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OUTER-CONVEX DOMINATION IN THE CORONA OF GRAPHS

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Abstract. Let G be a connected simple graph. A subset S of a vertex set $V(G)$ is called an outer-convex dominating set of G if for every vertex $v \in V(G)\backslash S$, there exists a vertex $x \in S$ such that xv is an edge of G and $V(G)\backslash S$ is a convex set. The outer-convex domination number of G, denoted by $\widetilde{\gamma}_{con}(G)$, is the minimum cardinality of an outer-convex dominating set of G. In this paper, we show that every integers a, b, c, and n with $a \leq b \leq c \leq n-1$ is realizable as domination number, outer-connected domination number, outer-convex domination number, and order of G respectively. Further, we give the characterization of the outer-convex dominating set in the corona of two graphs and give its corresponding outer-convex domination number.

Keywords: Domination, outer-connected domination, outer-convex domination.

AMS Subject Classification: 05C69.

1. Introduction

The theory of domination is an area in graph theory with numerous research activities. One of the domination parameters of interest is outer-convex domination which was introduced by Dayap and Enriquez in 2019 [1] and further investigated in [2]. In [1], the authors characterized the outer-convex domination in the join of two graphs and give some of its bounds. In [2], the authors characterized the parameter in the composition and Cartesian product of graphs. In this paper, we give the characterization of the outer-convex dominating set in the corona of two graphs and outer-convex domination number on the resulting graph. Further, we give some realization problems of the said domination parameter.

Let G be a simple graph. A subset S of a vertex set $V(G)$ is a *dominating set* of G if for every vertex $v \in V(G)\backslash S$, there exists a vertex $x \in S$ such that xv is an edge of G. The *domination number* $\gamma(G)$ of G is the smallest cardinality of a dominating set S of G. Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures and others. For further background on dominating sets, the reader may refer to [3]. Domination in graph was introduced by Claude Berge in 1958 [4] and Oystein Ore in 1962 [5].

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A graph G is *connected* if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is *disconnected*. A nonempty subset S of $V(G)$ is a *clique* in G if every two vertices in S are adjacent. For any two vertices u and v in a connected graph, the distance $d_G(u, v)$ between u and v is the length of a shortest path in G. A u-v path of length $d_G(u, v)$ is also referred to as u-v *geodesic*. The *closed interval* $I_G[u, v]$ consists of all those vertices lying on a u-v geodesic in G. For a subset S of vertices of G, the union of all sets $I_G[u, v]$ for $u, v \in S$ is denoted by $I_G[S]$. Hence $x \in I_G[S]$ if and only if x lies on some u-v geodesic, where $u, v \in S$. A set S is *convex* if $I_G[S] = S$. More specifically, if G is connected graph, then $V(G)$ is convex. If $V(G)\backslash S$ is convex, then S is an outer-convex set of G. Convexity in graphs was studied in [6, 7].

A dominating set S, which is also convex, is called a *convex dominating set* of G. The *convex domination number* $\gamma_{con}(G)$ of G is the smallest cardinality of a convex dominating set of G. A convex dominating set of cardinality $\gamma_{con}(G)$ is called a γ_{con}-*set* of G. Convex domination in graphs was studied in [8, 9, 10]. A set S of vertices of a graph G is an outer-connected dominating set if every vertex not in S is adjacent to some vertex in S and the sub-graph induced by $V(G) \setminus S$ is connected. The outer-connected domination number $\widetilde{\gamma}_c(G)$ is the minimum cardinality of the outer-connected dominating set S of a graph G. The concept of outer-connected domination in graphs was introduced by Cyman [12] and further investigated in [11].

A set S of vertices of a graph G is an outer-convex dominating set if every vertex not in S is adjacent to some vertex in S and $V(G) \setminus S$ is convex. The *outer-convex domination number* of G, denoted by $\widetilde{\gamma}_{con}(G)$, is the minimum cardinality of an outer-convex dominating set of G. An outer-convex dominating set of cardinality $\widetilde{\gamma}_{con}(G)$ will be called an $\widetilde{\gamma}_{con}$-*set*.

Let G and H be graphs of order m and n, respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H. The join of vertex v of G and a copy H^v of H in the corona of G and H is denoted by $v + H^v$.

2. Results

Theorem 2.1. *Given positive integers a, b, c, and n such that $n \geq 2$ and $a \leq b \leq c \leq n-1$, there exists a connected graph G with $\gamma(G) = a$, $\gamma_{con}(G) = b$, $\widetilde{\gamma}_{con}(G) = c$, and $|V(G)| = n$.*

Proof. Consider the following cases:
Case 1: Suppose $a = b = c = n - 1$.
Let $G = K_2$. Clearly, $\gamma(G) = 1, \gamma_{con}(G) = 1, \widetilde{\gamma}_{con} = 1$, and $|V(G)| = 2$.
Case 2: Suppose $a = b = c < n - 1$.
Let $G = P_a \circ K_1$ (see Figure 1) and let $n = 2a$.

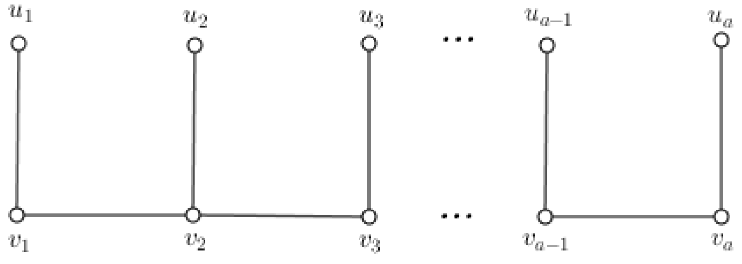

Figure 1: A graph G with $a = b = c < n - 1$

Clearly, the set $A = \{v_i : i = 1, 2, ..., a\}$ is a $\gamma - set$, and $\gamma_{con} - set$ of G. The set $B = \{u_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 2|A| = 2a = n$, $\gamma(G) = |A| = a$,

$\gamma_{con} = |A| = a = b$, and $\widetilde{\gamma}_{con} = |B| = a = b = c$.
Case 3: Suppose $a = b < c < n-1$.
Consider the graph G obtained from the graph in *Figure 1* by adding the vertex x_i and the edges $v_i x_i$ for $i = 1, 2, ..., a$ (see Figure 2) and let $2a = c$, and $3a = n$.

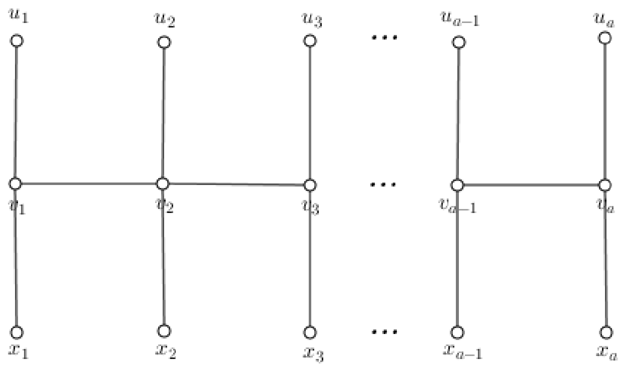

Figure 2: A graph G with $a = b < c < n-1$

The set $A = \{v_i : i = 1, 2, ..., a\}$ is a $\gamma - set$, and $\gamma_{con} - set$ of G. The set $B = \{u_i : i = 1, 2, ..., a\} \cup \{x_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 3|A| = 3a = n$, $\gamma(G) = |A| = a$, $\gamma_{con}(G) = |A| = a = b$, $\widetilde{\gamma}_{con}(G) = |B| = a + a = 2a = c$.
Case 4: Suppose $a < b = c < n-1$.
Consider the graph G obtained from the graph in *Figure 1* by adding the vertex y_i and the edges $y_i u_i$ for $i = 1, 2, ..., a$ (see Figure 3) and let $2a = b$, and $3a = n$.

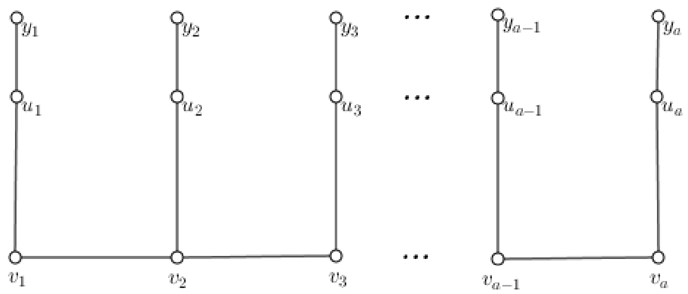

Figure 3: A graph G with $a < b = c < n-1$

Clearly, the set $A = \{u_i : i = 1, 2, ..., a\}$ is a $\gamma - set$, $B = A \cup \{v_i : i = 1, 2, ..., a\}$ is a $\gamma_{con} - set$, and $C = A \cup \{y_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 3|A| = 3a = n$, $\gamma(G) = |A| = a$, $\gamma_{con}(G) = |B| = |A| + a = a + a = 2a = b$, and $\widetilde{\gamma}_{con}(G) = |C| = |A| + a = a + a = 2a = b = c$.
Case 5: Suppose $a < b < c < n-1$.
Consider the graph G obtained from the graph in *Figure 3* by adding the vertices z, and x_i and edges $v_i z$, and $z x_i$ for $i = 1, 2, ..., a$ (see Figure 4) and let $b = 2a+1$, $c = 3a$, and $n = 4a + 1$.

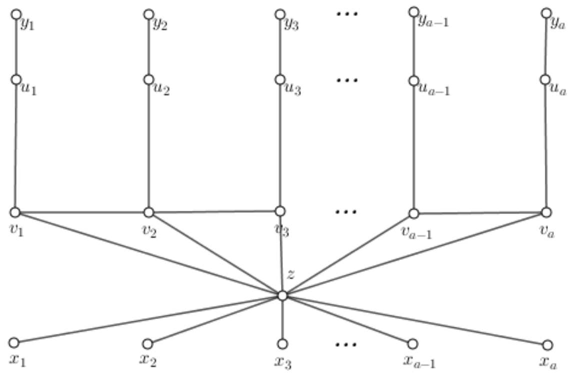

Figure 4: A graph G with $a < b < c < n - 1$

Clearly, the set $A = \{u_i : i = 1, 2, ..., a\} \cup z$ is a $\gamma - set$, $B = A \cup \{v_i : i = 1, 2, ..., a\}$ is $\gamma_{con} - set$, and $C = \{u_i : i = 1, 2, ..., a\} \cup \{x_i : i = 1, 2, ..., a\} \cup \{y_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 4|A| - 3 = 4(a + 1) - 3 = 4a + 4 - 3 = 4a + 1 = n$, $\gamma(G) = |A| = a + 1$, $\gamma_{con}(G) = |B| = |A| + a = (a + 1) + a = 2a + 1 = b$, $\widetilde{\gamma}_{con}(G) = a + a + a = 3a = c$.

Theorem 2.2. *Given positive integers a, b, c, and n such that $n \geq 2$ and $a \leq b \leq c \leq n-1$, there exists a connected graph G with $\gamma(G) = a$, $\widetilde{\gamma}_c(G) = b$, $\widetilde{\gamma}_{con}(G) = c$, and $|V(G)| = n$.*

Proof. Consider the following cases:
Case 1. Suppose $a = b = c = n - 1$.
Let $G = K_2$. Clearly, $\gamma(G) = 1, \widetilde{\gamma}_c(G) = 1, \widetilde{\gamma}_{con} = 1$, and $|V(G)| = 2$.
Case 2. Suppose $a = b = c < n - 1$.
Let $G = P_a \circ K_1$ (see Figure 1) and let $n = 2a$.
Clearly, the set $A = \{u_i : i = 1, 2, ..., a\}$ is a $\gamma - set$, $\widetilde{\gamma}_c - set$, and $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 2|A| = 2a = n$, $\gamma(G) = |A| = a$, $\widetilde{\gamma}_c = |A| = b$, and $\widetilde{\gamma}_{con} = |A| = c$.
Case 3. Suppose $a = b < c < n - 1$.
Let $G = P_a \circ P_3$ (See Figure 5) and let $c = 2a$ and $n = 4a$.

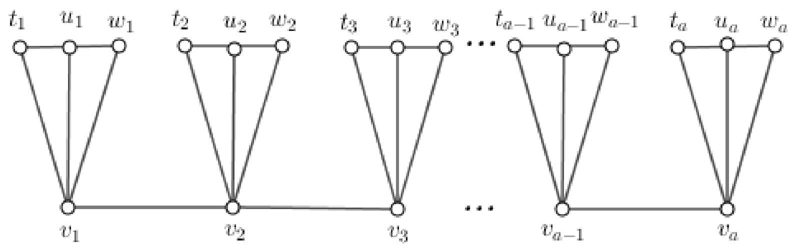

Figure 5: A graph G with $a = b < c < n - 1$

Clearly, the set $A = \{u_i : i = 1, 2, ..., a\}$ is a $\gamma - set$, and a $\widetilde{\gamma}_c - set$ and the set $B = A \cup \{t_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 4|A| = 4a = n$, $\gamma(G) = |A| = a$, $\widetilde{\gamma}_c(G) = |A| = a = b$, and $\widetilde{\gamma}_{con}(G) = |B| = |A| + a = a + a = 2a = c$.
Case 4. Suppose $a < b = c < n - 1$.
Consider the graph G obtained from the graph in *Figure 3* and let $b = 2a$, and $n = 3a$. The set $A = \{v_i : i = 1, 2, ..., a\}$ is a $\gamma - set$, and $B = \{u_i : i = 1, 2, ..., a\} \cup \{x_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_c - set$, and $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 3|A| = 3a = n$, $\gamma(G) = |A| = a$, $\widetilde{\gamma}_c(G) = |B| = a + a = 2a = b$, $\widetilde{\gamma}_{con}(G) = |B| = 2a = b = c$.
Case 5. Suppose $a < b < c < n - 1$.
Let $G = P_a \circ P_5$ (See Figure 6) and let $b = 2a$, $c = 3a$, and $n = 6a$.

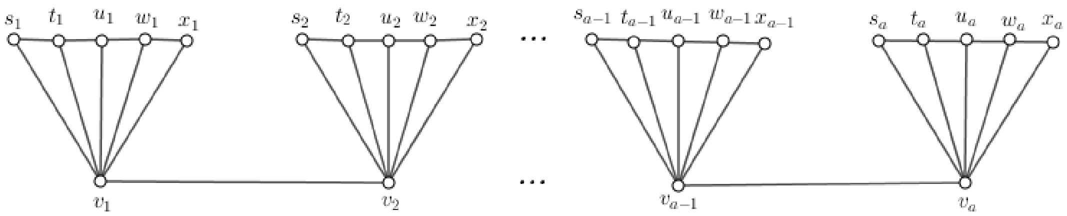

Figure 6: A graph G with $a < b < c < n-1$

Clearly, the set $A = \{v_i : i = 1, 2, ..., a\}$ is a $\gamma - set$, $B = \{s_i : i = 1, 2, ..., a\} \cup \{w_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_c - set$ and the set $C = B \cup \{x_i : i = 1, 2, ..., a\}$ is a $\widetilde{\gamma}_{con} - set$ of G. Thus, $|V(G)| = 6|A| = 6a = n$, $\gamma(G) = |A| = a$, $\widetilde{\gamma}_c(G) = |B| = a + a = 2a = b$, and $\widetilde{\gamma}_{con}(G) = |C| = |B| + a = 2a + a = 3a = c$.

This proves the assertion. □

Theorem 2.3. *Let G be a connected graph and H be a connected non-complete graph. Then a subset S of $V(G \circ H)$ is an outer-convex dominating set in $G \circ H$ if and only if one of the following statements is satisfied:*

(i) $S = \bigcup_{x \in V(G)} S_x$, where S_x is a dominating set in H^x and $V(H^x) \setminus S_x$ is convex in $x + H^x$ for all $x \in V(G)$.

(ii) $S = V(G) \bigcup S_x \bigcup \left(\bigcup_{z \in V(G) \setminus \{x\}} V(H^z) \right)$ for some $x \in V(G)$, where $S_x = V(H^x)$ or $V(H^x) \setminus S_x$ is a clique set in H^x.

(iii) $S = S_G \bigcup \left(\bigcup_{x \in V(G) \setminus S_G} S_x \right) \bigcup \left(\bigcup_{z \in S_G} V(H^z) \right)$, where S_G is an outer-convex set in G, S_x is a dominating set in H^x and $V(H^x) \setminus S_x$ is convex in $x + H^x$.

Proof. Suppose that a subset S of $V(G \circ H)$ is an outer-convex dominating set in $G \circ H$. Then S is a dominating set and $V(G \circ H) \setminus S$ is a convex set in $G \circ H$. Set $S_G = S \bigcap V(G)$ and $S_x = S \bigcap V(H^x)$. Consider the following cases:

Case 1: $S_G = \emptyset$

Then, obviously, S will be the one given in (i). Next, one needs to show that, for an arbitrary $x \in V(G)$, S_x is an outer-convex dominating set of $x + H^x$, that is, S_x is a dominating set in H^x and $V(H^x) \setminus S_x$ is convex in $x + H^x$. Suppose that S_x is not a dominating set in H^x for all $x \in V(G)$. Then $S = \bigcup_{x \in V(G)} S_x$ is clearly not a dominating set, contrary to our assumption. Thus, S_x must be a dominating set in H^x for all $x \in V(G)$. Now, suppose that $V(H^x) \setminus S_x$ is not convex in $x + H^x$ for all $x \in V(G)$. Then, $\bigcup_{x \in V(G)} (V(H^x) \setminus S_x)$, is not convex. Thus, $V(G) \cup \left(\bigcup_{x \in V(G)} (V(H^x) \setminus S_x) \right) = V(G \circ H) \setminus S$ is not convex, contrary to our assumption. Thus, $V(H^x) \setminus S_x$ must be convex in $G \circ H$ for all $x \in V(G)$. This proves statement (i).

Case 2: $S_G = V(G)$

Set $S = V(G)\bigcup S_x \bigcup \left(\bigcup_{z\in V(G)\setminus\{x\}} V(H^z)\right)$. If $S = V(G\circ H)$, then $S_x = V(H^x)$ for all $x \in V(G)$. Suppose, $S \neq V(G\circ H)$. Then, there exists $x \in V(G)$ such that $S_x \neq V(H^x)$. Suppose $V(H^x)\setminus S_x$ is not a clique set in H^x. Then, $V(H^x)\setminus S_x = V(G\circ H)\setminus \left(V(G)\bigcup S_x \bigcup \left(\bigcup_{z\in V(G)\setminus\{x\}} V(H^z)\right)\right)$. It follows that, $V(H^x)\setminus S_x = V(G\circ H)\setminus S$ is not a clique set. This implies that there exist $u, v \in V(G\circ H)\setminus S$ such that $uv \notin E(G\circ H)$. Since $ux, xv \in E(G\circ H)$ for some $x \in V(G)$, $x \in I_{G\circ H}[V(G\circ H)\setminus S]$. Now, $x \in V(G)$ implies that $x \in S$ and so $x \notin V(G\circ H)\setminus S$. Thus, $I_{G\circ H}[V(G\circ H)\setminus S] \neq V(G\circ H)\setminus S$, that is, $V(G\circ H)\setminus S$ is not convex contrary to our assumption. Hence, $V(H^x\setminus S_x)$ must be a clique set. Suppose, there exists $y \neq x \in V(G)$ such that $V(H^y)\setminus S_y$ is clique set in H^y. Since $x, y \in V(G)$ implies that $x, y \in S$, it follows that $x, y \neq V(G\circ H)\setminus S$. Clearly,$I_{G\circ H}[V(G\circ H)\setminus S] \neq V(G\circ H)\setminus S$ that is, $V(G\circ H)\setminus S$ is not convex, contrary to our assumption. Hence, $V(H^x)\setminus S_x$ is clique set in H^x for some $x \in V(G)$, showing statement (ii).

Case 3: $S_G \neq \emptyset$ and $S_G \neq V(G)$

Set $S = S_G \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} S_x\right) \bigcup \left(\bigcup_{z\in S_G} V(H^z)\right)$. Suppose S_G is not an outer-convex set in G. Then, $V(G)\setminus S_G$ is not convex in G. Consequently, $I_G[V(G)\setminus S_G] \neq V(G)\setminus S_G$. Now, pick distinct elements x and y in $V(G)\setminus S_G$. Then, $x, y \notin S_G$, implies $x, y \notin S$. This means, $x, y \in V(G\circ H)\setminus S$. Obviously, $I_{G\circ H}[V(G\circ H)\setminus S] \neq V(G\circ H)\setminus S$.This implies that $V(G\circ H)\setminus S$ is not convex, contrary to our assumption. Hence, S_G must be an outer-convex set in G. Suppose S_x is not a dominating set in H^x for all $x \in V(G)\setminus S_G$, then there exists $w \in V(H^x)\setminus S_x$ such that $wy \notin E(H^x)$ for all $y \in S_x$. Since for all $z \in S_G$, $wz \notin E(G\circ H)$ and for all $u \in V(H^z)$, $wu \notin E(G\circ H)$, it follows that $wv \notin E(G\circ H)$ for all $v \in \left(S_G \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} S_x\right) \bigcup \left(\bigcup_{z\in S_G} V(H^z)\right)\right)$. Thus, there exists $w \in V(G\circ H)\setminus S$ such that $wv \notin E(G\circ H)$ for all $v \in S$, contrary to our assumption that S is a dominating set in $G\circ H$. Thus, S_x must be a dominating set in H^x for all $x \in V(G)\setminus S_G$.Now, let distinct elements $x, y \in V(G\circ H)\setminus S$ with $S_x \subseteq V(H^x)$. Then

$$\begin{aligned}
y &\in V(G\circ H)\setminus \left(S_G \bigcup S_x \bigcup \left(\bigcup_{z\in S_G} V(H^z)\right)\right)\\
y &\notin \left(S_G \bigcup S_x \bigcup \left(\bigcup_{z\in S_G} V(H^z)\right)\right)\\
y &\notin S_x\\
y &\in V(H^x)\setminus S_x
\end{aligned}$$

Thus, $V(G\circ H)\setminus S \subseteq V(H^x)\setminus S_x$. Since $V(H^x)\setminus S_x \subseteq V(G\circ H)\setminus S$ for all $x \in V(G)\setminus S_G$ is clear, it follows that $V(G\circ H)\setminus S = V(H^x)\setminus S_x$. Hence, $V(x+H^x)\setminus S_x$ is convex set in

$G \circ H$. Therefore, $S = S_G \bigcup \left(\bigcup_{x \in V(G) \setminus S_G} S_x \right) \bigcup \left(\bigcup_{z \in S_G} V(H^z) \right)$, where S_G is an outer-convex set in G, S_x is a dominating set in H^x and $(V(H^x) \setminus S_x)$ is convex in $x + V(H^x)$ proving (iii).

For the converse, suppose that statement (i), (ii), or (iii) is satisfied. Consider first that statement (i) holds. Since for each $x \in V(G)$, S_x is a dominating set in H^x, it follows that $S = \bigcup_{x \in V(G)} S_x$ is a dominating set in $G \circ H$. Let $r, s \in V(G \circ H) \setminus S$ such that $r \neq s$.Then,

$$\begin{aligned} r, s &\in V(G \circ H) \setminus \left(\bigcup_{x \in V(G)} S_x \right) \\ &\in V(G) \bigcup \left(\bigcup_{x \in V(G)} (V(H^x) \setminus S_x) \right) \end{aligned}$$

To show that $V(G \circ H) \setminus S$ is convex, it is enough to show that $I_G[r, s] \subseteq V(G \circ H) \setminus S$. Now, suppose $r, s \in V(G)$. Clearly, $I_G[r, s] \subseteq V(G) \subseteq V(G \circ H) \setminus S$. Thus, $V(G \circ H) \setminus S$ is convex in $G \circ H$. Accordingly, S is an outer-convex dominating set in $G \circ H$.

Next, suppose that statement (ii) holds. Since $V(G)$ is a dominating set in $G \circ H$, $S = V(G) \bigcup S_x \bigcup \left(\bigcup_{z \in V(G) \setminus \{x\}} V(H^z) \right)$ is a dominating set in $G \circ H$. Now, $V(G \circ H) \setminus S$

$$\begin{aligned} &= V(G \circ H) \setminus \left(V(G) \bigcup S_x \bigcup \left(\bigcup_{z \in V(G) \setminus \{x\}} V(H^z) \right) \right) \\ &= \left(V(G) \bigcup \left(\bigcup_{z \in V(G)} V(H^z) \right) \right) \setminus \left(V(G) \bigcup S_x \bigcup \left(\bigcup_{z \in V(G) \setminus \{x\}} V(H^z) \right) \right) \\ &= \left(V(G) \bigcup \left(\bigcup_{z \in V(G) \setminus \{x\}} V(H^z) \right) \bigcup V(H^x) \right) \setminus \left(V(G) \bigcup S_x \bigcup \left(\bigcup_{z \in V(G) \setminus \{x\}} V(H^z) \right) \right) \\ &= V(H^x) \setminus S_x. \end{aligned}$$

Since $V(H^x) \setminus S_x$ is a clique set, it follows that $V(G \circ G) \setminus S$ is a clique set. Thus, $V(G \circ H) \setminus S$ is convex in $G \circ H$. Accordingly, S is an outer-convex dominating set in $G \circ H$.

Finally, suppose that statement (iii) holds. Since S_x is a dominating set in H^x, $S = S_G \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} S_x \right) \bigcup \left(\bigcup_{z\in S_G} V(H^z) \right)$ is a dominating set in $G\circ H$. Now, $V(G\circ H)\setminus S$

$$
\begin{aligned}
&= \left(V(G) \bigcup \left(\bigcup_{z\in V(G)} V(H^z) \right) \right) \setminus S \\
&= \left(V(G) \bigcup \left(\bigcup_{z\in V(G)} V(H^z) \right) \right) \setminus \left(S_G \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} S_x \right) \bigcup \left(\bigcup_{z\in S_G} V(H^z) \right) \right) \\
&= \left(V(G) \bigcup \left(\bigcup_{z\in S_G} V(H^z) \right) \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} V(H^x) \right) \right) \setminus \left(S_G \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} S_x \right) \bigcup \left(\bigcup_{z\in S_G} V(H^z) \right) \right) \\
&= \left(V(G) \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} V(H^x) \right) \right) \setminus \left(S_G \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} S_x \right) \right) \\
&= (V(G)\setminus S_G) \bigcup \left(\bigcup_{x\in V(G)\setminus S_G} V(H^x)\setminus S_x \right).
\end{aligned}
$$

To show that $V(G\circ H)\setminus S$ is convex, it is enough to show that for all $r, s \in V(G)\setminus S_G$, $I_G[r,s] \subseteq V(G\circ H)\setminus S$. Suppose, $r, s \in V(G)\setminus S_G$. Since $V(G)\setminus S_G$ is convex in G, then $I_G[r,s] \subseteq V(G)\setminus S_G \subseteq V(G\circ H)\setminus S$. It follows that $V(G\circ H)\setminus S$ is convex in $G\circ H$. Accordingly, S is an outer-convex dominating set in $G\circ H$. $\square$

Corollary 2.1. *Let G be a connected graph of order $m \geq 2$ and H be any graph order n. If $S_x \subseteq V(H^x)$ is a minimum outer-convex dominating set of $x + H^x$ for each $x \in V(G)$, then $\widetilde{\gamma}_{con}(G\circ H) = m\widetilde{\gamma}_{con}(x + H^x)$.*

Proof. Suppose that $S_x \subseteq V(H^x)$ is a minimum outer-convex dominating set of $x + H^x$ for each $x \in V(G)$. Then S_x is a dominating set of H^x and $V(H^x)\setminus S_x$ is convex in $x + H^x$. Then by Theorem 2.3, $S = \bigcup_{x\in V(G)} S_x$ is an outer-convex dominating set in $G\circ H$. Thus, $\widetilde{\gamma}_{con}(G\circ H) \leq |S| = |\bigcup_{x\in V(G)} S_x| = |V(G)||S_x| = m\widetilde{\gamma}_{con}(x + H^x)$. Now, let S^* be a minimum outer-convex dominating set of $G\circ H$. Since $m \geq 2$, by Theorem 2.3, it is clear that $S^* \subseteq \bigcup_{x\in V(G)} S_x$. Since S_x is the minimum outer-convex dominating set of $x + H^x$, it follows that,

$$
\begin{aligned}
\widetilde{\gamma}_{con}(G\circ H) = |S^*| = |\bigcup_{x\in V(G)} S^*_x| &\geq |\bigcup_{x\in V(G)} S_x| \\
&= |V(G)||S_x| \\
&= m\widetilde{\gamma}_{con}(x + H^x).
\end{aligned}
$$

Therefore, $\widetilde{\gamma}_{con}(G\circ H) = m\widetilde{\gamma}_{con}(H)$.

In view of *Theorem 2.3* and *Corollary 2.1*, the following corollary is immediate.

Corollary 2.2. *Let G be a connected graph of order $m \geq 2$ and H be any graph of order n. The set $S \subset V(G \circ H)$ is a minimum outer-convex dominating set in $G \circ H$ if $S = \bigcup_{x \in V(G)} S_x$, where $S_x \subseteq V(H^x)$ is a minimum outer-convex dominating set in $x + H^x$.*

The following result is due to Dayap and Enriquez [2].

Remark 2.1. *Let G be a nontrivial connected graph of order n. Then $1 \leq \gamma(G) \leq \widetilde{\gamma}_{con}(G) \leq n - 1$*

The following result is due to Canoy and Go [13].

Corollary 2.3. *Let G be a connected graph of order m and let H be any graph of order n. Then $\gamma(G \circ H) = m$.*

Corollary 2.4. *Let G be a connected graph and H be a complete graph. Then $\widetilde{\gamma}_{con}(G \circ H) = |V(G)|$*

Proof. Suppose that S_x is a minimum dominating set of H^x. Since H is complete, it follows that $V(H^x) \backslash S_x$ is convex in $x + H^x$ and $|S_x| = 1$. Then, $S = \bigcup_{x \in V(G)} S_x$ is an outer-convex dominating set in $G \circ H$. Thus, $\widetilde{\gamma}_{con}(G \circ H) \leq |S| = |\bigcup_{x \in V(G)} S_x| = |V(G)||S_x| = m$. By Corollary 2.3, $\gamma(G \circ H) = m$. Also, by Remark 2.1, $\gamma(G \circ H) \leq \widetilde{\gamma}_{con}(G \circ H)$. This implies that $m = \gamma(G \circ H) \leq \widetilde{\gamma}_{con}(G \circ H) \leq m$. Therefore, $\widetilde{\gamma}_{con}(G \circ H) = m$.

3. Conclusion

An outer-convex domination is a new variant of domination in graphs and the corona of two graphs is one of the graph operations that plays a very important role in mathematical chemistry. Hence, this paper is a contribution to the development of the application of domination theory in the field of mathematical chemistry. Since this is new, further investigations must be promoted to come up with coherent and substantial results of the parameter, an outer-convex domination number. Thus, the characterization of the outer-convex dominating set on some special graphs and some binary operations such as the sequential join and Square of Normal Product of two graphs are recommended for further study. Moreover, the applications of the characterization of the said parameter in the corona of two graphs are further to be looked into. The aforementioned characterization might be used as a tool in finding simpler graphs on some chemically interesting graphs and a tool in developing a symmetric encryption algorithm. Finally, domination in graphs is rich with immediate applications in the real world such as routing problems in the Internet, problems in electrical networks, data structures, neural and communication networks, protection and location strategies and many others. The outer-convex domination in graphs is not far from these applications.

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