TWMS J. App. and Eng. Math. V.12, N.2, 2022, pp. 650-661

TOTAL GLOBAL DOMINATOR CHROMATIC NUMBER OF GRAPHS

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Abstract. Let $G = (V, E)$ be k-colorable (k-vertex colorable) graph and $V_i \subseteq V$ be the class of vertices with color i. Then we assume that $f = (V_1, V_2, \cdots, V_k)$ is a coloring of G. A vertex $v \in V(G)$ is a dominator of f if v dominates all the vertices of at least one color class such as V_i (V_i is called a dom-color class respected to v) and v is said to be an anti dominator of f if v dominates none of the vertices of at least one color class such as V_i (V_i is called a anti dom-color class respected to v). A vertex $v \in V(G)$ is a total dominator of f, if v dominates all the vertices of at least one color class such as V_i not including v (V_i is called a total dom-color class respected to v). A total global dominator coloring of a graph G is a proper coloring f of G in which each vertex of the graph has a total dom-color class and an anti dom-color class in f. The minimum number of colors required for a total global dominator coloring of G is called the total global dominator chromatic number and is denoted by $\chi^t_{gd}(G)$. In this paper we initiates a study on this notion of total global dominator coloring. The complexity of total global dominator coloring is studied. Some basic results and some bounds in terms of order, chromatic number, domination parameters are investigated. Finally we classify the total global dominator coloring of trees.

Keywords: Total global domination, coloring, dominator coloring, total global dominator coloring.

AMS Subject Classification: 05C69, 05C15.

1. Introduction

Let $G = (V, E)$ be a simple, connected, finite, and undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For graph theoretic terminology, we refer to the references [3, 17]. Graph coloring and domination are two major fields in graph theory that have been well studied by many authors. An excellent treatment of domination is given in the scientific book by Haynes et al. [11], although after that and in the present, several papers on the domination parameters have been published. Several variations of graph coloring have been introduced and studied by many researchers. One of them is

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§ Manuscript received: March 08, 2020; accepted: May 30, 2020.
TWMS Journal of Applied and Engineering Mathematics, Vol.12, No.2

the book authored by Chartrand and Zhang [4] that gives an extensive survey of various graph coloring.

For each vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V | uv \in E\}$, while its cardinality is the *degree* of v, and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *minimum degree* of G is denoted by $\delta = \delta(G)$ and the *maximum degree* is denoted by $\Delta = \Delta(G)$. A vertex of degree zero in G is an *isolated vertex* and a vertex of degree one is a *pendant vertex* or a *leaf.* Any vertex which is adjacent to a pendant vertex is called a *support vertex.* The *complement of a graph* G is denoted by $\overline{G}$ and is a graph with the vertex set $V(G)$ and for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. The clique number $\omega(G)$ of a graph G is the maximum order among the complete subgraphs of G. A graph G is called a split graph if its vertex set can be partitioned into a clique and an independent set. The join of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union of two graphs G and H, by adding the edges $\{xy : x \in V(G), y \in V(H)\}$. For two graphs G and H, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V(G)|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H.

A set $S \subseteq V$ is a *dominating set* of G if $N_G[S] = V$. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G. In a graph G with free isolated vertices, a set $S \subseteq V(G)$ is a *total dominating set* of G if every vertex of $V(G)$ is adjacent to at least one vertex of S, in the other words $N_G(S) = V$. The cardinality of a total dominating set of G with minimum size, denoted by $\gamma_t(G)$, is called the *total domination number* of G. Total domination is now well studied in graph theory [16]. The literature on the subject on total domination in graphs has been surveyed and recently detailed in the book [12]. A set $S \subseteq V$ is a *global dominating set* of G if S is a dominating set of G and $\overline{G}$. The minimum cardinality of a global dominating set of G, denoted by $\gamma_g(G)$, is called the *global domination number* of G. Global domination is also studied [2, 7, 14].

A total dominating set T of G is a *total global dominating set* if T is also a total dominating set of $\overline{G}$. The *total global domination number* $\gamma_{tg}(G)$ is the minimum cardinality of a total global dominating set. We note that $\gamma_{tg}(G)$ is only defined for G with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$ [1].

A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way, no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors required for a proper coloring of G. A color class is the set of all vertices, having the same color. The color class corresponding to the color i is denoted by V_i. If f is a proper coloring of G with the color classes $V_1, V_2, ..., V_k$, we write simply $f = (V_1, V_2, ..., V_k)$.

A *dominator coloring* of a graph G is a proper coloring of G such that every vertex in $V(G)$ dominates all vertices of at least one color class. The *dominator chromatic number* $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. The concept of dominator coloring was introduced and studied by Gera et al. [9] and studied further, for example, by Gera [8] and Chellali et al. [5] and Mojdeh et al. [15].

Recently, some authors studied the notion of dominated coloring which is defined as follows: A *dominated coloring* of a graph G is a proper coloring of the graph in a way that, each color class of it is dominated by at least one vertex. The *dominated chromatic* *number* $\chi_{dom}(G)$ of G is the minimum number of colors needed for a dominated coloring of G [6].

In this paper, this invariant (dominated coloring) is not studied and we only study the dominator coloring and verify some properties of total global dominator coloring.

Recently, the concept of total dominator coloring in a graph were studied by Kazemi, in [13] as follows.
A *total dominator coloring* of a graph G with no isolated vertex is a proper coloring of G in which every vertex v of the graph G is adjacent to every vertex of some color class not including v. The *total dominator chromatic number* χ_d^t of G is the minimum number of color classes in a total dominator coloring of G. A χ_d^t-coloring of G is a total dominator coloring with $\chi_d^t(G)$ colors.

Let G be a k-colorable graph with color classes $f = (V_1, V_2, \cdots, V_k)$ of G. A vertex $v \in V(G)$ is a *dominator* of f, if v dominates all the vertices of at least one color class like V_i and the color class V_i is called a *dom-color class* respected to v. A vertex v is said to be an *anti dominator* of f, if v dominates none of the vertices of at least one color class like V_j and the color class V_j is called an *anti dom-color class* respected to v. A vertex $v \in V(G)$ is a *total dominator* of f, if v dominates all the vertices of at least one color class like V_k not including v and the color class V_k is called a *total dom-color class* respected to v.
Recently Sahul Hamid et al. [10] have introduced global dominator coloring of G as follows. The coloring f is called a global dominator coloring of G if every vertex of G has a *dom-color class* and an *anti dom-color class* in f. The minimum number of colors required for a global dominator coloring of G is called the global dominator chromatic number and is denoted by $\chi_{gd}(G)$.

Now we define a variant of total and global dominator coloring namely, total global dominator coloring, that is a generalization of total dominator coloring and global dominator coloring that have already been mentioned. The concept total global dominator coloring will be given in the next section.

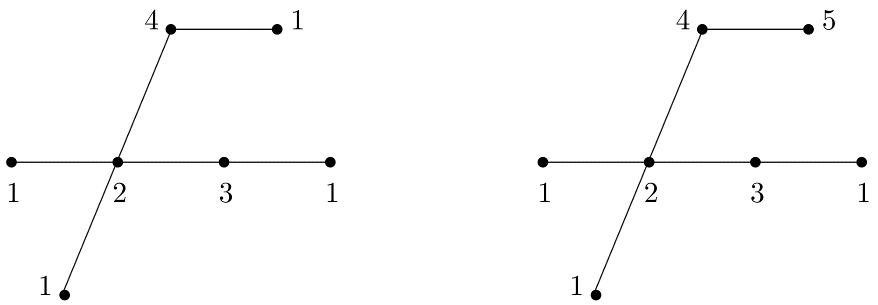
2. Preliminary results

In this section, we first define the total global dominator coloring of a graph and show that this notion is different from the notion of total dominator coloring of a graph. Then the total global dominator coloring of some graphs are investigated.

Definition 2.1. *Let G be a graph with no isolated vertices and G be a k-colorable graph. A total global dominator coloring of the graph G is a proper coloring $f = (V_1, V_2, \cdots, V_k)$ of G in which, each vertex of G has a total dom-color class and an anti dom-color class in f. The minimum number of colors required for a total global dominator coloring of G is called the total global dominator chromatic number and is denoted by $\chi_{gd}^t(G)$.*

A graph G does not admit a total global dominator coloring when $\Delta(G) = n - 1$ or $\delta(G) < 1$. When $\Delta(G) < n - 1$, and $\delta(G) \geq 1$, the trivial coloring (that assigns distinct colors to distinct vertices) would serve as a total global dominator coloring. Thus, a graph G admits a total global dominator coloring if and only if $\Delta(G) < n - 1$, and $\delta(G) \geq 1$. So throughout this paper, all the graphs G for which $\chi_{gd}^t(G)$ is discussed are assumed to have maximum degree at most $|V(G)| - 2$ and $\delta(G) \geq 1$.

In Figure 1, the total dominator coloring and the total global dominator coloring of the graph G are shown.



$\chi_d^t(G) = 4$ $\chi_{gd}^t(G) = 5$

Figure 1

Our aim is to study the total global dominator chromatic number of some graphs and trees, as well as finding general bounds and characterization. In particular the complexity of total global dominator coloring is studied. Some basic results and some bounds in terms of order, chromatic number, domination parameters are investigated. Finally we classify the total global dominator coloring of trees.

Now we determine the value of $\chi_{gd}^t(G)$ for some common classes of graphs such as paths, cycles, complete multipartite graphs. The following theorem from [13] is useful.

Theorem 2.1. ([13] Theorems 4.2, 4.3) *Let C_n be a cycle of order $n \geq 3$ and P_n be a path of order $n \geq 2$. Then*

$$\text{(i) } \chi_d^t(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ 4\lfloor \frac{n}{6} \rfloor + r & \text{if } n \neq 4 \text{ and } n \equiv r \pmod 6,\ r \in \{0,1,2,4\} \\ 4\lfloor \frac{n}{6} \rfloor + r - 1 & \text{if } n \equiv r \pmod 6,\ r = 3,5. \end{cases}$$

$$\text{(ii) } \chi_d^t(P_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil - 1 & \text{if } n \equiv 1 \pmod 3 \\ 2\left\lceil \frac{n}{3} \right\rceil & \text{otherwise.} \end{cases}$$

Note that a total global dominator coloring of a graph G is obviously a total dominator coloring of G therefore $\chi_d^t(G) \leq \chi_{gd}^t(G)$. The following theorem is useful for determining the value of $\chi_{gd}^t(G)$ of graphs.

Theorem 2.2. *If G is a connected graph such that neither G nor $\overline{G}$ have an isolated vertex with $\chi_d^t(G) \geq \Delta(G) + 2$, then $\chi_{gd}^t(G) = \chi_d^t(G)$.*

Proof. Let $(V_1, V_2, ..., V_{\chi_d^t})$ be a χ_d^t-coloring of G. If u is an arbitrary vertex of G, then it has a total dom-color class in this coloring. Since $\chi_d^t(G) \geq \Delta(G) + 2$, and $\deg(u) \leq \Delta(G)$, there is at least one color class such that vertex u has no neighbor in it. This class would serve as an anti dom-color class of u. Hence this χ_d^t-coloring is also a $\chi_{gd}^t(G)$. Thus $\chi_{gd}^t(G) = \chi_d^t(G)$. □

From Theorem 2.2 we will have the following.

Theorem 2.3. *Let C_n and P_n be a cycle and a path of order $n \geq 4$, respectively. Then we have:*

(i) $\chi^t_{gd}(C_n) = \begin{cases} 4\lfloor \frac{n}{6} \rfloor + r & \text{if } n \equiv r \pmod 6),\ r \in \{0,1,2,4\}, \\ 4\lfloor \frac{n}{6} \rfloor + r - 1 & \text{if } n \equiv r \pmod 6),\ r = 3,5. \end{cases}$

(ii) $\chi^t_{gd}(P_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil - 1 & \text{if } n \neq 4 \text{ and } n \equiv 1 \pmod 3) \\ 2\left\lceil \frac{n}{3} \right\rceil & \text{otherwise.} \end{cases}$

Proof. For $n = 4$, if $\chi^t_{gd}(G) < 4$ then there exists at least one vertex that has no anti-dom color class, that is a contradiction, thus we have $\chi^t_{gd}(G) = 4$. Further, if G is either a cycle or a path on $n \geq 5$ vertices, then Theorem 2.1 follows that $\chi^t_d(G) \geq \Delta(G) + 2$ and Theorem 2.2 completes the proof. □

Theorem 2.4. *The total global dominator chromatic number of a complete k-partite graph is $2k$.*

Proof. Let G be a complete k-partite graph with partition $(X_1, X_2, ..., X_k)$. By our convention, $\Delta(G) < |V(G)-1|$; that is, $|X_i| \geq 2$, for each $i = 1, 2, ..., k$. For each i with $1 \leq i \leq k$, choose a vertex in X_i, say x_i. Let $f = (\{x_1\}, \{x_2\}, ..., \{x_k\}, V_1 \setminus \{x_1\}, V_2 \setminus \{x_2\}, ..., V_k \setminus \{x_k\})$ be a coloring of G. Since every vertex x_i in G has at least one total dom-color class in jth partite set with $j \neq i$ and one anti dom-color class in the itself partite set in $V_i \setminus \{x_i\}$ and every vertex $y_i \in V_i \setminus \{x_i\}$ has also at least one total dom-color class in jth partite set with $j \neq i$ and one anti dom-color class in the itself partite set in $\{x_i\}$. thus the coloring f is a χ^t_{gd}-coloring of G with $2k$ colors. Therefore $\chi^t_{gd}(G) \leq 2k$.
Since each vertex in one partite set is adjacent to all vertices of other partite sets, and every partite set cannot be an anti dom-color class of a vertex lying in a different partite set, therefore at least two colors are needed to color the vertices of any partite set and $\chi^t_{gd}(G) \geq 2k$. Thus the proof ends. □

3. Complexity

In this section, we establish the complexity of the total global dominator chromatic number of an arbitrary graph. First, we define some relevant decision problems.

TOTAL GLOBAL DOMINATOR CHROMATIC NUMBER (TGDCN)

INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a $TGDCN$ of cardinality at most k?

To show that $TGDCN$ problem is NP-complete for arbitrary graphs, we use the NP-completeness result for total dominator chromatic number which is investigated by Kazemi [13].

Theorem 3.1. ([13] Theorem 2.1) TOTAL DOMINATOR CHROMATIC NUMBER (TDCN) *is NP-complete.*

Theorem 3.2. *$TGDCN$ is NP-complete for general graphs.*

Proof. TGDCN belongs to $\mathcal{NP}$ since we can check in polynomial time that an assignment of colors to the vertices of a graph G is both a proper coloring and that every vertex v has a total dom-color class and anti dom-color class. Now we show that, how to transform any instance of $TDCN$ in to an instance G of $TGDCN$ so that the solution one of them is equivalent to the solution of the other one. Let (G, k) be an arbitrary instance of $TDCN$. We construct an instance (H, k') of $TGDCN$ as follows. Suppose that $H = P_4 \circ G$ where $V(P_4) = \{v_1, v_2, v_3, v_4\}$ and set $k' = k + 4$.

Suppose G has a proper total dominator coloring using k colors. Then H has a total dominator coloring if the vertices v_i, $(1 \leq i \leq 4)$ are assigned with 4 distinct colors and further these colors must be different from the k colors of G. Therefore each vertex v_i, $(1 \leq i \leq 4)$ and each vertex $u \in G$ has an anti dom-color class and a total dom-color class in the color classes $V_i = \{v_i\}$ for $1 \leq i \leq 4$. Thus H is a graph with $TGDCN$ $k + 4$.
Now let H be a graph with $TGDCN$ k'. Since every total global dominator coloring of a graph is a total dominator coloring. So existing a total global dominator coloring of G guarantees the total dominator coloring G. □

4. Some basic results

In this section, we determine the value of $\chi^t_{gd}(G)$ of some general graphs.

Theorem 4.1. *Let G be a disconnected graph such that neither G nor $\overline{G}$ have an isolated vertex. Then, $\chi^t_{gd}(G) = \chi^t_d(G)$.*

Proof. Under any χ^t_d-coloring of a disconnected graph G, for each component of G, there exists a color class that intersects only the vertex set of this component. On the other hand, for a vertex v of G belonging to a component G_i, there will be a color class V_i which does not intersect $V(G_i)$ (of other component) so that V_i is an anti dom-color class of v. As a result, every χ^t_d-coloring of G is a total global dominator coloring as well. Hence $\chi^t_{gd}(G) = \chi^t_d(G)$. □

Theorem 4.2. *For every connected graph G, we have.*

$$\chi^t_{gd}(G \circ H) = \begin{cases} \mid V(G) \mid + \chi(H) & \text{if } |V(G)| \geq 4 \text{ and } \Delta(G) \leq \mid V(G) \mid -2, \\ \mid V(G) \mid + \chi(H) + 1 & \text{if } |V(G)| \geq 2 \text{ and } \Delta(G) = \mid V(G) \mid -1. \end{cases}$$

Proof. Case 1. Let $\Delta(G) \leq \mid V(G) \mid -2$ and $V = \{v_1, v_2, \cdots, v_n\}$ be the set of vertices of G. If we color any copy of H with $\chi(H)$ colors and the vertices of G with $\mid V(G) \mid$ distinct colors, then every vertex of the i^{th} copy of H has $\{v_i\}$ as a total dom-color class and has $\{v_j\}$ $(j \neq i)$ as an anti dom-color class. Also every vertex v_i of G has $\{v_t\}$ as a total dom-color class for a neighbor v_t of v_i and has an anti dom-color class $\{v_l\}$ for a non adjacent vertex v_l of v_i. Therefore $\chi^t_{gd}(G \circ H) \leq \mid V(G) \mid + \chi(H)$.
For equality, in any χ^t_{gd}-coloring of $G \circ H$, we need at least $\chi(H)$ colors for any copy of H. Since the color of vertex v_i must be different from the color of every vertex of H, so at least $\chi(G)$ colors need for coloring of G. On the other hand, if k vertices of G like $v_{i_1}, v_{i_2}, \cdots, v_{i_k}$ have same color, then the vertices of the ${i_j}^{th}$-copy for $(1 \leq j \leq k)$ of H have no total dom-color class. Thus $\chi^t_{gd}(G \circ H) \geq \mid V(G) \mid + \chi(H)$. Therefore $\chi^t_{gd}(G \circ H) = \mid V(G) \mid + \chi(H)$ if $\Delta(G) \leq \mid V(G) \mid -2$.

Case 2. Let $\Delta(G) = \mid V(G) \mid -1$ and $V = \{v_1, v_2, \cdots, v_n\}$ be the set of vertices of G. Then we color every copy of H with $\chi(H)$ colors with same color classes, except one of them, and for this copy we use $\chi(H)$ colors but one color class of this copy is different from others, for this order every vertex of G has an anti dom-color class. We also color the vertices of G with n distinct colors and different from colors used in H. This coloring is a χ^t_{gd}- coloring of $G \circ H$. Therefore $\chi^t_{gd}(G \circ H) \leq\mid V(G) \mid + \chi(H) + 1$. On the other hand, using the method of proof of the second part of Case 1 we can show that $\chi^t_{gd}(G \circ H) \geq\mid V(G) \mid + \chi(H) + 1$. Therefore $\chi^t_{gd}(G \circ H) =\mid V(G) \mid + \chi(H) + 1$ if $\Delta(G) =\mid V(G) \mid - 1$. □

As a consequence of Theorem 4.2 we have.

Corollary 4.1. $\chi^t_{gd}(G \circ \overline{K_r}) = \begin{cases} \mid V(G) \mid + 1 & \textit{if } \Delta(G) \leq\mid V(G) \mid - 2, \\ \mid V(G) \mid + 2 & \textit{if } \Delta(G) =\mid V(G) \mid - 1. \end{cases}$

Theorem 4.3. *Let G and H be two connected graphs, with $\delta(G), \delta(H) \geq 1$ and $\Delta(G), \Delta(H) \leq n - 2$, then*

$$\chi^t_{gd}(G \vee H) \leq \chi^t_{gd}(G) + \chi^t_{gd}(H).$$

The bound is sharp.

Proof. For the total global dominator coloring of $G \vee H$, the colors of vertices of G cannot be used for the coloring of vertices of H, and the colors of the vertices of H cannot be used for the coloring of vertices G. Now we color all vertices of graph G with colors $\{1, 2, ..., \chi^t_{gd}(G)\}$ and all vertices of graph H with colors $\{\chi^t_{gd}(G) + 1, \chi^t_{gd}(G) + 2, ..., \chi^t_{gd}(G) + \chi^t_{gd}(H)\}$. Therefore, we have the result. The bound is sharp as it can be seen for $P_4 \vee P_4$ or $C_4 \vee C_4$. □

5. Some bounds

In this section, we will present some sharp lower and upper bounds for the total global dominator chromatic number of a graph.

Theorem 5.1. ([1] Lemma 8) *For any graph G, $\gamma_{tg}(G) \geq 4$.*

Theorem 5.2. *Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then $\gamma_{tg}(G) \leq \chi^t_{gd}(G)$.*
The equality holds for $mC_6 \cup nP_6$ where $m + n \geq 1$ and $rP_2 \cup sP_3$ where $r + s \geq 2$ and union of at least two stars.

Proof. Let $\{V_1, V_2, ..., V_k\}$ be a χ^t_{gd}-coloring of G. Choose exactly one vertex u_i from each V_i for $1 \leq i \leq k$. Let S be the set of these vertices. Now, it is enough to prove that S is a total global dominating set of G. Every vertex $v \in V(G)$ has both a total dom-color class like V_i and an anti dom-color class like V_j, therefore v is adjacent to $v_i \in S$ and is not adjacent to $v_j \in S$. This means that S is a total dominating set of G and is global dominating set of G. That is, S is a total global dominating set of G.
For equality, it is well known that $\gamma_{tg}(C_6) = \gamma_{tg}(P_6) = 4$ and by Theorem 2.3, $\chi^t_{gd}(C_6) = \chi^t_{gd}(P_6) = 4$. Therefore $\gamma_{tg}(mC_6 \cup nP_6) = \gamma_{tg}(mC_6) + \gamma_{tg}(nP_6) = 4(m + n)$ and it is easy to see that $\chi^t_{gd}(mC_6 \cup nP_6) = \chi^t_{gd}(mC_6) + \chi^t_{gd}(nP_6) = 4(m + n)$. If $r + s \geq 2$, then $\gamma_{tg}(rP_2 \cup sP_3) = 2(r + s)$. Because of every P_2 or P_3 must be colored with two different colors till the graph are total global dominator colored, so $\chi^t_{gd}(rP_2 \cup sP_3) = 2(r + s)$. This reason is satisfied for the union of at least two stars. □

Observation 5.1. *Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then,*

$$max\{\gamma_{tg}(G), \chi(G) + 1\} \leq \chi^t_{gd}(G) \leq n.$$

Theorem 5.3. *Let G be a connected graph of order n such that neither G nor $\overline{G}$ have an isolated vertex. Then, $4 \leq \chi^t_{gd}(G) \leq n$.*

The equality in lower bound holds if and only if G is a 4-partite graph with this property, every vertex x, has at least one partite set as a total dom-color class and has one partite set as an anti dom-color class.

The equality in upper bound holds if and only if for every vertex x, at least one of the following holds:
1. *$x \in V_{n-2}$, where $V_{n-2} = \{v \in V(G) | \deg(v) = n-2\}$.*
2. *$V_{n-2} \nsubseteq N(x)$.*
3. *The set of vertices with the none of two properties 1 and 2, form a clique in G.*

Proof. Theorem 5.2 implies that $\gamma_{tg}(G) \leq \chi^t_{gd}(G)$, and since by Theorem 5.1 the total global domination number of graph G is at least 4, we obtain $4 \leq \chi^t_{gd}(G) \leq n$.
For seeing the equality of lower bound, let G be satisfied in the condition. Now we assign color i to the ith partite set for $1 \leq i \leq 4$. Since every vertex has a partite set as total dom-color class and a partite set as an anti dom-color class, therefore $\chi^t_{gd}(G) \leq 4$ and we deduce $\chi^t_{gd}(G) = 4$.
Conversely, let G be a graph with $\chi^t_{gd}(G) = 4$. Then G is a 4-partite graph. Since every vertex has a total dom-color class, this total dom-color class is a partite set. Since every vertex has an anti dom-color class, this class is also a partite set. Therefore the condition holds.

For seeing the equality upper bound, let G be a graph for which the given conditions hold. let x be an arbitrary vertex of G. If x satisfies in properties 1 or 2, then in any χ^t_{gd}-coloring of G the color of x is different from the color of all vertices and since graphs G and $\overline{G}$ have no isolated vertex, it is obvious that, x has a total dom-color class and anti dom-color class. Let x_i $(1 \leq i \leq k)$ be the vertices for which $\deg(x_i) \leq n-3$ and satisfy in the property 3. Then it is clear that the colors of all x_i are distinct and different from the color of the the vertices satisfy in the properties 1, or 2. Therefore $\chi^t_{gd}(G) = n$.
Conversely, let $\chi^t_{gd}(G) = n$. Let x be an arbitrary vertex which does not satisfy in the properties of 1, and 2. So $V_{n-2} \subseteq N(x)$ and $x \notin V_{n-2}$. If x is an only vertex with this property, then the proof is complete, otherwise suppose $x_1, x_2, \cdots, x_k$ $(k \geq 2)$ are the vertices such that do not satisfy the properties of 1, and 2. Let two vertices, without loss of generality, x_1 and x_2 are not adjacent. We assign same color to x_1 and x_2 and $n-2$ distinct colors to the rest vertices, different with the color x_1 and x_2. For each i, $V_{n-2} \subseteq N(x_i)$, thus every x_i has total dom-color class. Since every x_i has degree at most $n-3$, so x_i is not adjacent to a vertex with the property 2, or is not adjacent with two vertices like x_r or x_s. By the way, the vertex x_i has an anti dom-color class. Since each vertex with property 1, 2 has a total dom-color class and an anti dom color class, we have a contradiction. Therefore the two vertices x_1 and x_2 are adjacent and we deduce the set of vertices with the none of two properties 1 and 2, form a clique. Thus the proof ends. □

Theorem 5.4. *Let G be a connected graph of order n such that neither G nor $\overline{G}$ have an isolated vertex. Then, $8 \leq \chi^t_{gd}(G) + \chi^t_{gd}(\overline{G}) \leq 2n$, and these bounds are sharp.*

Proof. Since $\gamma_{tg}(G) = \gamma_{tg}(\overline{G})$, therefore $4 \leq \chi^t_{gd}(\overline{G}) \leq n$ and the inequalities are trivial. If $G = P_4$, then $\chi^t_{gd}(G) + \chi^t_{gd}(\overline{G}) = 8$. For the sharpness of the upper bound, consider the graph $\overline{mK_2}$ for $m \geq 2$. □

Theorem 5.5. *Let G be a split graph with split partition (K, I), and $|K| = w \geq 3$ such that neither G nor $\overline{G}$ have an isolated vertex. Then $\chi_{gd}^t(G) = w + 2$.*

Proof. Since neither G nor $\overline{G}$ have an isolated vertex hence the clique have at least two support vertices. Let $v_1, v_2 \in V(K)$ be two support vertices and U be all vertices of $V(G) \setminus V(K)$ that are adjacent to v_1. The coloring of G given by $f = (\{\{v\} : v \in K\} \cup \{U\} \cup \{I \setminus U\})$ is a χ_{gd}^t-coloring of G, and hence $\chi_{gd}^t \leq w + 2$.
On the other hand, in any χ_{gd}^t-coloring of G we assign an unique color to each vertex of K, a new color to all vertices of $V(G) \setminus V(K)$ that are adjacent to one support vertex and one color to the rest vertices of I. Therefore $\chi_{gd}^t(G) = w + 2$. □

6. Trees

In this section, we discuss on the total global dominator chromatic number of a tree. First, we present some needed definitions. In a connected graph G the distance between two vertices u and v, written $d_G(u, v)$ or simply $d(u, v)$, is the least length of a u, v-path, and the diameter of G, written $diam(G)$, is $max_{u,v \in V(G)} d(u, v)$. The eccentricity of a vertex u, written $\epsilon(u)$, is $max_{v \in V(G)} d(u, v)$, while the radius of G, written $rad(G)$, is $min_{v \in V(G)} \epsilon(u)$. The center G is the subgraph induced by the vertices of minimum eccentricity. The following classic theorem describes the center of trees.

Theorem 6.1. [17] *The center of a tree is a vertex or two adjacent vertices.*

The set of leaves in a tree T is denoted by L, and the set of its support vertices is denoted by S, while their cardinalities are denoted by l and s, respectively. Set $S = \{v_j : 1 \leq j \leq s\}$, and $L = \{u_i : 1 \leq i \leq l\}$. Let σ be a function from $\{1, 2, \cdots, l\}$ to $\{1, 2, \cdots, s\}$ in which $\sigma(i) = j$ if u_i is adjacent to v_j. Then $v_{\sigma(i)}$ denotes the support vertex corresponding to the leaf u_i. We start our discussion for trees with maximum degree at most $| V(T) | -2$ with the following theorem.

Observation 6.1. *Let T be a tree with the set of support vertices S and the set of leaves L. If tree T is of order $n \geq 4$, then $\chi_{gd}^t(T) \geq s + 1$.*

Proof. Let $|S| = s$ and $|L| = l$. In any total global dominator coloring of T, every support vertex of T must be contained in a color class of cardinality one, because every leaf has its corresponding support vertex as a total dom-color class and at least one different color needs to color the leaves of T, therefore $\chi_{gd}^t(T) \geq s + 1$. □

Theorem 6.2. *Let T be a tree with the set of support vertices S of size $|S| = s$ and the set of leaves L of size $|L| = l$.*
1. *If $diam(T) = 3$ or $diam(T) = 4$, then $\chi_{gd}^t(T) = s + 2$.*
2. *$\chi_{gd}^t(T) = s + 1$ if and only if T has diameter at least 5 and every vertex of T is a support vertex, a leaf or a vertex $v \in V(T) \setminus S \cup L$ in which $N(v) \subseteq S$ and moreover every vertex $w \in N(v)$ has a neighbor in S.*

Proof. 1. By Observation 6.1 we have $\chi_{gd}^t(T) \geq s+1$. Let $diam(T) = 3$. Then T is a bistar $S_{m,n}$ with support vertices v_1, v_2 and m, n leaves corresponding to v_1, v_2 respectively. It is clear that $\chi_{gd}^t(T) = 4 = s + 2$.

Let $diam(T) = 4$. Then the center C of T is a vertex set like $C = \{w\}$. If w is a support vertex, then $N(w)$ is consisted of all other support vertices and some leaves, and if w is not a support vertex, then $N(w)$ is consisted of all support vertices. Let S be the set of support vertices and L be the set of leaves. If $w \in S$, then for having anti dom-color class for w we need at least two different colors for vertices in L. If $w \notin S$, then the color of w is different from all colors of support vertices and the color of leaves. Therefore, in any case $\chi_{gd}^t(T) \geq s+2$. On the other hands, we have a total global dominator coloring of T with $s+2$ colors as follows. If $w \in S$, we assign color 1 to $N[w]-S$, color 2 to the rest of leaves and s distinct colors to all support vertices, and if $w \notin S$, we assign color 1 to the center vertex w, color 2 to all leaves and s distinct colors to s vertices in S, then $\chi_{gd}^t(T) \leq s+2$. Therefore $\chi_{gd}^t(T) = s+2$.

2. For seeing the equality, let T be a tree such that the given conditions hold. Then the color classes $(\{v_1\}, \{v_2\}, \cdots, \{v_s\}, V(T) \setminus S)$ can be a χ_{gd}^t-coloring of T. Thus $\chi_{gd}^t(T) = s+1$.
Conversely, let the equality holds. Then diameter of T is at least 5 by part 1. Let v be a vertex which is neither leaf nor support vertex in T. Since $\chi_{gd}^t(T) = s+1$, and v cannot be in any support vertex-color class, so v must be in L-color class. Now let $w \in N(v)$ and on the contrary, let $N(w) \cap S = \emptyset$. Then w has no total dom-color class, a contradiction. □

It is shown in [13] that every tree T admits a χ_d^t-coloring in which every support vertex of T must be contained in a color class of cardinality one as a singleton color class and one new color is required to color all the pendant vertices of T.

Theorem 6.3. *If T is a tree, then $\chi_d^t(T) \leq \chi_{gd}^t(T) \leq \chi_d^t(T)+1$. These bounds are sharp.*

Proof. Let f be a χ_d^t-coloring of T in which every support vertex appears as a singleton color class and all leaves of T have the same color. Let v be any support vertex of T. If we recolor all the pendant neighbors of v with the new color χ_d^t+1 and keep the colors of remaining vertices unchanged. Let f' be the resultant coloring. we claim that f' is a total global dominator coloring of T. Clearly, ev ery vertex of T has a total dom-color class in f'. Also, for every vertex of T other than v has the χ_d^t+1-color class as an anti dom-color class. For the vertex v, if there exists a support vertex u that is not adjacent to v, then $\{u\}$ is an anti dom-color class of v in f'. On the other hand, if every support vertex of T is adjacent to v, then every non-neighbor of v is a pendant vertex in T. This shows the color classes of f' which contains the pendant vertices of T other than the pendant neighbors of v can be an anti dom-color class for v. Therefore f' is a total global dominator coloring of T and so $\chi_{gd}^t(T) \leq \chi_d^t(T)+1$.
For seeing the sharpness of lower bound, let T be a tree with at least four support vertices in which $V(T) = S \cup L$. Then $\chi_d^t(T) = \chi_{gd}^t(T)$. For upper bound, let T be a bistar $S_{a,b}$ or a tree with diameter 3 named T_3. Then $\chi_{gd}^t(S_{a,b}) = 4 = 3+1 = \chi_d^t(S_{a,b})+1$, and $\chi_{gd}^t(T_3) = 4 = 3+1 = \chi_d^t(T_3)+1$. □

Theorem 6.4. *Let T be a tree with the center C. If $diam(T) \geq 5$, then $\chi_{gd}^t(T) = \chi_d^t(T)$.*

Proof. In any tree with $diam(T) \geq 5$ exist at least two support vertices with the property that the distance between the support vertices is at least three. Let u and v be two support vertices in T such that $d(u,v) \geq 3$. Then there is no vertex in T which is adjacent to both u and v. Consider a χ_d^t-coloring f of T in which u and v have distinct colors. Then f is a total global dominator coloring of T. If a vertex x of T does not lie on $N[u]$, then $\{u\}$ is an anti dom-color class of x and if x does lie on $N[v]$, then $\{v\}$ is an anti dom-color class of x. Therefore the result follows. □

7. Conclusions and discussions

The relation between total global dominator coloring and total dominator coloring of trees were investigated in Theorems 6.3 and 6.4. May we have such discussions on the arbitrary graphs?

In Theorem 5.2, we showed that, G and $\overline{G}$ have no isolated vertex. Then $\gamma_{tg}(G) \leq \chi^t_{gd}(G)$ and the equality holds for $mC_6 \cup nP_6$ where $m + n \geq 1$ and $rP_2 \cup sP_3$ where $r + s \geq 2$ and union of at least two stars.
One can find a necessary and sufficient conditions for family of graphs G for which $\gamma_{tg}(G) = \chi^t_{gd}(G)$.

Acknowledgement. The authors would like to extend their gratitude to the anonymous reviewer(s) for the careful reading of our manuscript, and for all the valuable suggestions.

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