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NEW SOLUTION OF CONFORMABLE FORNBERG-WHITHAM DIFFERENTIAL EQUATION VIA CONFORMABLE SUMUDU DECOMPOSITION METHOD

S. ALFAQEIH[1], G. BAKICIERLER[1], E. MISIRLI[1], §

Abstract. In this work, a new analytical method called the conformable Sumudu decomposition method is introduced to obtain approximate solutions of fractional Fornberg-Whitham differential equation. The proposed method is a combination between conformable Sumudu integral transform and the Adomian decomposition method. The fractional derivatives are taken in terms of the conformable sense. In order to demonstrate the applicability, efficiency and simplicity of the presented method, we compare the behavior of the obtained approximate solutions with the exact solution given in the literature.

Keywords: Conformable fractional derivative (CFD), Fornberg-Whitham equation, Adomian decomposition method (ADM), Sumudu transform (ST), Laplace transform (LT).

AMS Subject Classification: 26A33, 46F12, 35A22.

1. Introduction

In the past and present decades, nonlinear fractional differential equations have a great deal of interest due to their substantial contributions in life science and engineering [27, 24, 20, 10], therefore, many researchers have turned their attentions to solve such equations. In literature, many powerful methods have been used to obtain the approximate or the exact solutions of nonlinear fractional partial differential equations. For instance, variation iteration method [17, 18], Adomian decomposition method (ADM) [2, 6], homotopy perturbation method (HPM) [29, 19], reduce differential iteration method [22, 23], reliable methods [25], simplest equation method [31], and many others.
(ADM) was first introduced by G. Adomian in 1980, and it was applied to solve many nonlinear problems [5, 14, 26, 3, 4] in applied science and engineering. The main idea of this method is to solve partial differential equations by expressing the solution in terms of an infinite series, moreover, separate the linear and nonlinear terms. The nonlinear parts can be expressed in terms of Adomian polynomials and the initial approximation solution

[1] Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey.
e-mail: alfaqeihsuliman@gmail.com; ORCID: https://orcid.org/0000-0002-2115-8806.
e-mail: gizelbakicierler@gmail.com; ORCID:https://orcid.org/0000-0002-1789-0842.
e-mail: emine.misirli@ege.edu.tr; ORCID:https://orcid.org/0000-0001-5370-6283.
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can be come from the initial condition and the terms of independent variables, then by a recurrence relation, we can find other terms of the series.
Integral transformation method [7, 8, 9, 13, 11, 16] is considered to be one of the most attractive and effective methods to solve fractional differential equations cause it transforms the differential equation to an algebraic equation. The main disadvantage of integral transformation method that it is not able to solve nonlinear problems so, to overlap this problem, we must combine the integral transform with other analytical methods like (ADM), (HPM). Among the integral transformations, the Sumudu transform, which was first introduced by Watugala in 1993 [30] and it has been implemented to obtain the solution of many problems in real-life science and engineering. In order to solve conformable differential equations, the idea of single Sumudu transform was extended in [12] to the so-called conformable Sumudu transform (CST).
Now we feel compelled to combine the Adomian decomposition method with conformable Sumudu transform, in what is known conformable Sumudu decomposition method (CSDM). The pivotal aim of this article is to propose a new analytical technique namely, conformable Sumudu decomposition method (CSDM) to get an approximate analytical solution of the nonlinear conformable fractional Fornberg-Whitham equation which can be written in operator form as,

$$\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}-\frac{\partial^{2\nu}}{\partial x^{2\nu}}\left(\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}\right)+\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}=\varphi\frac{\partial^{3\nu}\varphi}{\partial x^{3\nu}}-\varphi\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+3\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}}, \tag{1}$$

with the initial condition

$$\varphi(x,0)=ke^{0.5\left(\frac{x^{\nu}}{v}\right)},$$

where, $\varphi(x,\tau)$ is the fluid velocity, x is the spatial coordinate, τ is the time, β and ν are the parameters defining the structure of the conformable fractional derivatives $(0<\nu,\ \beta\leq 1)$, and k is constant.

2. Preliminaries

In this section, we present basic notations about the conformable fractional derivatives (CFD) and the conformable Sumudu transform (CST).

Definition 2.1. [1, 21] *Let* $\frac{\partial^{s}\varphi}{\partial x^{s}},\ s=1,2,\cdots m-1,$ *be defined on* $\varphi(x,\tau): I\times(0,\infty)\to\mathbb{R}$*, then the (CFD) of a function* $\varphi(x,\tau): I\times(0,\infty)\to\mathbb{R}$ *of order* ν *is defined by:*

$$\frac{\partial^{\nu}\varphi(x,\tau)}{\partial x^{\nu}}=\lim_{\vartheta\to 0}\frac{\varphi_{x}^{(m-1)}(x+\vartheta x^{m-\nu},\tau)-\varphi_{x}^{(m-1)}(x,\tau)}{\vartheta},\ \nu\in(m-1,m],\ x,\ \tau\geq 0.$$

Definition 2.2. [1] *Let* $\frac{\partial^{s}\varphi}{\partial\tau^{s}},\ s=1,2,\cdots m-1,$ *be defined on* $\varphi(x,\tau): I\times(0,\infty)\to\mathbb{R}$*, then the (CFD) of a function* $\varphi(x,\tau): I\times(0,\infty)\to\mathbb{R}$ *of order* β *is defined by:*

$$\frac{\partial^{\beta}\varphi(x,\tau)}{\partial\tau^{\beta}}=\lim_{\varepsilon\to 0}\frac{\varphi_{\tau}^{(m-1)}\left(x,\tau+\varepsilon\tau^{m-\beta}\right)-\varphi_{\tau}^{(m-1)}(x,\tau)}{\varepsilon},\ \beta\in(m-1,m],\ x,\ \tau\geq 0.$$

Definition 2.3. [1] *Let* $\nu\in(m-1,m]$*, if* φ *is m-differentiable at* $x>0$*, then*

$$\frac{\partial^{\nu}\varphi(x,\tau)}{\partial x^{\nu}}=x^{m-\nu}\frac{\partial^{m}\varphi(x,\tau)}{\partial x^{m}}.$$

2.1. **(CFDs) of some functions:**

Example 2.1. *We have the following*

(1) $\frac{\partial^{\nu}(k)}{\partial x^{\nu}}=0,\ \frac{\partial^{\beta}(k)}{\partial\tau^{\beta}}=0,\ k$ is constant.
(2) $\frac{\partial^{\nu}}{\partial x^{\nu}}\left(k\left(\frac{x^{\nu}}{\nu}\right)^{n}\left(\frac{\tau^{\beta}}{\beta}\right)^{m}\right)=nk\left(\frac{x^{\nu}}{\nu}\right)^{n-\nu}\left(\frac{\tau^{\beta}}{\beta}\right)^{m}$,
(3) $\frac{\partial^{\beta}}{\partial\tau^{\beta}}\left(k\left(\frac{x^{\nu}}{\nu}\right)^{n}\left(\frac{\tau^{\beta}}{\beta}\right)^{m}\right)=mk\left(\frac{x^{\nu}}{\nu}\right)^{r}\left(\frac{\tau^{\beta}}{\beta}\right)^{m-\beta}$, $\forall k,m,\mathrm{n}\in\mathbb{R}$.
(4) $\frac{\partial^{\nu}}{\partial x^{\nu}}\left(e^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}\right)=ce^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}$,
(5) $\frac{\partial^{\beta}}{\partial\tau^{\beta}}\left(e^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}\right)=de^{c\left(\frac{x^{\nu}}{\nu}\right)+d\left(\frac{\tau^{\beta}}{\beta}\right)}$,$\forall\mathrm{c},\mathrm{d}\in\mathbb{R}$,
(6) $\frac{\partial^{\nu}}{\partial x^{\nu}}\left(\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=c.\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)$,
(7) $\frac{\partial^{\beta}}{\partial\tau^{\beta}}\left(\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=d\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\cos\left(\lambda\left(\frac{\tau^{\beta}}{\beta}\right)\right)$, $\forall\mathrm{c},\mathrm{d}\in\mathbb{R}$,
(8) $\frac{\partial^{\nu}}{\partial x^{\nu}}\left(\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\cos\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=-c.\sin\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)$,
(9) $\frac{\partial^{\beta}}{\partial\tau^{\beta}}\left(\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\cos\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)\right)=-d\cos\left(c\left(\frac{x^{\nu}}{\nu}\right)\right)\sin\left(d\left(\frac{\tau^{\beta}}{\beta}\right)\right)$, $\forall\mathrm{c},\mathrm{d}\in\mathbb{R}$.

Definition 2.4. [12] *The (CST) of a piecewise continuous function $\varphi:[0,\infty)\to\mathbb{R}$ of exponential order is defined on the set;*

$$\Omega_{\beta}=\left\{\varphi\left(\tau\right):\exists\lambda_{1},\ \lambda_{2}>0,\ \left|\varphi\left(\tau\right)\right|<K\exp\left(\frac{\left|\tau^{\beta}\right|}{\beta\lambda_{j}}\right),\ j=1,2\text{ and }\tau^{\beta}\in(0,\infty]\right\},$$

by the following integral

$$S_{\tau}^{\beta}\left(\varphi\left(\tau\right):u\right)=\int_{0}^{\infty}e^{-\frac{\tau^{\beta}}{\beta}}\varphi\left(u\tau\right)\tau^{\beta-1}d\tau. \tag{2}$$

Definition 2.5. [12] *Let $\varphi\left(x,\tau\right)$ be m times $\beta-$ differentiable and $\beta\in(0,1]$, then the (CST) of $\frac{\partial^{m\beta}\varphi(x,\tau)}{\partial\tau^{m\beta}}$ with respect to τ can be calculated as*

$$S_{\tau}^{\beta}\left(\frac{\partial^{m\beta}\varphi\left(x,\tau\right)}{\partial\tau^{m\beta}}\right)=\frac{S_{\tau}^{\beta}\left(\varphi\left(x,\tau\right)\right)}{u^{m}}-\frac{\varphi\left(x,0\right)}{u^{m}}-\sum_{i=1}^{m-1}u^{i-m}\left(\frac{\partial^{i\beta}}{\partial x^{i\beta}}\varphi\left(x,0\right)\right). \tag{3}$$

In particular for $\beta\in(0,1]$

$$S_{\tau}^{\beta}\left(\frac{\partial^{\beta}\varphi\left(x,\tau\right)}{\partial\tau^{\beta}}\right)=\frac{S_{\tau}^{\beta}\left(\varphi\left(x,\tau\right)\right)}{u}-\frac{\varphi\left(x,0\right)}{u}. \tag{4}$$

Example 2.2. *Let $k\in\mathbb{R}$ and $\beta\in(0,1]$, then the (CST) for certain functions is calculated by:*

(1) $S_{\tau}^{\beta}\left(k\right)=k,\ k$ is constant.
(2) $S_{\tau}^{\beta}\left(\left(\frac{\tau^{\beta}}{\beta}\right)^{s}\right)=\Gamma(s+1)u^{s}$,
(3) $S_{\tau}^{\beta}\left(e^{k\left(\frac{\tau^{\beta}}{\beta}\right)}\right)=\frac{1}{(1-ku)},\ \mathrm{k}u>1$,
(4) $S_{\tau}^{\beta}\left(\sin\left(k\frac{\tau^{\beta}}{\beta}\right)\right)=\frac{k}{(1+k^{2}u^{2})},\left|k\right|u>1$,
(5) $S_{\tau}^{\beta}\left(\cos\left(k\frac{\tau^{\beta}}{\beta}\right)\right)=\frac{1}{(1+k^{2}u^{2})},\left|k\right|u>1$.

3. The procedure of (CSDM)

In this section, the (CSDM) is discussed for the solutions of conformable fractional Fornberg-Whitham equation, we first recall the conformable fractional Fornberg-Whitham partial differential equation.

$$\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}-\frac{\partial^{2\nu}}{\partial x^{2\nu}}\left(\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}\right)+\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}=\varphi\frac{\partial^{3\nu}\varphi}{\partial x^{3\nu}}-\varphi\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+3\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}}, \tag{5}$$

with the initial condition

$$\varphi(x,0)=e^{0.5\left(\frac{x^{\nu}}{v}\right)}, \tag{6}$$

with the exact solution [28] when ν, $\beta=1$, $\varphi(x,\tau)=e^{\left(-\frac{2}{3}\tau+\frac{x}{2}\right)}$.

Taking the (CST) S_{τ}^{β}, on both sides of (5), we have

$$S_{\tau}^{\beta}\left[\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}\right]=S_{\tau}^{\beta}\left[\frac{\partial^{2\nu}}{\partial x^{2\nu}}\left(\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}\right)-\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+\varphi\frac{\partial^{3\nu}\varphi}{\partial x^{3\nu}}-\varphi\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+3\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}}\right], \tag{7}$$

using the differentiation property of the (CST), we obtain

$$S_{\tau}^{\beta}\left(\varphi(x,\tau)\right)=\varphi(x,0)+uS_{\tau}^{\beta}\left[\frac{\partial^{2\nu}}{\partial x^{2\nu}}\left(\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}\right)-\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+\varphi\frac{\partial^{3\nu}\varphi}{\partial x^{3\nu}}-\varphi\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+3\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}}\right], \tag{8}$$

operating with the inverse (CST) both sides of (8), we get

$$\varphi(x,\tau)=\varphi(x,0)+S_{\tau}^{-1}\left(uS_{\tau}^{\beta}\left[\frac{\partial^{2\nu}}{\partial x^{2\nu}}\left(\frac{\partial^{\beta}\varphi}{\partial\tau^{\beta}}\right)-\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+\varphi\frac{\partial^{3\nu}\varphi}{\partial x^{3\nu}}-\varphi\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}+3\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}}\right]\right). \tag{9}$$

Now, Adomian solution is

$$\varphi(x,\tau)=\sum_{i=0}^{\infty}\varphi_i(x,\tau), \tag{10}$$

and we can decompose the nonlinear terms by the series of Adomian polynomials as

$$N_1(\varphi)=\varphi\frac{\partial^{3\nu}\varphi}{\partial x^{3\nu}}=\sum_{i=0}^{\infty}A_i, \tag{11}$$

$$N_2(\varphi)=\varphi\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}=\sum_{i=0}^{\infty}B_i,$$

$$N_3(\varphi)=\frac{\partial^{\nu}\varphi}{\partial x^{\nu}}\frac{\partial^{2\nu}\varphi}{\partial x^{2\nu}}=\sum_{i=0}^{\infty}C_i,$$

where,

$$A_i=\tfrac{1}{i!}\tfrac{d^i}{dq^i}\left[N_1\left(\textstyle\sum_{j=0}^{\infty}q^j\varphi_j\right)\right]_{q=0},$$
$$B_i=\tfrac{1}{i!}\tfrac{d^i}{dq^i}\left[N_2\left(\textstyle\sum_{j=0}^{\infty}q^j\varphi_j\right)\right]_{q=0},$$
$$C_i=\tfrac{1}{i!}\tfrac{d^i}{dq^i}\left[N_3\left(\textstyle\sum_{j=0}^{\infty}q^j\varphi_j\right)\right]_{q=0}.$$

Substituting (10) and (11) in (9), we get

$$\sum_{i=0}^{\infty}\varphi_i=\varphi(x,0)+S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{\partial^{2\nu}\partial^\beta}{\partial x^{2\nu}\partial\tau^\beta}\left(\sum_{i=0}^{\infty}\varphi_i\right)-\frac{\partial^\nu}{\partial x^\nu}\left(\sum_{i=0}^{\infty}\varphi_i\right)+\sum_{i=0}^{\infty}A_i-\sum_{i=0}^{\infty}B_i+3\sum_{i=0}^{\infty}C_i\right]\right), \tag{12}$$

comparing both sides of (12), we get

$$\varphi_0(x,\tau)=\varphi(x,0), \tag{13}$$

$$\varphi_n(x,\tau)=S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{\partial^{2\nu}\partial^\beta}{\partial x^{2\nu}\partial\tau^\beta}(\varphi_{n-1})-\frac{\partial^\nu}{\partial x^\nu}(\varphi_{n-1})+A_{n-1}-B_{n-1}+3C_{n-1}\right]\right)\ n=1,2,3,\ldots.$$

Hence,

$$\varphi_0(x,\tau)=e^{0.5\left(\frac{x^\nu}{v}\right)},$$

$$\begin{aligned}\varphi_1(x,\tau)&=S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{\partial^{2\nu}\partial^\beta}{\partial x^{2\nu}\partial\tau^\beta}(\varphi_0)-\frac{\partial^\nu}{\partial x^\nu}(\varphi_0)+A_0-B_0+3C_0\right]\right),\\&=S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{\partial^{2\nu}\partial^\beta}{\partial x^{2\nu}\partial\tau^\beta}(\varphi_0)-\frac{\partial^\nu}{\partial x^\nu}(\varphi_0)+\varphi_0\frac{\partial^{3\nu}\varphi_0}{\partial x^{3\nu}}-\varphi_0\frac{\partial^\nu\varphi_0}{\partial x^\nu}+3\frac{\partial^\nu\varphi_0}{\partial x^\nu}\frac{\partial^{2\nu}\varphi_0}{\partial x^{2\nu}}\right]\right)\\&=S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{-1}{2}e^{0.5\left(\frac{x^\nu}{v}\right)}\right]\right)=\frac{-1}{2}e^{0.5\left(\frac{x^\nu}{v}\right)}S_\tau^{-1}(u)\\&=\frac{-1}{2}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}.\end{aligned}$$

$$\begin{aligned}\varphi_2(x,\tau)&=S_\tau^{-1}\left(uS_\tau^\beta\left[\begin{array}{c}\frac{\partial^{2\nu}\partial^\beta}{\partial x^{2\nu}\partial\tau^\beta}(\varphi_1)-\frac{\partial^\nu}{\partial x^\nu}(\varphi_1)+\varphi_1\varphi_0\frac{\partial^{4\nu}\varphi_0}{\partial x^{4\nu}}+\varphi_1\frac{\partial^\nu\varphi_0}{\partial x^\nu}\frac{\partial^{3\nu}\varphi_0}{\partial x^{3\nu}}-\\\varphi_1\varphi_0\frac{\partial^{2\nu}\varphi_0}{\partial x^{2\nu}}-\varphi_1\left(\frac{\partial^\nu\varphi_0}{\partial x^\nu}\right)^2+3\varphi_1\frac{\partial^\nu\varphi_0}{\partial x^\nu}\frac{\partial^{3\nu}\varphi_0}{\partial x^{3\nu}}+3\varphi_1\left(\frac{\partial^{2\nu}\varphi_0}{\partial x^{2\nu}}\right)^2\end{array}\right]\right)\\&=S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{-1}{8}e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{4}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}\right]\right)\\&=\frac{-1}{8}e^{0.5\left(\frac{x^\nu}{v}\right)}S_\tau^{-1}(u)+\frac{1}{4}e^{0.5\left(\frac{x^\nu}{v}\right)}S_\tau^{-1}\left(u^2\right)\\&=\frac{-1}{8}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{8}\left(\frac{\tau^\beta}{\beta}\right)^2e^{0.5\left(\frac{x^\nu}{v}\right)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\varphi_3(x,\tau)&=S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{\partial^{2\nu}\partial^\beta}{\partial x^{2\nu}\partial\tau^\beta}(\varphi_2)-\frac{\partial^\nu}{\partial x^\nu}(\varphi_2)+A_2-B_2+3C_2\right]\right)\\&=\frac{-1}{32}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{16}\left(\frac{\tau^\beta}{\beta}\right)^2e^{0.5\left(\frac{x^\nu}{v}\right)}-\frac{1}{48}\left(\frac{\tau^\beta}{\beta}\right)^3e^{0.5\left(\frac{x^\nu}{v}\right)},\end{aligned}$$

$$\begin{aligned}\varphi_4(x,\tau)&=S_\tau^{-1}\left(uS_\tau^\beta\left[\frac{\partial^{2\nu}\partial^\beta}{\partial x^{2\nu}\partial\tau^\beta}(\varphi_3)-\frac{\partial^\nu}{\partial x^\nu}(\varphi_3)+A_3-B_3+3C_3\right]\right)\\&=\frac{-1}{128}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{32}\left(\frac{\tau^\beta}{\beta}\right)^2e^{0.5\left(\frac{x^\nu}{v}\right)}-\frac{1}{64}\left(\frac{\tau^\beta}{\beta}\right)^3e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{384}\left(\frac{\tau^\beta}{\beta}\right)^4e^{0.5\left(\frac{x^\nu}{v}\right)}.\end{aligned}$$

Consequently, the approximate solution of the (5) is given by

$$\begin{aligned}\varphi(x,\tau) &= \varphi_0(x,\tau)+\varphi_1(x,\tau)+\varphi_2(x,\tau)+\varphi_3(x,\tau)+\varphi_4(x,\tau)+\cdots\\ &= e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{-1}{2}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{-1}{8}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{8}\left(\frac{\tau^\beta}{\beta}\right)^2e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{-1}{32}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}\\ &+\frac{1}{16}\left(\frac{\tau^\beta}{\beta}\right)^2e^{0.5\left(\frac{x^\nu}{v}\right)}-\frac{1}{48}\left(\frac{\tau^\beta}{\beta}\right)^3e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{-1}{128}\left(\frac{\tau^\beta}{\beta}\right)e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{32}\left(\frac{\tau^\beta}{\beta}\right)^2e^{0.5\left(\frac{x^\nu}{v}\right)}\\ &-\frac{1}{64}\left(\frac{\tau^\beta}{\beta}\right)^3e^{0.5\left(\frac{x^\nu}{v}\right)}+\frac{1}{384}\left(\frac{\tau^\beta}{\beta}\right)^4e^{0.5\left(\frac{x^\nu}{v}\right)}+\cdots\end{aligned}$$

Simplifying,

$$\varphi_{CSDM}(x,\tau)=e^{0.5\left(\frac{x^\nu}{v}\right)}\left[1+\frac{-85}{128}\left(\frac{\tau^\beta}{\beta}\right)+\frac{7}{32}\left(\frac{\tau^\beta}{\beta}\right)^2+\frac{-7}{192}\left(\frac{\tau^\beta}{\beta}\right)^3+\frac{1}{384}\left(\frac{\tau^\beta}{\beta}\right)^4+\cdots\right]. \quad (14)$$

4. Results and discussion

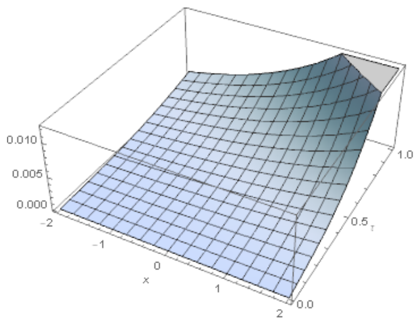
In this section, we illustrate the efficiency of the (CSDM) by comparing the exact solution and approximate solutions. First, in Table.1 and Table.2 we compare the approximate φ_{CSDM} with the exact solution φ_{exact}, at some point in case of $\nu=1,\ \beta=1$ and $\nu=1,\ \beta=0.75$. Figure.1a and Figure.1b show the absolute error between the exact and approximate solutions for $\nu=1,\ \beta=1$ and $\nu=1,\ \beta=0.75$. The obtained results illustrate that the (CSDM) is highly accurate. The exact solution φ_{exact} is presented by Figure.2a for $-2\le x\le 2,\ 0\le\tau\le 2$, Figure.2b shows the surface graph of φ_{CSDM} in case $-2\le x\le 2,\ 0\le\tau\le 2,\ \nu=1,\ \beta=1$. Figure.3a, Figure.3b, show the approximate solutions φ_{CSDM} in case $\nu=1,\ \beta=0.75$ and $\nu=0.1,\ \beta=0.98$, respectively, we observe that when both x and τ increase the value of φ_{CSDM} increases for $\beta=0.75, 0.98$ and $\beta=1$. In Figure.4, we present the the exact and approximate solutions graphically at $x=0.75$ for different values of $\tau,\ \nu,\ \beta$. It is clear from Figure.4 that the approximate solution φ_{CSDM} is very close to the exact solution as the values of $\nu,\ \beta$ increasing to 1.

Table 1. Comparison between the approximate solution φ_{CSDM} and the exact solution φ_{exact} for $\nu=\beta=1$.

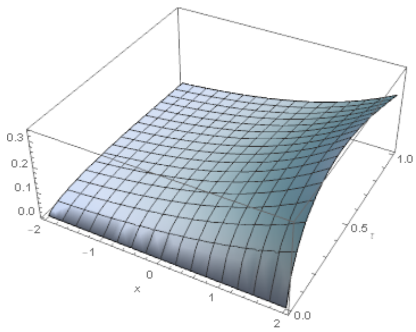
x	τ	φ_{exact}	φ_{CSDM}	Absolute error
-2	0.2	0.321958	0.322134	0.000176
-1	0.4	0.464559	0.465275	0.000716
0	0.6	0.670320	0.672775	0.002455
1	0.8	0.967216	0.974642	0.007426
2	1	1.39561	1.415770	0.020160

TABLE 2. Comparison between the approximate solution φ_{CSDM} and the exact solution φ_{exact} for $\nu = 1,\ \beta = 0.75$.

x	τ	φ_{exact}	φ_{CSDM}	Absolute error
-2	0.2	0.321958	0.282434	0.039524
-1	0.4	0.464559	0.389739	0.07482
0	0.6	0.670320	0.551519	0.118801
1	0.8	0.967216	0.793367	0.173849
2	1	1.395610	1.156040	0.239570

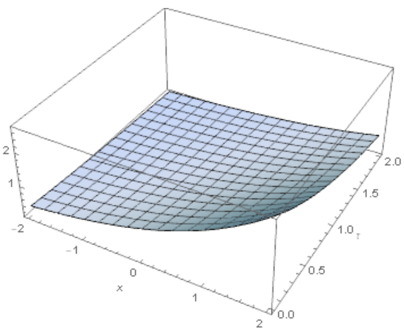


(A)

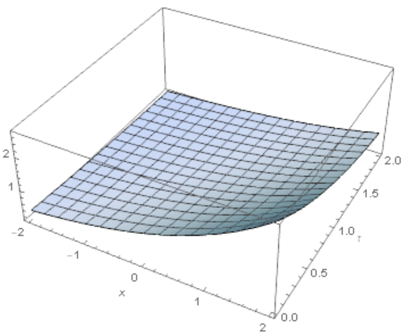


(B)

FIGURE 1. (A) The absolute error for $\nu = \beta = 1$. (B) The absolute error for $\nu = 1,\ \beta = 0.75$.



(A)

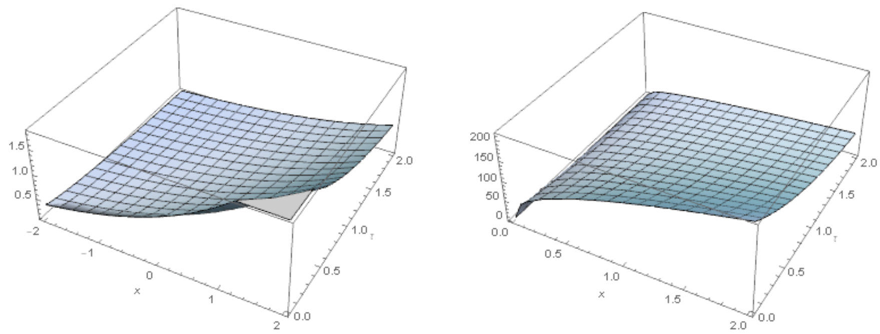


(B)

FIGURE 2. (A) The behavior of the exact solution $\varphi_{exact}(x, \tau)$. (B) The behavior of the approximate solution $\varphi_{CSDM}(x, \tau)$ in case $\nu, \beta = 1$.

5. CONCLUSIONS

In this article, we have successfully implemented a novel computational method called the conformable Sumudu decomposition method (CSDM) to get the approximate solutions of the conformable fractional Fornberg-Whitham equation. (CSDM) is based on the conformable Sumudu transform method and the Adomian decomposition method. To



(A) (B)

FIGURE 3. (A) The behavior of the approximate solution $\varphi_{CSDM}(x,\tau)$ in case $\nu = 1,\ \beta = 0.75$. (B) The behavior of the approximate solution $\varphi_{CSDM}(x,\tau)$ in case $\nu = 0.1,\ \beta = 0.98$.

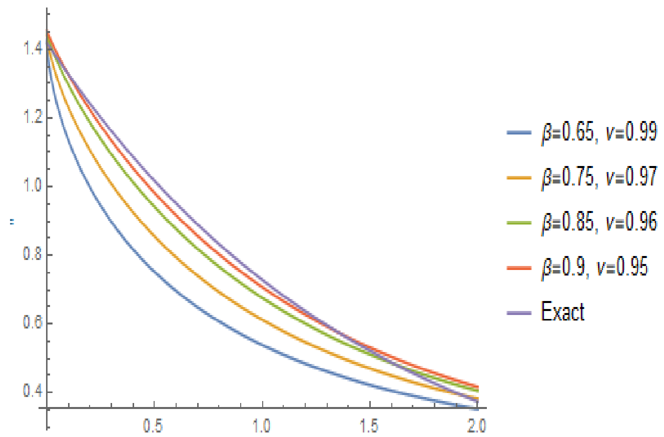


FIGURE 4. The behavior of the approximate solution $\varphi_{CSDM}(x,\tau)$ in case $x = 0.75$ for different values of $\nu,\ \beta$.

show the good agreement of the obtained approximate solutions and the exact solution, we compare our results with the exact solution obtained in the literature. Moreover, we have discussed and drawn the absolute error. The solution graphs for the problem show that the proposed method has good agreement with the exact solution. The obtained results reveal that the proposed approach is considered to be an attractive, easy and straightforward to solve the nonlinear conformable partial differential equations and a system of conformable fractional differential equations.

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Suliman Alfaqeih obtained his B.Sc. degree in Mathematics from Al-Quds University in 2009 and M.Sc. in Applied Mathematics in 2011 from the same University, from Palestine. Presently, he is a Ph.D. scholar in Applied Mathematics at Ege University, Izmir, Turkey. His area of interest includes numerical methods, fractional calculus, integral transforms, partial differential equations.

Gizel Bakıcıerler is a Ph.D. candidate in the Department of Mathematics, Ege University, Izmir, Turkey. She got her B.Sc. and M.Sc. degrees in Applied Mathematics from Ege University, Turkey. She presented some research studies at various international conferences in Turkey. Her major research interest includes analytical and numerical methods, nonlinear partial differantial equations, fractional calculus and physical equations in applied sciences and engineering.

Emine Mısırlı has been a professor in the Department of Mathematics, Ege University, Izmir, Turkey since 2011. She received her Ph.D. degree in Applied Mathematics, Dokuz Eylül University, Izmir, Turkey. Her research interests are differential equations, functional equations, general mathematics, fractional calculus, multiplicative calculus, numerical analysis and natural sciences.