(x,y) = d(y,x)\ \ \forall\ x,y \in X$,

(3) $d(x,y) \leq d(x,z) + d(z,y)\ \ \forall\ x,y,z \in X$.

Then d is a cone metric on X, and (X,d) is known as cone metric space.

The concept of b- metric space introduced by Czerwik in [11]. For more details the following definitions, we refer the reader to [11].

Definition 2.2. [11] *Let X be a nonempty set and $s \geq 1$ be a real number. A function $d : X \times X \to R^+$ is a b-metric on X if, for $x,y,z \in X$, the following conditions hold:*

(1) $d(x,y) = 0 \iff x = y$,

(2) $d(x,y) = d(y,x)$,

(3) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

Here, the pair (X,d) is known as b-metric space.

For more details about the following definitions, we refer the reader to [2].

Definition 2.3. [2] *Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is called an A-metric on X if, for any $x_i, a \in X, i = 1,2,\cdots,n$, the following conditions hold:*

(A1) $A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) \geq 0$,

(A2) $A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) = 0 \iff x_1 = x_2 = x_3 = \cdots = x_{n-1} = x_n$,

(A3)

$$\begin{aligned} A(x_1,x_2,x_3,\cdots,x_{n-1},x_n) \leq\ & A(x_1,x_1,x_1,\cdots,(x_1)_{n-1},a) \\ & + A(x_2,x_2,x_2,\cdots,(x_2)_{n-1},a) \\ & \vdots \\ & + A(x_{n-1},x_{n-1},x_{n-1},\cdots,(x_{(n-1)})_{n-1},a) \\ & + A(x_n,x_n,x_n,\cdots,(x_n)_{n-1},a). \end{aligned}$$

The pair (X,A) is called an A-metric space.

3. Y-cone Metric Spaces

In the following we always suppose E is a Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 3.1. *Let X be a nonempty set and $k \geq 1$ be a given real number. Suppose a mapping $Y : X^n \to E$ is called a Y-cone metric on X if, for any $s_i, a \in X, i = 1, 2, \cdots, n$, the following conditions hold:*

(Y1) $Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) \geq \theta$,
(Y2) $Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) = \theta \iff s_1 = s_2 = s_3 = \cdots = s_{n-1} = s_n$,
(Y3)

$$\begin{aligned} Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) \leq k[&Y(s_1, s_1, s_1, \cdots, (s_1)_{n-1}, a) \\ &+ Y(s_2, s_2, s_2, \cdots, (s_2)_{n-1}, a) \\ &\vdots \\ &+ Y(s_{n-1}, s_{n-1}, s_{n-1}, \cdots, (s_{(n-1)})_{n-1}, a) \\ &+ Y(s_n, s_n, s_n, \cdots, (s_n)_{n-1}, a)]. \end{aligned}$$

The pair (X, Y) is called an Y-cone metric space.

Note that cone b- metric space is a special case of Y-cone metric space with $n = 2$.

Proposition 3.1. *If (X, Y) is Y-cone metric space, then for all $s, u \in X$, we have $Y(s, s, \cdots, s, u) = Y(u, u, \cdots, u, s)$*

Example 3.1. *Let $X = \{1, 2, 3, 4, 5\}, E = R$ and $P = \{s \in E : s \geq 0\}$. Define $Y : X^n \to E$ by:*

$$Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) = \begin{cases} |s_1 - s_2|^{-1} + |s_2 - s_3|^{-1} + \cdots + |s_{n-1} - s_n|^{-1} & \text{if } s_i \neq s_j, \\ \theta & \text{if } s_i = s_j. \end{cases}$$

$\forall\, i, j = 1, 2, \cdots, n$. *Then (X, Y) is a Y-cone metric space with the coefficient $k = \frac{12}{7}$.*

Example 3.2. *Let $X = [0, 1]$ and $E = C^1_{\mathbb{R}}$ with $\| u \| = \| u \|_\infty + \| u' \|_\infty, u \in E$ and let $P = \{u \in E : u(t) \geq 0$ on $[0, 1]\}$. It is well known that this cone is solid but it is not normal. Define a Y-cone metric $Y : X^n \to E$ by*

$$\begin{aligned} Y(s_1, s_2, s_3, \cdots, s_{n-1}, s_n) =& [|s_1 - s_2|^2 + |s_1 - s_3|^2 + \cdots + |s_1 - s_n|^2 \\ &+ |s_2 - s_3|^2 + |s_2 - s_4|^2 + \cdots + |s_2 - s_n|^2 \\ &+ \cdots + |s_{n-1} - s_n|^2]e^t \\ =& \sum_{i=1}^{n} \sum_{i<j} |s_i - s_j|^2 e^t. \end{aligned}$$

Then (X, Y) is a complete Y-cone metric space with the coefficient $k = 2$.

Lemma 3.1. *Let X be a Y-cone metric space, then for all $s, u \in X$ we have, $Y(s, s, \cdots, s, z) \leq k[(n-1)Y(s, s, \cdots, s, u) + Y(z, z, \cdots, z, u)]$ and $Y(s, s, \cdots, s, z) \leq k[(n-1)Y(s, s, \cdots, s, u) + Y(u, u, \cdots, u, z)]$.*

Proof. Applying Proposition (3.1) and condition (Y3) of the Y-cone metric, we have

$$\begin{aligned} Y(s, s, \cdots, sz) &\leq k[Y(s, s, \cdots, s, u) + Y(s, s, \cdots, s, u) + \cdots + Y(z, z, \cdots, z, u)] \\ &= k[(n-1)Y(s, s, \cdots, s, u) + Y(z, z, \cdots, z, u)] \\ &= k[(n-1)Y(s, s, \cdots, s, u) + Y(u, u, \cdots, u, z)]. \end{aligned}$$

Hence the result. □

Definition 3.2. *Let (X, Y) is a Y-cone metric space. Then, for $s \in X$ and $\theta \ll p$, the Y-balls with center s and radius $\theta \ll p$ are*

$$B_Y(s,p) = \{u \in X : Y(s, s, \cdots, s, u) \ll p\}.$$

4. Topological Y-cone Metric Spaces

In this section, we define the topology of Y-cone metric space and study its topological properties.

Definition 4.1. *Let (X, Y) be a Y-cone metric space with coefficient $k \geq 1$. For each $s \in X$ and each $\theta \ll p$, put $B_Y(s,p) = \{u \in X : Y(s, s, \cdots, s, y) \ll p\}$ and put $B = \{B_Y(s,p) : s \in X \text{ and } \theta \ll p\}$. Then, B is a subbase for some topology τ on X.*

Remark 4.1. *Let (X, Y) be a Y-cone metric space. In this paper, τ denotes the topology on X, B denotes a subbase for the topology on τ and $B_Y(x,p)$ denotes the Y-ball in (X, Y), which are described in Definition (4.1). In addition, U denotes the base generated by subbase B.*

Theorem 4.1. *Let (X, Y) be a Y-cone metric space and let P be a solid cone in E. Let $m \in P$ be an arbitrarily given vector, then (X, Y) is a Hausdorff space.*

Proof. Let (X, Y) be a Y-cone metric space and let $s, u \in X$ with $s \neq u$.

Let $Y(s, s, \cdots, s, u) = p$. Let $U = B_Y(s, \frac{p}{2(n-1)})$ and $V = B_Y(u, \frac{p}{2})$. Then, $s \in U$ and $u \in V$.

We claim that $U \cap V = \emptyset$.
If not, there exist $z \in U \cap V$.

But then $Y(s, s, \cdots, s, z) \ll \frac{p}{2k(n-1)}$, $Y(u, u, \cdots, u, z) = \frac{p}{2k}$.
We get

$$\begin{aligned} p = Y(s, s, \cdots, s, u) &\leq k[(n-1)Y(s, s, \cdots, s, z) + Y(u, u, \cdots, u, z)] \\ &\ll k[(n-1)\frac{p}{2k(n-1)} + \frac{p}{2k}] \\ &\ll p. \end{aligned}$$

Which is a contradiction. Hence, $U \cap V = \emptyset$ and X is Hausdorff space. □

Definition 4.2. *Let (X, Y) be a Y-cone metric space. A sequence $\{s_n\}$ in X converges to s if for every $c \in E$ with $\theta \ll c$, there is a natural number N such that for all $n > N$, $Y(s_n, s_n, \cdots, s_n, s) \ll c$ for some fixed s in X. Hence s is called the limit of a sequence $\{s_n\}$ and is denoted by $\lim_{n\to\infty} s_n = s$ or $s_n \to s$ as $n \to \infty$.*

Definition 4.3. *Let (X, Y) be a Y-cone metric space. A sequence $\{s_n\}$ in X is called a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is a natural number N such that for all $n, m > N$, we have $Y(s_n, s_n, \cdots, s_n, s_m) \ll c$.*

Definition 4.4. *The Y-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.*

Definition 4.5. *Let (X, Y) and (X', Y') be a Y-cone metric space.Then a function $F : X \to X'$ is said to be continuous at a point $s \in X$ if and only if it is sequentially continuous at s, that is, whenever $\{s_n\}$ is convergent to s, we have $\{fs_n\}$ is convergent to $f(s)$.*

Lemma 4.1. *Let (X, Y) be a Y-cone metric space. If $\{s_n\}$ be a sequence in X converges to a point s, then s is unique.*

Lemma 4.2. *Let (X, Y) be a Y-cone metric space. If $\{s_n\}$ be a sequence in X converges to s, then $\{s_n\}$ is a Cauchy sequence.*

Remark 4.2. *The converse of Lemma 4.2 does not hold in general. A Cauchy sequence in an Y-cone metric space does not need to be convrgent. To see this we consider the space $(X = \mathbb{Q}, Y)$ with the Y-cone metric defined as in Example (3.2). Let $\{s_n\}$ be a sequence defined by $s_n = (1 + \frac{1}{n})^n$. Observe that $s_n \in \mathbb{Q}, \forall n \in \mathbb{N}$. Furthermore,*

$$Y(s_n, s_n, \cdots, s_n, s_m) = (n-1)|s_n - s_m|^2 e^t$$
$$= (n-1)|(1 + \frac{1}{n})^n - (1 + \frac{1}{m})^m|^2 e^t \to 0,$$

as $n, m \to \infty$. Thus, $\{s_n\}$ is Cauchy. But $s_n \to e$ as $n \to \infty$ and e is not in $\mathbb{Q}$. Hence $\{s_n\}$ does not converge.

Lemma 4.3. *Let (X, Y) be a Y-cone metric space. If there exist sequences $\{s_n\}, \{u_n\}$ such that $s_n \to s, u_n \to u$, then $\lim_{n\to\infty} Y(s_n, s_n, \cdots, s_n, u_n) = Y(s, s, \cdots, s, u)$.*

Remark 4.3. *Let (X, Y) be a Y-cone metric space over the ordered real Banach space E with a cone P. Then the following properties are often used:*

(1) *If $a \leq b$ and $b \ll c$, then $a \ll c$.*
(2) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(3) *If $\theta \leq u \ll c$ for each $c \in intP$, then $u = \theta$.*
(4) *If $c \in intP$, $\theta \leq a_n$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.*
(5) *If $\theta \leq a_n \leq b_n$ and $a_n \to a$, $b_n \to b$, thena $\leq b$, for each cone P.*
(6) *If E is a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P$ and $\theta \leq \lambda < 1$, then $a = \theta$.*
(7) *$\alpha\ intP \subseteq intP$ for $\alpha > 0$.*
(8) *For each $\delta > 0$ and $s \in intP$ there is $0 < \gamma < 1$, such that $\| \gamma s \| < \delta$.*
(9) *For each $\theta \ll c_1$ and $c_2 \in P$, there is an element $\theta \ll d$ such that $c_1 \ll d$, $c_2 \ll d$.*
(10) *For each $\theta \ll c_1$ and $\theta \ll c_2$, there is an element $\theta \ll e$ such that $e \ll c_1$, $e \ll c_2$.*

Especially properties (1), (3), (4) and (6) of this remark are often used, so we give their proofs.

Proof. (1) We have to prove that $c - a \in intP$ if $b - a \in P$ and $c - b \in intP$. There exists a neighbourhood of V of θ in E such that $c - b + V \subset P$. Then, from $b - a \in P$ it follows that

$$c - a + V = (c - b) + V + (b - a) \subset P + P \subset P,$$

Since P is convex.

(3) Since $c - u \in intP$ for each $c \in intP$, it follows that $\frac{1}{n}c - u \in intP$ for each $n \in \mathbb{N}$. Thus,

$$\lim_{n\to\infty} (\frac{1}{n}c - u) = \theta - u \in \overline{P} = P.$$

Hence $u \in -P \cap P = \{\theta\}$, that is $u = \theta$.

(4) Let $\theta \ll c$ be given. Choose a symmetric neighborhood V such that $c + V \subset P$. Since $a_n \to \theta$, there is n_0 such that $a_n \in V = -V$ for $n > n_0$. This means that $c \pm a_n \in c + V \subset P$ for $n > n_0$, that is, $a_n \ll c$.

(6) The condition $a \leq \lambda a$ means that $\lambda a - a \in P$, that is, $-(1-\lambda)a \in P$. Since $a \in P$ and $1-\lambda > 0$, we have also $(1-\lambda)a \in P$. Thus we have $(1-\lambda)a \in P \cap (-P) = \{\theta\}$, and $a = \theta$.

□

5. Coupled Common Fixed Point Results

Bhaskar and Lakshmikantham [6] initiated the study of a coupled fixed point theorem in ordered metric spaces and applied their results to prove the existence and uniqueness of a solutions for a periodic boundary value problem. Many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the framework of partial metric spaces [2, 7, 8, 9, 10, 13, 17, 15, 19, 22, 23]. For more delineate, we recommend the reader to [6]

Definition 5.1. [6] *An element $(s,u) \in X \times X$ is called coupled fixed point for the mapping $F : X \times X \to X$ if $F(s,u) = s, F(u,s) = u$.*

Recently, Gordji et al.[13] introduced the concepts of coupled common fixed points and mixed weakly monotone pair of mappings as follows.

Definition 5.2. [13] *Let $(X, \leq)$ be a partially ordered set and $F, H : X \times X \to X$ be two maps. The pair (F, H) is said to have the mixed weakly monotone property on X if, for all $s, u \in X$*

$$\begin{aligned} &s \leq F(s,u), u \geq F(u,s) \\ &\quad \Longrightarrow F(s,u) \leq H(F(s,u), F(u,s)), \;\; F(u,s) \geq H(F(u,s), F(s,u)) \end{aligned}$$

and

$$\begin{aligned} &s \leq H(s,u), u \geq H(u,s) \\ &\quad \Longrightarrow H(s,u) \leq F(H(s,u), H(u,s)), \;\; H(u,s) \geq F(H(u,s), H(s,u)). \end{aligned}$$

Remark 5.1. [13] *Let $(X, \leq)$ be a partially ordered set, $F : X \times X \to X$ be a map with the mixed monotone property on X. Then for all $n \in N$, the pair (F^n, F^n) has the mixed weakly monotone property on X.*

Now, we obtain common coupled fixed point results of mappings satisfying more general contractive conditions in the framework of partially ordered Y-cone metric spaces. we start with the following result.

Lemma 5.1. *Let (X, Y) be an Y-cone metric space, then $X \times X$ is an Y-cone metric space with the Y-cone metric D given by*

$$D((s_1,u_1),(s_2,u_2),\cdots,(s_n,u_n)) = Y(s_1,s_2,\cdots,s_n) + Y(u_1,u_2,\cdots,u_n)$$

for all $s_i, u_j \in X, i, j = 1, 2, 3, \cdots, n$.

Proof. For all $s_i, u_j \in X, i, j = 1, 2, 3, \cdots, n$, we have $D((s_1,u_1),(s_2,u_2),\cdots,(s_n,u_n)) \geq 0$.

Note that

$$\begin{aligned}
&D((s_1,u_1),(s_2,u_2),\cdots,(s_n,u_n))=0\\
&\iff Y(s_1,s_2,\cdots,s_n)+Y(u_1,u_2,\cdots,u_n)=0\\
&\iff Y(s_1,s_2,\cdots,s_n)=0, Y(u_1,u_2,\cdots,u_n)=0\\
&\iff s_1=s_2=\cdots=s_n, u_1=u_2=\cdots=u_n\\
&\iff (s_1,u_1)=(s_2,u_2)=\cdots=(s_n,u_n).
\end{aligned}$$

Consider

$$\begin{aligned}
D((s_1,u_1),(s_2,u_2),\cdots,(s_n,u_n)) &= Y(s_1,s_2,\cdots,s_n)+Y(u_1,u_2,\cdots,u_n)\\
&\le k[Y(s_1,s_1,\cdots,s_1,a)+Y(s_2,s_2,\cdots,s_2,a)\\
&\quad \vdots\\
&\quad +Y(s_n,s_n,\cdots,s_n,a)+Y(u_1,u_1,\cdots,u_1,b)\\
&\quad +Y(u_2,u_2,\cdots,u_2,b)\\
&\quad \vdots\\
&\quad +Y(u_n,u_n,\cdots,u_n,b)]\\
&= k[D((s_1,u_1),(s_1,u_1),\cdots,(s_1,u_1),(a,b))\\
&\qquad +D((s_2,u_2),(s_2,u_2),\cdots,(s_2,u_2),(a,b))\\
&\qquad \cdots+D((s_n,u_n),(s_n,u_n),\cdots,(s_n,u_n),(a,b))].
\end{aligned}$$

By the above D is an Y-cone metric on $X\times X$. □

Theorem 5.1. *Let $(X,\le,Y)$ be a partially ordered complete Y-cone metric space with the coefficient $k\ge 1$ relative to a solid cone P. Let $F,H:X\times X\to X$ be the mappings such that a pair (F,H) has the mixed weakly monotone property on X. Suppose that there exist $a_i\ge 0, i=1,2,\cdots,6$ with $a_1+a_2+a_3+2ka_4<1$ and $\sum_{i=1}^{6}a_i<1$ such that*

$$\begin{aligned}
&Y(F(s,u),F(s,u),\cdots,F(s,u),H(r,v))+Y(F(u,s),F(u,s),\cdots,F(u,s),H(v,r))\\
&\quad\le a_1D((s,u),(s,u),\cdots,(s,u),(r,v))\\
&\qquad +a_2D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s)))\\
&\qquad +a_3D((r,v),(r,v),\cdots,(r,v),(H(r,v),H(v,r)))\\
&\qquad +a_4D((s,u),(s,u),\cdots,(s,u),(H(r,v),H(r,v)))\\
&\qquad +a_5D((r,v),(r,v),\cdots,(r,v),(F(s,u),F(u,s)))\\
&\qquad +a_6(min\{D((r,v),(r,v),\cdots,(r,v),(H(r,v),H(v,r))),\\
&\qquad\qquad D((s,u),(s,u),\cdots,(s,u),(H(r,v),H(r,v))),\\
&\qquad\qquad D((r,v),(r,v),\cdots,(r,v),(F(s,u),F(u,s)))\})
\end{aligned}\tag{1}$$

$\forall\ s,u,r,v\in X$ with $s\le r$ and $u\ge v$, where D is defined as in Lemma 5.1. Suppose that there exists $s_0,u_0\in X$ such that $s_0\le F(s_0,u_0)$, $u_0\ge F(u_0,s_0)$ or $s_0\le H(s_0,u_0)$, $u_0\ge H(u_0,s_0)$, then F and H have a coupled common fixed point in X.

Proof. Choose $s_0,u_0\in X$ and set $s_1=F(s_0,u_0), u_1=F(u_0,s_0), s_2=H(s_1,u_1)$ and $u_2=H(u_1,s_1)$.

From the condition $s_0\le F(s_0,u_0)$, $u_0\ge F(u_0,s_0)$ and the fact that (F,H) has the mixed weakly monotone property we have

$$s_1 = F(s_0,u_0) \le H(F(s_0,u_0),F(u_0,s_0)) = H(s_1,u_1) \implies s_1 \le s_2,$$
$$s_2 = H(s_1,u_1) \le F(H(s_1,u_1),H(u_1,s_1)) = F(s_2,u_2) \implies s_2 \le s_3.$$

Thus,

$$u_1 = F(u_0,s_0) \ge H(F(u_0,s_0),F(s_0,u_0)) = H(u_1,s_1) \implies u_1 \ge u_2,$$
$$u_2 = H(u_1,s_1) \ge F(H(u_1,s_1),H(s_1,u_1)) = F(u_2,s_2) \implies u_2 \ge u_3.$$

Repeating this process, we obtain

$$s_{2n+1} = F(s_{2n},u_{2n}), \;\; u_{2n+1} = F(u_2n, s_2n),$$
$$s_{2n+2} = H(s_{2n+1},u_{2n+1}), \;\; u_{2n+2} = H(u_{2n+1},s_{2n+1}) \text{ for all } n \in N.$$

Therefore the sequences $\{s_n\}$ and $\{u_n\}$ are monotone:

$$s_0 \le s_1 \le \cdots \le s_n \le s_{n+1} \le \cdots$$
$$u_0 \ge u_1 \ge \cdots \ge u_n \ge u_{n+1} \ge \cdots .$$

Similarly, from the condition $s_0 \le H(s_0,u_0)$, $u_0 \ge H(u_0,s_0)$, one can show that the sequences $\{s_n\}$ and $\{u_n\}$ are increasing and decreasing respectively. Then by (1), we have

$$\begin{aligned}
&Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2}) + Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})\\
&= Y\big(F(s_{2n},u_{2n}),F(s_{2n},u_{2n}),\cdots,F(s_{2n},u_{2n}),H(s_{2n+1},u_{2n+1})\big)\\
&\quad + Y\big(F(u_{2n},s_{2n}),F(u_{2n},s_{2n}),\cdots,F(u_{2n},s_{2n}),H(u_{2n+1},s_{2n+1})\big)\\
&\le a_1 D\big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),(s_{2n+1},u_{2n+1})\big)\\
&\quad + a_2 D\Big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),\big(F(s_{2n},u_{2n}),F(u_{2n},s_{2n})\big)\Big)\\
&\quad + a_3 D\Big((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),\big(H(s_{2n+1},u_{2n+1}),H(u_{2n+1},s_{2n+1})\big)\Big)\\
&\quad + a_4 D\Big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),\big(H(s_{2n+1},u_{2n+1}),H(u_{2n+1},s_{2n+1})\big)\Big)\\
&\quad + a_5 D\Big((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),\big(F(s_{2n},u_{2n}),F(u_{2n},s_{2n})\big)\Big)\\
&\quad + a_6\Big[min\Big\{D\Big((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),\big(H(s_{2n+1},u_{2n+1}),H(u_{2n+1},s_{2n+1})\big)\Big),\\
&\qquad\quad D\Big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),\big(H(s_{2n+1},u_{2n+1}),H(u_{2n+1},s_{2n+1})\big)\Big),\\
&\qquad\quad D\big((s_{2n+1},u_{2n+1})),(s_{2n+1},u_{2n+1})),\cdots,(s_{2n+1},u_{2n+1}),\big(F(s_{2n},u_{2n}),F(u_{2n},s_{2n})\big)\Big)\Big\}\Big].
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2}) + Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})\\
&\le a_1 D\big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),(s_{2n+1},u_{2n+1})\big)\\
&\quad + a_2 D\big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),(s_{2n+1},u_{2n+1})\big)\\
&\quad + a_3 D\big((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(s_{2n+2},u_{2n+2})\big) \qquad (2)\\
&\quad + a_4 D\big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),(s_{2n+2},u_{2n+2})\big)\\
&\le (a_1+a_2+ka_4) D\big((s_{2n},u_{2n}),(s_{2n},u_{2n}),\cdots,(s_{2n},u_{2n}),(s_{2n+1},u_{2n+1})\big)\\
&\quad + (a_3+ka_4) D\big((s_{2n+1},u_{2n+1})),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(s_{2n+2},u_{2n+2})\big).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})+Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2})\\
&\quad\leq (a_1+a_2+ka_4)[Y(u_{2n},u_{2n},\cdots,u_{2n},u_{2n+1})+Y(s_{2n},s_{2n},\cdots,s_{2n},s_{2n+1})]\\
&\qquad+(a_3+ka_4)[Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})+Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2})].
\end{aligned} \tag{3}$$

It follows from (2) and (3) that

$$\begin{aligned}
&[Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2})+Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})]\\
&\quad\leq \frac{a_1+a_2+ka_4}{1-(a_3+ka_4)}[Y(s_{2n},s_{2n},\cdots,s_{2n},s_{2n+1})+Y(u_{2n},u_{2n},\cdots,u_{2n},u_{2n+1})].
\end{aligned}$$

Let $\delta=\frac{(a_1+a_2+ka_4)}{1-(a_3+ka_4)}$, then $0\leq\delta<1$ and

$$\begin{aligned}
&[Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2})+Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})]\\
&\quad\leq \delta[Y(s_{2n},s_{2n},\cdots,s_{2n},s_{2n+1})+Y(u_{2n},u_{2n},\cdots,u_{2n},u_{2n+1})].
\end{aligned} \tag{4}$$

For all $n\in N$, by interchanging the roles of F and H and using (1), we have

$$\begin{aligned}
&Y(s_{2n+2},s_{2n+2},\cdots,s_{2n+2},s_{2n+3})+Y(u_{2n+2},u_{2n+2},\cdots,u_{2n+2},u_{2n+3})\\
&\quad\leq \delta[Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2})+Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})].
\end{aligned} \tag{5}$$

It follows from (4) that

$$\begin{aligned}
&[Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2})+Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})]\\
&\quad\leq \delta[Y(s_{2n},s_{2n},\cdots,s_{2n},s_{2n+1})+Y(u_{2n},u_{2n},\cdots,u_{2n},u_{2n+1})]\\
&\quad\leq \delta\Big(\delta\big(Y(s_{2n-1},s_{2n-1},\cdots,s_{2n-1},s_{2n})+Y(u_{2n-1},u_{2n-1},\cdots,u_{2n-1},u_{2n})\big)\Big)\\
&\quad\leq \delta\Big(\delta\big(\delta(Y(s_{2n-2},s_{2n-2},\cdots,s_{2n-2},s_{2n-1})+Y(u_{2n-2},u_{2n-2},\cdots,u_{2n-2},u_{2n-1}))\big)\Big).
\end{aligned}$$

This implies

$$\begin{aligned}
&[Y(s_{2n+1},s_{2n+1},\cdots,s_{2n+1},s_{2n+2})+Y(u_{2n+1},u_{2n+1},\cdots,u_{2n+1},u_{2n+2})]\\
&\quad\leq \delta^3[Y(s_{2n-2},s_{2n-2},\cdots,s_{2n-2},s_{2n-1})+Y(u_{2n-2},u_{2n-2},\cdots,u_{2n-2},u_{2n-1})]\\
&\qquad\vdots\\
&\quad\leq \delta^{2n+1}(Y(s_0,s_0,\cdots,s_0,s_1)+Y(u_0,u_0,\cdots,u_0,u_1).
\end{aligned} \tag{6}$$

and similarly, by (5), we get

$$\begin{aligned}
&Y(s_{2n+2},s_{2n+2},\cdots,s_{2n+2},s_{2n+3})+Y(u_{2n+2},u_{2n+2},\cdots,u_{2n+2},u_{2n+3})\\
&\quad\leq \delta^{2n+2}[Y(s_0,s_0,\cdots,s_0,s_1)+Y(u_0,u_0,\cdots,u_0,u_1)].
\end{aligned} \tag{7}$$

By Lemma 3.1 we have for all $n, m \in N$ with $n \leq m$

$$
\begin{aligned}
&Y(s_{2n+1}, s_{2n+1}, \cdots, s_{2n+1}, s_{2m+1}) + Y(u_{2n+1}, u_{2n+1}, \cdots, u_{2n+1}, u_{2m+1})\\
&\quad \leq k[(n-1)Y(s_{2n+1}, s_{2n+1}, \cdots, s_{2n+1}, s_{2n+2}) + Y(s_{2n+2}, s_{2n+2}, \cdots, s_{2n+2}, s_{2m+1})]\\
&\qquad + k[(n-1)Y(u_{2n+1}, u_{2n+1}, \cdots, u_{2n+1}, u_{2n+2})) + Y(u_{2n+2}, u_{2n+2}, \cdots, u_{2n+2}, u_{2m+1})]\\
&\quad \leq (k(n-1)Y(s_{2n+1}, s_{2n+1}, \cdots, s_{2n+1}, s_{2n+2}) + k(n-1)Y(u_{2n+1}, u_{2n+1}, \cdots, u_{2n+1}, u_{2n+2}))\\
&\qquad + (k^2(n-1)Y(s_{2n+2}, s_{2n+2}, \cdots, s_{2n+2}, s_{2n+3}) + k^2(n-1)Y(u_{2n+2}, u_{2n+2}, \cdots, u_{2n+2}, u_{2n+3}))\\
&\qquad \cdots + (k^{2m-1}(n-1)Y(s_{2m-1}, s_{2m-1}, \cdots, s_{2m-1}, s_{2m}) + k^{2m-1}(n-1)Y(u_{2m-1}, u_{2m-1}, \cdots, u_{2m-1}, u_{2m}))\\
&\qquad + k^{2m-1}(Y(s_{2m}, s_{2m}, \cdots, s_{2m}, s_{2m+1}) + Y(u_{2m}, u_{2m}, \cdots, u_{2m}, u_{2m+1}))\\
&\quad \leq (k(n-1)Y(s_{2n+1}, s_{2n+1}, \cdots, s_{2n+1}, s_{2n+2}) + k(n-1)Y(u_{2n+1}, u_{2n+1}, \cdots, u_{2n+1}, u_{2n+2}))\\
&\qquad + (k^2(n-1)Y(s_{2n+2}, s_{2n+2}, \cdots, s_{2n+2}, s_{2n+3}) + k^2(n-1)Y(u_{2n+2}, u_{2n+2}, \cdots, u_{2n+2}, u_{2n+3}))+\\
&\qquad \vdots\\
&\qquad + (k^{2m}(n-1)Y(s_{2m}, s_{2m}, \cdots, s_{2m}, s_{2m+1}) + k^{2m}(n-1)Y(u_{2m}, u_{2m}, \cdots, u_{2m}, u_{2m+1}))\\
&\quad \leq k(n-1)\delta^{2n+1}(1 + k\delta + k^2\delta^2 + \cdots)(Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1)).
\end{aligned}
$$

This implies

$$
\begin{aligned}
&Y(s_{2n+1}, s_{2n+1}, \cdots, s_{2n+1}, s_{2m+1}) + Y(u_{2n+1}, u_{2n+1}, \cdots, u_{2n+1}, u_{2m+1})\\
&\quad \leq (n-1)\frac{k\delta^{2n+1}}{1-k\delta}(Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1)).
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&Y(s_{2n}, s_{2n}, \cdots, s_{2n}, s_{2m+1}) + Y(u_{2n}, u_{2n}, \cdots, u_{2n}, u_{2m+1})\\
&\quad \leq (n-1)\frac{k\delta^{2n}}{1-k\delta}(Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1)),
\end{aligned}
$$

and

$$
\begin{aligned}
&Y(s_{2n}, s_{2n}, \cdots, s_{2n}, s_{2m}) + Y(u_{2n}, u_{2n}, \cdots, u_{2n}, u_{2m})\\
&\quad \leq (n-1)\frac{k\delta^{2n}}{1-k\delta}(Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1))
\end{aligned}
$$

and

$$
\begin{aligned}
&Y(s_{2n+1}, s_{2n+1}, \cdots, s_{2n+1}, s_{2m}) + Y(u_{2n+1}, u_{2n+1}, \cdots, u_{2n+1}, u_{2m})\\
&\quad \leq (n-1)\frac{k\delta^{2n+1}}{1-k\delta}(Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1)).
\end{aligned}
$$

Hence, for all $n, m \in N$ with $n \leq m$, we have

$$
\begin{aligned}
&Y(s_n, s_n, \cdots, s_n, s_m) + Y(u_n, u_n, \cdots, u_n, u_m)\\
&\quad \leq (n-1)\frac{k\delta^{n}}{1-k\delta}(Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1))
\end{aligned}
$$

According to Remark 3.12(4), and for any $c \in E$ with $\theta \ll c$, there exists n_0 such that for any $> n_0$, $(n-1)\frac{k\delta^n}{1-k\delta}(Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1)) \ll c$. Furthermore, for any $m > n > n_0$, Remark (4.3) (1) shows that $Y(s_0, s_0, \cdots, s_0, s_1) + Y(u_0, u_0, \cdots, u_0, u_1) \ll c$. Hence, by Definition (4.3), $\{s_n\}$ and $\{u_n\}$ are Cauchy sequences in X. By the completeness of X, $\exists\ s, u \in X$ such that

$$\lim_{n\to\infty} s_n = s \text{ and } \lim_{n\to\infty} u_n = u.$$

Now we show that (s, u) is a coupled common fixed point of F and H.

Suppose F is continuous, then we have

$$\begin{aligned} s &= \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} F(s_n, u_n) \\ &= F(\lim_{n\to\infty} s_n, \lim_{n\to\infty} u_n) \\ &= F(s, u), \end{aligned}$$

and

$$\begin{aligned} u &= \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} F(u_n, s_n) \\ &= F(\lim_{n\to\infty} u_n, \lim_{n\to\infty} s_n) \\ &= F(u, s). \end{aligned}$$

Taking $s = r$ and $u = v$ in (1), we get

$$\begin{aligned} &Y(F(s,u), F(s,u), \cdots, F(s,u), H(s,u)) + Y(F(u,s), F(u,s), \cdots, F(s,u), H(u,s)) \\ &\quad \le a_1 D((s,u),(s,u),\cdots,(s,u),(s,u)) + a_2 D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s))) \\ &\qquad + a_3 D((s,u),(s,u),\cdots,(s,u),(H(s,u),H(u,s))) \\ &\qquad + a_4 D((s,u),(s,u),\cdots,(s,u),(H(s,u),H(u,s))) \\ &\qquad + a_5 D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s))) \\ &\qquad + a_6 (min\{D((s,u),(s,u),\cdots,(s,u),(H(s,u),H(u,s))), \\ &\qquad\qquad D((s,u),(s,u),\cdots,(s,u),(H(s,u),H(u,s))), \\ &\qquad\qquad D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s)))\}) \\ &\quad = (a_3 + a_4) D((s,u),(s,u),\cdots,(s,u),(H(s,u),H(u,s))) \\ &\quad = (a_3 + a_4)(Y(s,s,\cdots,s,H(s,u)) + Y(u,u,\cdots,u,H(u,s))). \end{aligned}$$

Therefore

$$\begin{aligned} &Y(s,s,\cdots,s,H(s,u)) + Y(u,u,\cdots,u,H(u,s)) \\ &\quad \le (a_3 + a_4)(Y(s,s,\cdots,s,H(s,u)) + Y(u,u,\cdots,u,H(u,s))). \end{aligned}$$

Since $0 \le (a_3 + a_4) < 1$, Remark 3.12(6) shows that $Y(s, s, \cdots, s, H(s, u)) + Y(u, u, \cdots, u, H(u, s)) = \theta$. Hence, $Y(s, s, \cdots, s, H(s, u)) = \theta$ and $Y(u, u, \cdots, u, H(u, s)) = \theta$. That is $H(s, u) = s$ and $H(u, s) = u$. This implies (s, u) is a coupled fixed point of H. Similarly, we can prove that (s, u) is a coupled fixed point of F when H is a continuous mapping. This completes the proof. □

Theorem 5.2. *Suppose all the assumptions of Theorem 5.1 are satisfied . Moreover, assume that X has the following properties*

(a) *if a non-decreasing sequence$\{s_n\}$ in X converges to some point $s \in X$, then $s_n \le s$, $\forall n$,*

(b) *if a non-increasing sequence $\{u_n\}$ in X converges to some point $u \in X$, then $u_n \ge u$, $\forall n$.*

Then the conclusion of Theorem 5.1 also hold.

Proof. Following the proof of Theorem 5.1 we only have to check that (s, u) is a coupled fixed point of F.

In fact, since $\{s_n\}$ is non-decreasing and $s_n \to s$ and $\{u_n\}$ is non-increasing and $u_n \to u$, by our assumption, $s_n \le s$ and $u_n \ge u$ $\forall$ n.

Applying the contractive condition we have

$$\begin{aligned}
&Y(s,s,\cdots,s,F(s,u))+Y(u,u,\cdots,u,F(u,s))\\
&\quad\le k[(n-1)Y(s,s,\cdots,s,s_{2n+2})+(n-1)Y(u,u,\cdots,u,u_{2n+2})\\
&\qquad+Y(s_{2n+2},s_{2n+2},\cdots,s_{2n+2},F(s,u))+Y(u_{2n+2},u_{2n+2},\cdots,u_{2n+2},F(u,s))]\\
&\quad= k[(n-1)Y(s,s,\cdots,s,s_{2n+2})+(n-1)Y(u,u,\cdots,u,u_{2n+2})\\
&\qquad+Y(H(s_{2n+1},u_{2n+1}),H(s_{2n+1},u_{2n+1}),\cdots,H(s_{2n+1},u_{2n+1}),F(s,u))\\
&\qquad+Y(H(u_{2n+1},s_{2n+1}),H(u_{2n+1},s_{2n+1}),\cdots,H(u_{2n+1},s_{2n+1}),F(u,s))].
\end{aligned}\tag{8}$$

By using (1) and interchanging the roles of F and H we obtain

$$\begin{aligned}
&Y(H(s_{2n+1},u_{2n+1}),H(s_{2n+1},u_{2n+1}),\cdots,H(s_{2n+1},u_{2n+1}),F(s,u))\\
&\quad+Y(H(u_{2n+1},s_{2n+1}),H(u_{2n+1},s_{2n+1}),\cdots,H(u_{2n+1},s_{2n+1}),F(u,s))\\
&\quad\le a_1D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(s,u))\\
&\qquad+a_2D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(s_{2n+2},u_{2n+2}))\\
&\qquad+a_3D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s)))\\
&\qquad+a_4D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(F(s,u),F(u,s)))\\
&\qquad+a_5D((s,u),(s,u),\cdots,(s,u),(s_{2n+2},u_{2n+2}))\\
&\qquad+a_6(min\{D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s))),\\
&\qquad\qquad D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(F(s,u),F(u,s))),\\
&\qquad\qquad D((s,u),(s,u),\cdots,(s,u),(s_{2n+2},u_{2n+2})).
\end{aligned}\tag{9}$$

It follows (8) and (9) that

$$\begin{aligned}
&Y(s,s,\cdots,s,F(s,u))+Y(u,u,\cdots,u,F(u,s))\\
&\quad\le k[(n-1)Y(s,s,\cdots,s,s_{2n+2})+(n-1)Y(u,u,\cdots,u,u_{2n+2})\\
&\qquad+a_1D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(s,u))\\
&\qquad+a_2D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(s_{2n+2},u_{2n+2}))\\
&\qquad+a_3D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s)))\\
&\qquad+a_4D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(F(s,u),F(u,s)))\\
&\qquad+a_5D((s,u),(s,u),\cdots,(s,u),(s_{2n+2},u_{2n+2}))\\
&\qquad+a_6(min\{D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s))),\\
&\qquad\qquad D((s_{2n+1},u_{2n+1}),(s_{2n+1},u_{2n+1}),\cdots,(s_{2n+1},u_{2n+1}),(F(s,u),F(u,s))),\\
&\qquad\qquad D((s,u),(s,u),\cdots,(s,u),(s_{2n+2},u_{2n+2}))].
\end{aligned}\tag{10}$$

Taking the limit as $n \to \infty$ in above inequality, we have

$$\begin{aligned}
&Y(s, s, \cdots, s, F(s, u)) + Y(u, u, \cdots, u, F(u, s)) \\
&\quad \le k[(n-1)Y(s, s, \cdots, s, s) + (n-1)Y(u, u, \cdots, u, u) + a_1 D((s, u), (s, u), \cdots, (s, u), (s, u)) \\
&\qquad + a_2 D((s, u), (s, u), \cdots, (s, u), (s, u)) + a_3 D((s, u), (s, u), \cdots, (s, u), (F(s, u), F(u, s))) \\
&\qquad + a_4 D((s, u), (s, u), \cdots, (s, u), (F(s, u), F(u, s))) + a_5 D((s, u), (s, u), \cdots, (s, u), (s, u)) \\
&\qquad + a_6 (min\{D((s, u), (s, u), \cdots, (s, u), (F(s, u), F(u, s))), \\
&\qquad\quad D((s, u), (s, u), \cdots, (s, u), (F(s, u), F(u, s))) \\
&\qquad\quad D((s, u), (s, u), \cdots, (s, u), (s, u))\}) \\
&\quad = k[a_3 D((s, u), (s, u), \cdots, (s, u), (F(s, u), F(u, s))) \\
&\qquad\quad + a_4 D((s, u), (s, u), \cdots, (s, u), (F(s, u), F(u, s)))].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Y(s, s, \cdots, s, F(s, u)) + Y(u, u, \cdots, u, F(u, s)) \\
&\quad \le k(a_3 + a_4) D((s, u), (s, u), \cdots, (s, u), (F(s, u), F(u, s))) \\
&\quad = k(a_3 + a_4)[Y(s, s, \cdots, s, F(s, u)) + Y(u, u, \cdots, u, F(u, s))].
\end{aligned}$$

Since, $0 \le k(a_3+a_4) < 1$ Remark (4.3)(6) shows $Y(s, s, \cdots, s, F(s, u))+Y(u, u, \cdots, u, F(u, s)) = \theta$, that is, $F(s, u) = s$ and $F(u, s) = u$. Similarly, one can show that $H(s, u) = s$ and $H(u, s) = u$. This proves that (s, u) is a coupled common fixed point of F and H and this finishes the proof. □

Now, we give a sufficient condition for the uniqueness of the coupled fixed point in Theorem 5.1 and 5.2. This condition is

for $(s, u), (r, v) \in X \times X$ there exists $(z, t) \in X \times X$ which is comparable to (s, u) and (r, v). (11)

Note that in $X \times X$ we consider the partial order relation given by

$$(s, u) \le (r, v) \iff s \le r \text{ and } u \ge v.$$

Theorem 5.3. *Adding condition* (11) *to the hypotheses of Theorem 5.1 (resp. Theorem 5.2) we obtain uniqueness of the coupled common fixed point of F and H. Furthermore, any fixed point of F is a fixed point of H and conversely.*

Proof. Suppose (s, u) and (s', u') are coupled common fixed points of F and H, that is, $F(s, u) = s, F(u, s) = u, H(s', u') = s'$ and $H(u', s') = u'$. We shall prove that $s = s', u = u'$.

Let (s, u) and (s', u') are not comparable. By assumption there exist $(z, t) \in X \times X$ comparable with both of them. Suppose that $(s, u) \le (s', u')$ without loss of generality, then it follows from Theorem 5.1.

$$
\begin{aligned}
&Y(s,s,\cdots,s,s')+Y(u,u,\cdots,u,u')\\
&\quad=Y(F(s,u),F(s,u),\cdots,F(s,u),H(s',u'))+Y(F(u,s),F(u,s),\cdots,F(s,u),H(u',s'))\\
&\quad\le(a_1+a_4+a_5)D((s,u),(s,u),\cdots,(s,u),(s',u'))\\
&\quad=(a_1+a_4+a_5)Y(s,s,\cdots,s,s')+Y(u,u,\cdots,u,u').
\end{aligned}
$$

Since, $0\le(a_1+a_4+a_5)<1$, Remark (4.3)(6) shows that $Y(s,s,\cdots,s,s')+Y(u,u,\cdots,u,u')=\theta$, which implies $s=s'$ and $u=u'$.

Now, we show that any fixed point of F is a fixed point of H and conversely. Applying Theorem 5.1, we get

$$
\begin{aligned}
&Y(s,s,\cdots,s,u)+Y(u,u,\cdots,u,s)\\
&\quad=Y(F(s,u),F(s,u),\cdots,F(s,u),H(u,s))\\
&\quad\ \ +Y(F(u,s),F(u,s),\cdots,F(u,s),H(s,u))\\
&\quad\le(a_1+a_4+a_5)D((s,u),(s,u),\cdots,(s,u),(u,s))\\
&\quad=(a_1+a_4+a_5)Y(s,s,\cdots,s,u)+Y(u,u,\cdots,u,s).
\end{aligned}
$$

Since, $0\le(a_1+a_4+a_5)<1$, Remark (4.3)(6) shows $Y(s,s,\cdots,s,u)+Y(u,u,\cdots,u,s)=\theta$, which implies $s=u$. The coupled common fixed point of F and H is unique. This finishes the proof. □

Example 5.1. *Let $(X,\le,Y)$ be a totally ordered complete Y-cone metric space with Y-cone metric defined as in Example 3.2.Let $F,H:X\times X\to X$ as $F(s,u)=H(s,u)=\frac{(s+2u)}{7}$ for all $s,u\in X$.*

The pair (F,H) has the mixed weakly monotone property on X and

$$
\begin{aligned}
&Y(F(s,u),F(s,u),\cdots,F(s,u),H(r,v))+Y(F(u,s),F(u,s),\cdots,F(u,s),H(u,s))\\
&\quad=\Big[(n-1)|F(s,u)-H(r,v)|^2+(n-1)|F(u,s)-H(v,r)|^2\Big]e^t\\
&\quad=\Big[(n-1)|\frac{s+2u}{7}-\frac{r+2v}{7}|^2+(n-1)|\frac{u+2s}{7}-\frac{v+2r}{7}|^2\Big]e^t\\
&\quad=\Big[(n-1)|(\frac{s}{7}-\frac{r}{7})+(\frac{2u}{7}-\frac{2v}{7})|^2+(n-1)|(\frac{u}{7}-\frac{v}{7})+(\frac{2r}{7}-\frac{2s}{7})|^2\Big]e^t\\
&\quad\le2(n-1)\Big[|\frac{s}{7}-\frac{r}{7}|^2+|\frac{2u}{7}-\frac{2v}{7}|^2+|\frac{u}{7}-\frac{v}{7}|^2+|\frac{2r}{7}-\frac{2s}{7}|^2\Big]e^t\\
&\quad=\frac{2(n-1)}{49}\Big[|s-r|^2+|2u-2v|^2+|u-v|^2+|2r-2s|^2\Big]e^t\\
&\quad\le\frac{2(n-1)}{7}\Big[|s-r|^2+|u-v|^2\Big]e^t\\
&\quad=\frac{2}{7}D((s,u),(s,u),\cdots,(s,u),(r,v)).
\end{aligned}
$$

where $a_1=\frac{2}{7},a_2=a_3=a_4=a_5=a_6=0$. Hence, the conditions of Theorem 5.1 are satisfied. Moreover, $(0,0)$ is the unique coupled common fixed point of F and H.

Corollary 5.1. *Let $(X,\le,Y)$ be a partially ordered complete Y-cone metric space with the coefficient $k\ge1$ relative to a solid cone P. Let $F:X\times X\to X$ be the mappings such that F has the mixed monotone property on X. Suppose that there exist $a_i\ge0$ with*

$a_1 + a_2 + a_3 + 2ka_4 < 1$ and $\sum_i^6 a_i < 1$ such that

$$\begin{aligned} &Y(F(s,u), F(s,u), \cdots, F(s,u), F(r,v)) \\ &\quad + Y(F(u,s), F(u,s), \cdots, F(u,s), F(v,r)) \\ &\leq a_1 D(((s,u),(s,u),\cdots,(s,u),(r,v))) \\ &\quad + a_2 D((s,u),(s,u),\cdots,(s,u),(F(s,u),F(u,s))) \\ &\quad + a_3 D((r,v),(r,v),\cdots,(r,v),(F(r,v),F(v,r))) \\ &\quad + a_4 D((s,u),(s,u),\cdots,(s,u),(F(r,v),F(r,v))) \\ &\quad + a_5 D((r,v),(r,v),\cdots,(r,v),(F(s,u),F(u,s))) \\ &\quad + a_6 (min\{D((r,v),(r,v),\cdots,(r,v),(F(r,v),F(v,r))), \\ &\qquad D((s,u),(s,u),\cdots,(s,u),(F(r,v),F(r,v))), \\ &\qquad D((r,v),(r,v),\cdots,(r,v),(F(s,u),F(u,s)))\}) \end{aligned} \tag{12}$$

$\forall\ s,u,r,v \in X$ with $s \leq r$ and $u \geq v$, where D is defined as in Lemma 5.1.

Suppose either F is continous or X has the following properties

(a) *if a non-decreasing sequence$\{s_n\}$ in X converges to some point $s \in X$, then $s_n \leq s$, $\forall n$,*
(b) *if a non-increasing sequence $\{u_n\}$ in X converges to some point $u \in X$, then $u_n \geq u$, $\forall n$.*

If there exists $s_0, u_0 \in X$ such that $s_0 \leq F(s_0, u_0)$ and $u_0 \geq F(u_0, s_0)$, then F has a unique coupled fixed point.

Corollary 5.2. *Let $(X, \leq, Y)$ be a partially ordered complete Y-cone metric space with the coefficient $k \geq 1$ relative to a solid cone P. Let $F : X \times X \to X$ be the mappings such that F has the mixed monotone property on X. Suppose that there exist $K \in [0,1)$ such that*

$$\begin{aligned} &Y(F(s,u), F(s,u), \cdots, F(s,u), F(r,v)) \\ &\quad + Y(F(u,s), F(u,s), \cdots, F(u,s), F(v,r)) \\ &\leq K(Y(s,s,\cdots,s,r) + Y(u,u,\cdots,u,v)) \end{aligned} \tag{13}$$

$\forall\ s,u,r,v \in X$ with $s \leq r$ and $u \geq v$.

Suppose either F is continous or X has the following properties

(a) *if a non-decreasing sequence$\{s_n\}$ in X converges to some point $s \in X$, then $s_n \leq s$, $\forall n$,*
(b) *if a non-increasing sequence $\{u_n\}$ in X converges to some point $u \in X$, then $u_n \geq u$, $\forall n$.*

If there exists $s_0, u_0 \in X$ such that $s_0 \leq F(s_0, u_0)$ and $u_0 \geq F(u_0, s_0)$, then F has a unique coupled fixed point.

Proof. Applying $H = F$ and $a_1 = K, a_2 = a_3 = a_4 = a_5 = a_6 = 0$ in Theorems (5.1) and (5.2) and using Remark (5.1), we obtain the corollary. □

6. Application to integral equations

In this section, we study the existence of a unique solution to an initial solution to an initial value problem, as an application to Theorem 5.1.

Consider the initial value problem

$$\begin{aligned} s'(z) &= f(z, s(z), s(z)), \ \ z \in I = [0,1], \\ s(0) &= s_0, \end{aligned} \tag{14}$$

where $f : I \times [s_0, \infty] \times [s_0, \infty] \to \times [s_0, \infty]$ and $s_0 \in \mathbb{R}$.

Theorem 6.1. *Consider the initial value problem* (14) *with* $f \in C(I \times [s_0, \infty] \times [s_0, \infty])$ *for the initial value problem* (14) *and*

$$\int_0^z f(p, s(p), u(p))dp \leq \frac{1}{4}\int_0^z f(p, s(p), s(p))dp - \frac{1}{4}\int_0^z f(p, u(p), u(p))dp$$

Then there exists a unique solution in $C(I, [x_0, \infty])$ *for the initial value problem* (14).

Proof. The integral equation corresponding to initial value problem (14) is

$$s(z) = s_0 + \int_0^z f(p, x(p), x(p))dp. \tag{15}$$

Let $X = C(I, [s_0, \infty])$ and Y-cone metric defined as in Example 3.2 for $s, u \in X$.
Define $F : X \times X \to X$ by

$$F(s, u)(z) = \frac{s_0}{4} + \int_0^z f(p, s(p), u(p))dp.$$

Now

$$\begin{aligned} & Y(F(s,u), F(s,u), \cdots, F(s,u), H(r,v)) \\ & = \Big[(n-1)|F(s,u) - H(r,v)|^2\Big]e^t \\ & = \Big[(n-1)|(\frac{s_0}{4} + \int_0^z f(p, s(p), u(p))dp) - (\frac{s_0}{4} + \int_0^z f(p, r(p), v(p))dp)|^2\Big]e^t \\ & \leq \Big[(n-1)|(\frac{1}{4}\int_0^z f(p, s(p), s(p))dp - \frac{1}{4}\int_0^z f(p, u(p), u(p))dp) \\ & - (\frac{1}{4}\int_0^z f(p, r(p), r(p))dp - \frac{1}{4}\int_0^z f(p, v(p), v(p))dp)|^2\Big]e^t \\ & = \Big[(n-1)|(\frac{1}{4}\int_0^z f(p, s(p), s(p))dp - \frac{1}{4}\int_0^z f(p, r(p), r(p))dp) \\ & - (\frac{1}{4}\int_0^z f(p, u(p), u(p))dp - \frac{1}{4}\int_0^z f(p, v(p), v(p))dp)|^2\Big]e^t \\ & \leq \Big[\frac{(n-1)}{16}|(\int_0^z f(p, s(p), s(p))dp - \int_0^z f(p, r(p), r(p))dp)|^2 \\ & + (\int_0^z f(p, u(p), u(p))dp - \int_0^z f(p, v(p), v(p))dp)|^2\Big]e^t \\ & \leq \frac{1}{16}D((s,u), (s,u), \cdots, (s,u), (r,v)) \end{aligned} \tag{16}$$

In similar way, we can also find

$$Y(F(u,s), F(u,s), \cdots, F(u,s), H(u,s)) \leq \frac{1}{16}D((s,u), (s,u), \cdots, (s,u), (r,v)) \tag{17}$$

From (16) and (17), we attain

$$Y(F(s,u),F(s,u),\cdots,F(s,u),H(r,v))+Y(F(u,s),F(u,s),\cdots,F(u,s),H(u,s))$$
$$\leq \frac{1}{8}D((s,u),(s,u),\cdots,(s,u),(r,v))$$

where $a_1=\frac{1}{8}, a_2=a_3=a_4=a_5=a_6=0$. hence, the conditions of Theorem 5.1 are satisfied. From Theorem 5.1, we conclude that F possesses a unique coupled fixed point (s,u) with $s=u$. In particular $s(z)$ is the unique solution of the integral equation (14). □

7. Conclusions

As presented at the beginning of this work, many eminent researchers gave the new ideas of generalization of metric spaces. Obtaining results as concerns the existence and uniqueness of certain coupled common fixed point theorems involving a contractive condition for a map possesses the property of mixed weakly monotone in the new framework of Y-cone metric spaces. Lastly, we provide an example and application to an integral equation to support our result.

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References

[1] Abbas, M. ,Khan, M. A. and Radenovic, S., (2010), Common coupled fixed point theorem in cone metric space for w-compatible mappings, Appl. Math. Comput. 217, 195-202.

[2] Abbas, M. , Ali, B. and Suleiman, Y. I., (2015), Generalized coupled common fixed point results in A-metric spaces, Fixed Point Theory and Applications, 2015, Article ID 64.

[3] Abdellaoui, M. A. , Dahmani, Z. and Bedjaoui, N., (2015), Applications of fixed point theorems for coupled systems of fractional integro-differential equations involving convergent series, IAENG International Journal of Applied Mathematics, 45(4), 1-6.

[4] Al-Mazrooei, A. E., Ahmad J., (2017), Fixed point results for multivalued mappings in G_b-cone metric spaces, Journal of Nonlinear Sciences and Applications, 10(9), 4866-4875.

[5] Arshad, M. , Azam, A. and Vetro, P., (2009), Some common fixed point results in cone metric spaces, Fixed Point Theory and Applications, 2009, Article ID 493965.

[6] Bhaskar, T.G. and Lakshmikantham, V., (2006), Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis: Theory, Methods and Applications, 65(7), 1379-1393.

[7] Choudhury, B. S. and Kundu, A., (2010), A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Analysis: Theory, Methods and Applications, 73, 2524-2531.

[8] Chouhan, V. S. and Sharma, R., (2017), Coupled fixed point theorems involving rational expressions in partially ordered cone metric spaces, Commun. Optim. Theory, 2017, Article ID 14.

[9] Ciric, L. and Lakshmikantham, V., (2009), Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, Stochastic and Applications, 27, 1246-1259.

[10] Ciric, L.and Lakshmikantham, V., (2009), Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Methods and Applications, 70(12), 4341-4349.

[11] Czerwik, S., (1998), Nonlinear set- valued contraction mappings in b- metric spaces, Atti Sem. Mat. Fiz. Univ. Modena, 46, 263-276.

[12] Fairouz, T., Calogero, V. and Francesca, V., (2017), A coincidence- point problem of Perov type on rectangular cone metric spaces, J. Nonlinear Sci. Appl., 10(8), 4307-4317.

[13] Gordji, M. E. , Akbartabar, E. , Cho, Y. J. and Ramezani, M. , (2012), Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces, Fixed Point Theory and Applications, 2012, Article ID 95.

[14] Haung, L. G. and Zhang, X., (2007), Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332, 1468-1476.

[15] Luong, N. V. and Thuan, N. X. , (2011), Coupled fixed points in partially ordered metric spaces and application, Nonlinear Analysis: Theory, Methods and Applications 74, 983-992.
[16] Mustafa, Z. and Sims, B., (2006), A new approach to generalized metric spaces, J. Nonlinear Convex Anal., 7(2), 289-297.
[17] Razani, A. and Parvaneh, V. , (2012), Coupled coincidence point results for (ψ, α, β)-weak contractions in partially ordered metric spaces, Journal of Applied Mathematics, Article ID 496103.
[18] Rezapour, S. and Hamlbarani, R., (2008), Some notes on the paper " Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl., 345, 719-724.
[19] Sabetghadam, F. , Masiha, H.P. and Sanatpour, A.H., (2009), Some coupled fixed point theorems in cone metric spaces, Fixed Point Theory and Applications, Article ID 125426.
[20] Sedghi, S. , Shobe, N. and Aliouche, A., (2012), A generalization of fixed point theorems in S- metric spaces, Mat. Vesnik, 64 (3), 258-266.
[21] Sharma, R., Chouhan, V. S. and Mishra, S. , (2019), Coupled fixed point theorems for rational contractions in partially ordered cone metric spaces, AIP Conference Proceedings, 2095, 1-7.
[22] Sharma, R., Chouhan, V. S. and Mishra, S. , (2020), Some coupled coincidence point result using altering distances with ordered metric spaces, International Journal on Emerging Technologies, 11(2), 705-713.
[23] Shatanawi, W. , Samet, B. and Abbas, M., (2012), Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, Math Comput Model, 55, 680-687.
[24] Turkoglu, D. and Abuloha, M. , (2010), Cone metric spaces and fixed point theorems in diametrically contractive mappings, Acta Math. Sin. (Eng. Ser.), 26(3), 489-496.

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