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$(p,q)-$CHEBYSHEV POLYNOMIALS AND THEIR APPLICATIONS TO BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, a subclass of analytic and bi-univalent functions by means of $(p,q)-$Chebyshev polynomials is introduced. Certain coefficient bounds for functions belong to this subclass are obtained. Furthermore, the Fekete-Szegö problem in this subclass is solved.

Keywords: Analytic functions, bi-univalent functions, Fekete-Szegö problem, Chebyshev polynomials, coefficient bounds, subordination.

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{A}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Given two functions f, $g \in \mathcal{A}$. The function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$, analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [9] and [17]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Koebe one-quarter theorem [5] asserts that the image of $\mathbb{U}$ under each univalent function f in $\mathcal{A}$ contains a disk of radius $\frac{1}{4}$. According to this, every function $f \in \mathcal{A}$ has an inverse map f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

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and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \geq \tfrac{1}{4}\right).$$

In fact, the inverse function is given by

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(w)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and some intriguing examples of functions and characterization of the class Σ, see Srivastava et al. [11] and Amourah [6], we employ techniques similar to these used earlier by [1, 2, 3, 19, 12, 21], see also [15, 7, 8, 10, 13, 20].

For any integer $n \geq 2$ and $0 < q < p \leq 1$, $(p, q)-$Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$U_n(x, s, p, q) = (p^n + q^n)\, xU_{n-1}(x, s, p, q) + (pq)^{n-1} sU_{n-2}(x, s, p, q), \tag{3}$$

with the initial values $U_0(x, s, p, q) = 1$ and $U_1(x, s, p, q) = (p + q)\, x$ and s is a variable.

Very recently, Kızılateş et al. [14], defined $(p, q)-$Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

The generating function of the $(p, q)-$Chebyshev polynomials of the second kind is as follows:

$$H_{p,q}(z) = \frac{1}{1 - xpz\tau_p - xqz\tau_q - spqz^2\tau_{p,q}} = \sum_{n=0}^{\infty} U_n(x, s, p, q)z^n \quad (z \in \mathbb{U}).$$

where the Fibonacci operator τ_q was introduced in Mason and Handscomb (see [16]), by $\tau_q f(z) = f(qz)$.

Similarly, $\tau_{p,q} f(z) = f(pqz)$.

Definition 1.1. *For $\lambda \geq 1$, and $\mu \geq 0$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(\lambda, \mu, p, q)$ if the following subordinations hold for all $z, w \in \mathbb{U}$:*

$$(1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) + \mu z f''(z) \prec H_{p,q}(z) = \frac{1}{1 - xpz\tau_p - xqz\tau_q - spqz^2\tau_{p,q}} \tag{4}$$

and

$$(1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) + \mu w g''(w) \prec H_{p,q}(w) = \frac{1}{1 - xpw\tau_p - xqw\tau_q - spqw^2\tau_{p,q}}, \tag{5}$$

where the function $g(w) = f^{-1}(w)$ is defined by (2).

2. Coefficient bounds for the function class $\mathcal{B}_\Sigma(\lambda, \mu, p, q)$

We begin with the following result involving initial coefficient bounds for the function class $\mathcal{B}_\Sigma(\lambda, \mu, p, q)$.

Theorem 2.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_\Sigma(\lambda, \mu, p, q)$. Then*

$$|a_2| \leq \frac{(p+q)\,x\sqrt{(p+q)\,x}}{\sqrt{\left|\left[2\,(1+2\lambda+6\mu)\,(p+q)^2\,x^2 - 2\left[(p^2+q^2)\,x^2\,(p+q)+pqs\right](1+\lambda+2\mu)^2\right]\right|}} \tag{6}$$

and

$$|a_3| \leq \frac{(p+q)^3\,x^3}{\left|\left[2\,(1+2\lambda+6\mu)\,(p+q)^2\,x^2 - 2\left[(p^2+q^2)\,x^2\,(p+q)+pqs\right](1+\lambda+2\mu)^2\right]\right|} \tag{7}$$

$$+\frac{(p+q)\,x}{1+2\lambda+6\mu}. \tag{8}$$

Proof. Let $f \in \mathcal{B}_\Sigma(\lambda,\mu,p,q)$. From (4) and (5), we have

$$(1-\lambda)\,\frac{f(z)}{z}+\lambda f'(z)+\mu z f''(z) = H_{p,q}(w(z)) \tag{9}$$

and

$$(1-\lambda)\,\frac{g(w)}{w}+\lambda g'(w)+\mu w g''(w) = H_{p,q}(v(w)), \tag{10}$$

for some analytic functions

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (z \in \mathbb{U}),$$

and

$$v(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \quad (w \in \mathbb{U}),$$

such that $w(0) = v(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$ and $|v(w)| < 1$ $(w \in \mathbb{U})$.

It follows from (9) and (10) that

$$\begin{aligned}&(1-\lambda)\,\frac{f(z)}{z}+\lambda f'(z)+\mu z f''(z)\\ &= 1 + U_1(x,s,p,q)c_1 z + \left[U_1(x,s,p,q)c_2 + U_2(x,s,p,q)c_1^2\right] z^2 + \cdots\end{aligned}$$

and

$$\begin{aligned}&(1-\lambda)\,\frac{g(w)}{w}+\lambda g'(w)+\mu w g''(w)\\ &= 1 + U_1(x,s,p,q)d_1 w + \left[U_1(x,s,p,q)d_2 + U_2(x,s,p,q)d_1^2\right]) w^2 + \cdots.\end{aligned}$$

A short calculation shows that

$$(1+\lambda+2\mu)\,a_2 = U_1(x,s,p,q)c_1, \tag{11}$$

$$(1+2\lambda+6\mu)\,a_3 = U_1(x,s,p,q)c_2 + U_2(x,s,p,q)c_1^2, \tag{12}$$

and

$$-\,(1+\lambda+2\mu)\,a_2 = U_1(x,s,p,q)d_1, \tag{13}$$

$$(1+2\lambda+6\mu)\,(2a_2^2 - a_3) = U_1(x,s,p,q)d_2 + U_2(x,s,p,q)d_1^2. \tag{14}$$

From (11) and (13), we have

$$c_1 = -d_1, \tag{15}$$

and

$$2\,(1+\lambda+2\mu)^2\,a_2^2 = U_1^2(x,s,p,q)\left(c_1^2+d_1^2\right). \tag{16}$$

By adding (12) to (14), we get

$$2\,(1+2\lambda+6\mu)\,a_2^2 = U_1(x,s,p,q)\,(c_2+d_2) + U_2(x,s,p,q)\left(c_1^2+d_1^2\right). \tag{17}$$

By using (16) in (17), we obtain

$$\left[2\left(1+2\lambda+6\mu\right)-\frac{2U_2(x,s,p,q)}{U_1^2(x,s,p,q)}(1+\lambda+2\mu)^2\right]a_2^2=U_1(x,s,p,q)\left(c_2+d_2\right). \tag{18}$$

It is fairly well known [5] that if $|w(z)|<1$ and $|v(w)|<1$, then

$$|c_j|\leq 1 \text{ and } |d_j|\leq 1 \text{ for all } j\in\mathbb{N}. \tag{19}$$

By considering (3) and (19), we get from (18) the desired inequality (6).

Next, by subtracting (14) from (12), we have

$$2\left(1+2\lambda+6\mu\right)a_3-2(1+2\lambda+6\mu)a_2^2=U_1(x,s,p,q)\left(c_2-d_2\right)+U_2(x,s,p,q)\left(c_1^2-d_1^2\right). \tag{20}$$

Further, in view of (15), it follows from (20) that

$$a_3=a_2^2+\frac{U_1(x,s,p,q)}{2(1+2\lambda+6\mu)}\left(c_2-d_2\right). \tag{21}$$

By considering (16) and (19), we get from (21) the desired inequality (7). This completes the proof of Theorem 2.1. □

Taking $\lambda=1$ and $\mu=0$ in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_\Sigma\left(\lambda,\mu,p,q\right)$. Then*

$$|a_2|\leq\frac{(p+q)\,x\sqrt{(p+q)\,x}}{\sqrt{\left|\left[6\left(p+q\right)^2x^2-8\left[\left(p^2+q^2\right)x^2\left(p+q\right)+pqs\right]\right]\right|}},$$

and

$$|a_3|\leq\frac{\left(p+q\right)^3x^3}{\left|\left[6\left(p+q\right)^2x^2-8\left[\left(p^2+q^2\right)x^2\left(p+q\right)+pqs\right]\right]\right|}+\frac{(p+q)\,x}{3}. \tag{22}$$

3. Fekete-Szegö inequality for the function class $\mathcal{B}_\Sigma\left(\lambda,\mu,p,q\right)$

Now, we are ready to find the sharp bounds of Fekete-Szegö functional $a_3-\eta a_2^2$ defined for $f\in\mathcal{B}_\Sigma\left(\lambda,\mu,p,q\right)$ given by (1).

Theorem 3.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_\Sigma\left(\lambda,\mu,p,q\right)$. Then for some $\eta\in\mathbb{R}$,*

$$|a_3-\eta a_2^2|\leq\begin{cases}\frac{(p+q)x}{1+2\lambda+6\mu}, & |\eta-1|\leq M\\ \frac{2(p+q)^3x^3|1-\eta|}{\left|2(1+2\lambda+6\mu)(p+q)^2x^2-2(1+\lambda+2\mu)^2[(p^2+q^2)x^2(p+q)+pqs]\right|}, & |\eta-1|\geq M\end{cases} \tag{23}$$

where

$$M=\frac{\left|2\left[\left(1+2\lambda+6\mu\right)\left(p+q\right)^2x^2-\left(1+\lambda+2\mu\right)^2\left[\left(p^2+q^2\right)x^2\left(p+q\right)\right]\right]\right|}{2\left(1+2\lambda+6\mu\right)\left(p+q\right)^2x^2}.$$

Proof. Let $f \in \mathcal{B}_{\Sigma}(\lambda,\mu,p,q)$. By using (18) and (21) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2^2 = (1-\eta)\left[\frac{U_1^3(\lambda,\mu,p,q)(c_2+d_2)}{2(1+2\lambda+6\mu)U_1^2(\lambda,\mu,p,q) - 2(1+\lambda+2\mu)^2 U_2(\lambda,\mu,p,q)}\right] + \frac{U_1(\lambda,\mu,p,q)(c_2-d_2)}{2(1+2\lambda+6\mu)}$$
$$= U_1(\lambda,\mu,p,q)\left[\left(h(\eta)+\frac{1}{2(1+2\lambda+6\mu)}\right)c_2 + \left(h(\eta)-\frac{1}{2(1+2\lambda+6\mu)}\right)d_2\right],$$

where

$$h(\eta) = \frac{U_1^2(\lambda,\mu,p,q)(1-\eta)}{2\left[(1+2\lambda+6\mu)U_1^2(\lambda,\mu,p,q) - (1+\lambda+2\mu)^2 U_2(\lambda,\mu,p,q)\right]}.$$

Then, we easily conclude that

$$|a_3-\eta a_2^2| \le \begin{cases} \frac{(p+q)x}{1+2\lambda+6\mu}, & |h(\eta)| \le \frac{1}{2(1+2\lambda+6\mu)} \\ 2(p+q)|h(\eta)|x, & |h(\eta)| \ge \frac{1}{2(1+2\lambda+6\mu)} \end{cases}$$

This proves Theorem 3.1. □

We end this section with some corollaries concerning the sharp bounds of Fekete-Szegö functional $a_3-\eta a_2^2$ defined for $f \in \mathcal{B}_{\Sigma}(\lambda,\mu,p,q)$ given by (1).

Taking $\eta = 1$ in Theorem 3.1, we get the following corollary.

Corollary 3.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma}(\lambda,\mu,p,q)$. Then*

$$|a_3 - a_2^2| \le \frac{(p+q)x}{1+2\lambda+6\mu}.$$

Taking $\lambda = 1$ and $\mu = 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.2. *Let the function $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma}(t)$. Then for some $\eta \in \mathbb{R}$,*

$$|a_3-\eta a_2^2| \le \begin{cases} \frac{(p+q)x}{3}, & |\eta-1| \le M \\ \frac{2(p+q)^3x^3|1-\eta|}{\left|6(p+q)^2x^2 - 8[(p^2+q^2)x^2(p+q)+pqs]\right|}, & |\eta-1| \ge M \end{cases}$$

where

$$M = \frac{\left|2\left[3(p+q)^2x^2 - 4\left[\left(p^2+q^2\right)x^2(p+q)\right]\right]\right|}{6(p+q)^2x^2}.$$

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