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APPROXIMATION BY DE LA VALLÉE POUSSIN MEANS IN WEIGHTED GENERALIZED GRAND SMIRNOV CLASSES

AHMET TESTICI[1], §

ABSTRACT. Let G be a simple connected domain on complex plane such that $\Gamma := \partial G$ where Γ is a Carleson curve. In this work, we investigate the rate of approximation by De La Vallée Poussin mean constructed via $p-\varepsilon$ Faber series in the proper subclass of weighted generalized grand Smirnov classes $\mathcal{E}_{\omega}^{p),\theta}(G)$, $1<p<\infty$ where the ω satisfying Muckenhoupt's condition.

Keywords: De La Vallée Poussin mean, Faber series, rate of approximation, Generalized grand Smirnov classes, Muckenhoupt weights.

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1. INTRODUCTION AND MAIN RESULTS

Let $G\subset\mathbb{C}$ be a Jordan domain bounded rectifiable curve Γ and $G^{-} := Ext\ \Gamma$. Let $\mathbb{D} := \{w\in\mathbb{C} : |w|<1\}$, $\mathbb{T} := \partial\mathbb{D}$ and $\mathbb{D}^{-} := \overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$. We denote by $L^p(\Gamma)$, $1\le p<\infty$ the set of all measurable complex valued functions f on Γ such that $|f|^p$ is Lebesgue integrable with respect to arc length on Γ.

If there exist a sequence $(G_\nu)_{\nu=1}^{\infty}\subset G$ of domains G_ν which boundary rectifiable Jordan curve $(\Gamma_\nu)_{\nu=1}^{\infty}$ such that the domain G_ν contains each compact subset G^* of G where $\nu\ge\nu_0$ for some $\nu_0\in\mathbb{N}$ and

$$\limsup_{\nu\to\infty}\int\limits_{\Gamma_\nu}|f(z)|^p\,|dz|<\infty,$$

then we say that f belongs to the Smirnov class $E^p(G)$. Each function $f\in E^p(G)$ has the nontangential boundary values almost everywhere ($a.e.$) on Γ and the boundary function belongs to $L^p(\Gamma)$ (see, [5, p. 168] or [6, p. 438]).

Let $|\Gamma|$ be the Lebesgue measure of Γ and let $\omega:\Gamma\to[0,\infty]$ be a weight function. The weighted Lebesgue spaces $L_\omega^p(\Gamma)$, $1<p<\infty$ arises the set of all measurable functions f

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on Γ such that

$$\left(\frac{1}{|\Gamma|}\int_{\Gamma}|f(z)|^{p}\,\omega(z)\,|dz|\right)^{1/p}<\infty.$$

The set of all measurable on Γ functions f such that

$$\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^{\theta}}{|\Gamma|}\int_{\Gamma}|f(z)|^{p-\varepsilon}\,\omega(z)\,|dz|\right)^{1/(p-\varepsilon)}<\infty$$

for a $\theta\geq 0$ constitutes the weighted generalized grand Lebesgue space $L_{\omega}^{p),\theta}(\Gamma)$. The $L_{\omega}^{p),\theta}(\Gamma), 1<p<\infty$, $\theta\geq 0$ become a Banach space equipped with the norm

$$\|f\|_{L_{\omega}^{p),\theta}(\Gamma)}:=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon^{\theta}}{|\Gamma|}\int_{\Gamma}|f(z)|^{p-\varepsilon}\,\omega(z)\,|dz|\right)^{1/(p-\varepsilon)}.$$

Particularly, if we choose $\theta=1$, then $L_{\omega}^{p),1}(\Gamma)$ turns into the weighted grand Lebesgue space $L_{\omega}^{p)}(\Gamma)$ and if $\theta=0$, then $L_{\omega}^{p),0}(\Gamma)$ turns into the weighted Lebesgue space $L_{\omega}^{p}(\Gamma)$. The grand and generalized grand Lebesgue spaces were introduced in [15] and [7], respectively. The grand Lebesgue spaces have essential roles in the integrability problem of Jacobian under the minimal hypotheses. Furthermore, these spaces are appropriate for treating the existence and uniqueness, as well as the regularity problems for various nonlinear differential equations. Hence, some properties of grand Lebesgue space and integral operators with homogeneous kernel were investigated in [22, 25, 26, 27]. For these spaces the embedding $L_{\omega}^{p}(\Gamma)\subset L_{\omega}^{p),\theta}(\Gamma)$ holds. Although there is closely relation between $L_{\omega}^{p}(\Gamma)$ and $L_{\omega}^{p),\theta}(\Gamma)$, the generalized grand Lebesgue space has specific characteristics and advantages in comparison with the classical Lebesgue space.

Since the space $L_{\omega}^{p}(\Gamma)$ is not dense in $L_{\omega}^{p),\theta}(\Gamma)$, we denote by $\mathcal{L}_{\omega}^{p),\theta}(\Gamma)$ the closure of $L_{\omega}^{p}(\Gamma)$ in the $L_{\omega}^{p),\theta}(\Gamma)$ which consists of the functions f satisfying the condition

$$\lim_{\varepsilon\to 0}\left(\frac{\varepsilon^{\theta}}{|\Gamma|}\int_{\Gamma}|f(z)|^{p-\varepsilon}\,\omega(z)\,|dz|\right)=0.$$

We define the generalized grand Smirnov classes

$$E_{\omega}^{p),\theta}(G):=\left\{f\in E^{1}(G):f\in L_{\omega}^{p),\theta}(\Gamma)\right\}$$

of analytic functions f in G normed by $\|f\|_{E_{\omega}^{p),\theta}(G)}:=\|f\|_{L_{\omega}^{p),\theta}(\Gamma)}$. We denote the closure of Smirnov class $E_{\omega}^{p}(G)$ in the $E_{\omega}^{p),\theta}(G)$ by $\mathcal{E}_{\omega}^{p),\theta}(G)$.

Direct and inverse problems of approximation theory were studied in $E^{p}(G)$, $p\geq 1$, when Γ satisfied various geometric conditions. For instance, S. Y. Alper investigated [1] the order of approximation in $E^{p}(G)$, $p>1$, when G was sufficiently smooth domain. Later, this result was extended to domain with regular boundary for $p\geq 1$ by J. E. Andersson in [2]. In weighted Smirnov classes, direct and inverse theorems of approximation theory were proved in [10, 8, 11]. Firstly some results on the approximation theory in $\mathcal{E}_{\omega}^{p),\theta}(G)$ were proved in [14]. $\mathcal{L}_{\omega}^{p),\theta}(\Gamma)$ is reduced to weighted generalized grand Lebesgue spaces of 2π periodic functions defined on $[0,2\pi]$ when Γ is chosen as a circle particularly. In this case, some of the fundamental problems of approximation theory were studied in [3, 24, 4].

Later, using the more sensitive different type modulus of smoothness direct and inverse theorems of approximation theory in $\mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ were proved in [13]. Various approximation theorems in the subspace of grand Lebesgue space were investigated in [23, 16, 17, 18].

In this paper, we investigate approximation properties of De La Vallée Poussin means constructed via $p-\varepsilon$ Faber series of functions f belonged to $\mathcal{E}_{\omega}^{p),\theta}(G)$. Similar result in Smirnov classes with variable exponent and weighted Smirnov-Orlicz spaces were proved where Γ is Dini-smooth curve that more narrow curve than Carlson curves in [12] and [19], respectively.

Definition 1.1. *A rectifiable Jordan curve Γ is called a Carleson or regular curve if*

$$\sup_{z\in\Gamma}\sup_{r>0}\frac{|\Gamma(z,r)|}{r}<\infty$$

where $\Gamma(z,r)$ is portion of Γ in the open disk of radius r centered at z with length $|\Gamma(z,r)|$. We denote by $\mathcal{S}$ the set of all Carleson curves.

Definition 1.2. *Let ω be a weight function on $\Gamma\in\mathcal{S}$. We say that ω satisfy Muckenhoupt's A_p, $1<p<\infty$ condition on Γ if*

$$\sup_{z_0\in\Gamma}\sup_{r>0}\left(\frac{1}{r}\int\limits_{\Gamma(z_0,r)}\omega(z)\,|dz|\right)\left(\frac{1}{r}\int\limits_{\Gamma(z_0,r)}[\omega(z)]^{-1/(p-1)}\,|dz|\right)^{p-1}<\infty.$$

We denote by $A_p(\Gamma)$ the set of all weights ω satisfying Muckenhoupt's A_p condition on Γ.

Let $z_0\in\Gamma$ and let

$$S_\Gamma(f)(z_0):=\lim_{r\to 0}\int\limits_{\Gamma\setminus\Gamma(z_0,r)}\frac{f(z)}{z-z_0}dz \quad\text{and}\quad M_\Gamma(f)(z_0):=\sup_{r>0}\frac{1}{r}\int\limits_{\Gamma(z_0,r)}|f(z)|\,dz$$

be the Cauchy singular integral and the Hardy-Littlewood maximal function of $f\in L^1(\Gamma)$, respectively, existing for almost all $z_0\in\Gamma$.

Theorem 1.1 ([20])**.** *Let $\Gamma\in\mathcal{S}$, $1<p<\infty$ and $\theta>0$. The operators $S_\Gamma(f)$ and $M_\Gamma(f)$ are bounded in $L_{\omega}^{p),\theta}(\Gamma)$ if and only if $\omega\in A_p(\Gamma)$.*

Let $f\in\mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$, $1<p<\infty$ *and* $\theta>0$. For a given $r\in\mathbb{N}:=\{1,2,...\}$ we set

$$\Delta_t^r f(w):=\sum_{s=0}^{r}(-1)^{r+s+1}\binom{r}{s}f\left(we^{ist}\right),\ t>0.$$

We define the operator

$$\sigma_h^r f(w):=\frac{1}{h}\int_0^h|\Delta_t^r f(w)|\,dt,\ h>0.$$

If $\omega\in A_p(\mathbb{T})$, then by using Theorem 1.1 we get

$$\sup_{|h|\leq\delta}\|\sigma_h^r f(w)\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\leq c\,\|f\|_{L_{\omega}^{p),\theta}(\mathbb{T})}<\infty.$$

Definition 1.3. *Let $f\in L^{p),\theta}(\mathbb{T},\omega)$, $1<p<\infty$, $\omega\in A_p(\mathbb{T})$, and $\theta>0$. The function $\Omega_r(f,.)_{p),\theta,\omega}:[0,\infty)\to[0,\infty)$ defined as*

$$\Omega(f,\delta)_{p),\theta,\omega}:=\sup_{h\leq\delta}\|\Delta_h f(w)\|_{L_{\omega}^{p),\theta}(\mathbb{T})}$$

is called the modulus of f.

By φ we denote the conformal mappings of G^- onto $\mathbb{D}^-$ respectively, normalized by the conditions

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \varphi(z)/z > 0.$$

Let ψ be the inverse mappings of φ. The mappings φ and ψ have the continuous extensions to Γ and $\mathbb{T}$, respectively. Their derivatives φ' and ψ' have the definite nontangential limit values $a.e$ on Γ and $\mathbb{T}$, the limit functions are Lebesgue integrable on Γ and $\mathbb{T}$, respectively [6, pp. 419-438].

For any $f \in L_\omega^{p),\theta}(\Gamma)$ and $\omega \in A_p(\Gamma)$ we set

$$f_0(w) := f[\psi(w)]\left(\psi'(w)\right)^{1/(p-\varepsilon)} \quad , \quad \omega_0(w) := \omega[\psi(w)]$$

Obviously if $f \in L_\omega^{p),\theta}(\Gamma)$ then $f_0 \in L_{\omega_0}^{p),\theta}(\mathbb{T})$ (see [14]).

For a given function $f \in L_\omega^{p),\theta}(\Gamma)$ we define the Cauchy type integral

$$f_0^+(w) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(z)}{z-w}dz$$

which are analytic in $\mathbb{D}$.

We define rth mean modulus of smoothness for $f \in \mathcal{E}^{p),\theta}(G,\omega)$ as

$$\Omega_r(f,\delta)_{G,p),\theta,\omega} := \Omega_r\left(f_0^+,\delta\right)_{p),\theta,\omega_0}, \delta > 0.$$

The best approximation number for $f \in \mathcal{E}_\omega^{p),\theta}(G)$ in the class $\mathcal{P}_n$ of algebraic polynomials of degree not exceeding n is defined as

$$E_n(f)_{G,p),\theta,\omega} := \inf_{P_n\in\mathcal{P}_n} \|f - P_n\|_{L_\omega^{p),\theta}(\Gamma)}.$$

The set

$$Lip_\omega^{p),\theta}(G,\alpha) := \left\{ f \in \mathcal{E}_\omega^{p),\theta}(G) : \Omega_r(f,\delta)_{G,p),\theta,\omega} = \mathcal{O}(\delta^\alpha),\ \delta > 0 \right\}$$

is called the generalized grand Lipschitz class for a given $\alpha \in (0,1]$.

For construction of the approximation aggregates in $\mathcal{E}_\omega^{p),\theta}(G)$, we use the $p-\varepsilon$ Faber polynomials of $\overline{G}$. Since the conformal mapping φ is analytic in G^-, the function

$$[\varphi(z)]^k \left[\varphi'(z)\right]^{1/(p-\varepsilon)}$$

has a pole order k at ∞. Applying technique in [9] we obtain the representation

$$\frac{[\psi'(w)]^{1-1/(p-\varepsilon)}}{\psi(w)-z} = \sum_{k=0}^{\infty} \frac{F_{k,p-\varepsilon}(z)}{w^{k+1}}, \qquad z \in G, w \in \mathbb{D}^-; \tag{1}$$

for $1 < p < \infty$ and $0 < \varepsilon < p-1$ where $F_{k,p-\varepsilon}(z)$ is called $p-\varepsilon$ Faber polynomials of $\overline{G}$. $F_{k,p-\varepsilon}$ are algebraic polynomials with respect to z and have the following integral representation for every $k = 1,2,...,$:

$$F_{k,p-\varepsilon}(z) = \frac{1}{2\pi i}\int_{|w|=R} \frac{w^k\left(\psi'(w)\right)^{1-1/(p-\varepsilon)}}{\psi(w)-z}dw, \qquad z \in G \text{ and } R > 1.$$

Using (1) and Cauchy's integral formula

$$f(z) = \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int\limits_{|w|=1} \frac{f_0(w) \left[\psi'(w)\right]^{1-1/(p-\varepsilon)}}{\psi(w) - z} dw, \;\; z \in G,$$

which holds for every $f \in \mathcal{E}_{\omega}^{p),\theta}(G) \subset E^1(G)$, we have

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f) F_{k,p-\varepsilon}(z) \;\; , z \in G, \tag{2}$$

where

$$a_k(f) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw, \qquad k = 0, 1, 2, ..., \quad .$$

The series (2) are called the $p - \varepsilon$ Faber series of $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$, and the coefficients $a_k(f)$, $k = 0, 1, 2, ...$, are called $p - \varepsilon$ Faber coefficient of $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$. The nth partial sum of $p - \varepsilon$ Faber series of f is defined as

$$S_n^G(f) := S_n^G(f, z) = \sum_{k=0}^{n} a_k(f) F_{k,p-\varepsilon}(z) \;\; , n = 1, 2, 3, ..., \; . \tag{3}$$

De Vallée Poussin mean of $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$ is defined as

$$V_{n,m}^G(f) := V_{n,m}^G(f, z) = \frac{1}{n+1} \sum_{k=m}^{m+n} S_k^G(f, z)$$

for $m, n = 1, 2, ...$, and if $m = 0$, then $V_{n,m}^G(f)$ mean coincides with the Cèsaro mean

$$\sigma_n^G(f) := \sigma_n^G(f, z) = \frac{1}{n+1} \sum_{k=0}^{n} S_k^G(f, z).$$

We use the notations $m = \mathcal{O}(n)$, if there exist a positive constant c_2 such that $m \le c_2 n$ and we denote the different constants in the different relations in this work. Our main results are following.

Theorem 1.2. *Let $\Gamma \in \mathcal{S}$, $1 < p < \infty$ and $\theta > 0$. If $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$, then*

$$\left\| f - V_{n,m}^G(f) \right\|_{L_{\omega}^{p),\theta}(\Gamma)} = \mathcal{O}\left(\Omega_r(f, 1/n)_{G,p),\theta,\omega} \right).$$

Theorem 1.2 implies that following corollaries.

Corollary 1.1. *Let $\Gamma \in \mathcal{S}$, $1 < p < \infty$ and $\theta > 0$. If $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$, then*

$$\left\| f - \sigma_n^G(f) \right\|_{L_{\omega}^{p),\theta}(\Gamma)} = \mathcal{O}\left(\Omega_r(f, 1/n)_{G,p),\theta,\omega} \right).$$

Corollary 1.2. *Let $\Gamma \in \mathcal{S}$, $1 < p < \infty$, $\theta > 0$ and $\alpha \in (0, 1)$. If $f \in Lip_{\omega}^{p),\theta}(G, \alpha)$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$, then*

$$\left\| f - V_{n,m}^G(f) \right\|_{L_{\omega}^{p),\theta}(\Gamma)} = \mathcal{O}\left(n^{-\alpha} \right).$$

Corollary 1.1 implies that

Corollary 1.3. *Let $\Gamma \in \mathcal{S}$, $1 < p < \infty$, $\theta > 0$, and $\alpha \in (0,1)$. If $f \in Lip_{\omega}^{p),\theta}(G,\alpha)$, $\omega \in A_p(\Gamma)$ and $\omega_0 \in A_p(\mathbb{T})$, then*

$$\left\| f - \sigma_n^G(f) \right\|_{L_{\omega}^{p),\theta}(\Gamma)} = \mathcal{O}\left(n^{-\alpha}\right).$$

2. Auxiliary results

Let $f \in L^1(\mathbb{T})$ and

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k(f) e^{ikx}$$

be its Fourier series representation with the Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt, \quad k = 0, \pm 1, \pm 2, \ldots, \quad .$$

Let

$$S_n(f) := S_n(f,x) = \sum_{k=-n}^{n} c_k(f) e^{ikx}, \quad n = 1,2,...,$$

and

$$V_{n,m}(f) := V_{n,m}(f,x) = \frac{1}{n+1} \sum_{k=m}^{m+n} S_k(f,x), \quad m,n = 1,2,...,$$

be the nth partial sum of Fourier series and De Vallée Poussin mean of $f \in L^1(\mathbb{T})$ respectively.

Let Π_n be the set of the trigonometric polynomials of degree not exceeding n. The best approximation number of $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ is defined as

$$E_n(f)_{p),\theta,\omega} := \inf\left\{ \|f - T_n\|_{L_{\omega}^{p),\theta}(\mathbb{T})} : T_n \in \Pi_n \right\}, \quad n = 0,1,2,.. \quad ,$$

and if $E_n(f)_{p),\theta,\omega} = \|f - T_n^*\|_{L_{\omega}^{p),\theta}(\mathbb{T})}$, then $T_n^* \in \Pi_n$ is called the best approximation trigonometric polynomial to $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$.

Theorem 2.1 ([24])**.** *Let $1 < p < \infty$ and $\theta > 0$. Then there exists a positive constant c_3 such that the inequality*

$$\left\| \sup_n |S_n(f)| \right\|_{L_{\omega}^{p),\theta}(\mathbb{T})} \le c_3 \|f\|_{L_{\omega}^{p),\theta}(\mathbb{T})}, \quad n = 1,2,...,$$

holds for every $f \in L_{\omega}^{p),\theta}(\mathbb{T})$ if and only if $\omega \in A_p(\mathbb{T})$.

Theorem 2.1 implies that the operator $S_n : f \to S_n(f)$ is uniformly bounded in $L_{\omega}^{p),\theta}(\mathbb{T})$ with respect to n. Moreover, $S_n(f)$ convergences to f in norm with respect to $L_{\omega}^{p),\theta}(\mathbb{T})$, that is

$$\lim_{n\to\infty} \|S_n(f) - f\|_{L_{\omega}^{p),\theta}(\mathbb{T})} = 0$$

for $f \in L_{\omega}^{p),\theta}(\mathbb{T})$ where $1 < p < \infty$, $\theta > 0$ and $\omega \in A_p(\mathbb{T})$.

Theorem 2.2 ([13])**.** *Let $1 < p < \infty$, $\theta > 0$ and $\omega \in A_p(\mathbb{T})$. If $f \in \mathcal{L}_{\omega}^{p),\theta}(\mathbb{T})$ then there exists a positive constant c_4 such that the inequality*

$$E_n(f)_{p),\theta,\omega} \le c_4 \Omega_r(f, 1/n)_{p),\theta,\omega}, \quad n = 1,2,\ldots,$$

holds.

We define the operator $T_{p-\varepsilon}: E_{\omega_0}^{p),\theta}(\mathbb{D}) \to E_{\omega}^{p),\theta}(G)$ as

$$T_{p-\varepsilon}(f)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(w) [\psi'(w)]^{1-1/(p-\varepsilon)}}{\psi(w) - z} dw, \ z \in G,$$

then we have

$$T_{p-\varepsilon}\left(\sum_{k=0}^{n} \alpha_k w^k\right) = \sum_{k=0}^{n} \alpha_k F_{k,p-\varepsilon}(z)$$

where $\sum_{k=0}^{n} \alpha_k w^k$ is an algebraic polynomials with respect to w on $\mathbb{D}$.

Theorem 2.3 ([14])**.** *Let $\Gamma \in \mathcal{S}$, $1 < p < \infty$, $\omega \in A_p(\Gamma)$ and $\theta > 0$. If $\omega_0 \in A_p(\mathbb{T})$ then the following assertions hold:*
i) The operator $T_{p-\varepsilon}$ is linear and bounded,
ii) The operator $T_{p-\varepsilon}: \mathcal{E}_{\omega_0}^{p),\theta}(\mathbb{D}) \to \mathcal{E}_{\omega}^{p),\theta}(G)$ is one-to-one and onto and we have $T_{p-\varepsilon}(f_0^+) = f$ for a given $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$.

For a given $f \in L_{\omega}^{p),\theta}(\Gamma)$, $\omega \in A_p(\Gamma)$, $1 < p < \infty$ and $0 < \varepsilon < p-1$ we set

$$f^+(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G$$

and

$$f^-(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in G^-.$$

The functions $f^+(z)$ and $f^-(z)$ have the non-tangential limit *a.e.* on Γ and the following formulas

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2} f(z) \quad \text{and} \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2} f(z)$$

hold, hence we have

$$f(z) = f^+(z) - f^-(z) \tag{4}$$

a.e. on Γ [6, p. 431].

Lemma 2.1 ([14])**.** *Let $\Gamma \in \mathcal{S}$, $1 < p < \infty$, $\omega \in A_p(\mathbb{T})$, $\theta > 0$. If $f \in L_{\omega}^{p),\theta}(\Gamma)$ then $f^+ \in E_{\omega}^{p),\theta}(G)$ and $f^- \in E_{\omega}^{p),\theta}(G^-)$.*

3. Proof of Main Result

Proof Theorem 1.2. Let $\Gamma \in \mathcal{S}$, $1 < p < \infty$, $\omega \in A_p(\Gamma)$, $\omega_0 \in A_p(\mathbb{T})$ and $\theta > 0$. If $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$ then $f_0 \in \mathcal{L}_{\omega_0}^{p),\theta}(\mathbb{T})$ and by Lemma 2.1 we have $f_0^- \in \mathcal{E}_{\omega_0}^{p),\theta}(\mathbb{D}^-)$ and $f_0^+ \in \mathcal{E}_{\omega_0}^{p),\theta}(\mathbb{D}) \subset E^1(\mathbb{D})$. Hence the boundary function f_0^+ belongs to $\mathcal{L}_{\omega_0}^{p),\theta}(\mathbb{T})$. Theorem 2.1 implies that there exists a positive constant such that

$$\begin{aligned} \left\| V_{n,m}(f_0^+) \right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} &= \left\| \frac{1}{n+1} \sum_{k=m}^{m+n} S_k(f_0^+) \right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \\ &\leq \frac{1}{n+1} \sum_{k=m}^{m+n} \left\| S_k(f_0^+) \right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \leq c_5 \left\| f_0^+ \right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})}. \end{aligned} \tag{5}$$

Let $T_n^*\left(f_0^+\right) \in \Pi_n$ be the best approximation polynomial to $f_0^+ \in \mathcal{L}_{\omega_0}^{p),\theta}(\mathbb{T})$. Since $V_{n,m}(f)$ is a quasi projector on Π_n we have $V_{n,m}\left(T_n^*\left(f_0^+\right)\right) = T_n^*\left(f_0^+\right)$. Applying the technique in [21, p. 207], (5) and Theorem 2.2 we can obtain that

$$\begin{aligned}
\left\|f_0^+ - V_{n,m}\left(f_0^+\right)\right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} &\leq \left\|f_0^+ - T_n^*\left(f_0^+\right)\right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \\
&\quad + \left\|T_n^*\left(f_0^+\right) - V_{n,m}\left(f_0^+\right)\right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \\
&= E_n\left(f_0^+\right)_{p),\theta,\omega_0} + \left\|V_{n,m}\left(T_n^*\left(f_0^+\right) - f_0^+\right)\right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \\
&\leq E_n\left(f_0^+\right)_{p),\theta,\omega_0} + c_6\left\|T_n^*\left(f_0^+\right) - f_0^+\right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \\
&\leq c_7 E_n\left(f_0^+\right)_{p),\theta,\omega_0} \leq c_8 \Omega_r\left(f_0^+, 1/n\right)_{p),\theta,\omega_0}
\end{aligned} \tag{6}$$

where the constants independent of n. Since f_0^+ is analytic function on $\mathbb{D}$, it has the Taylor series expansion

$$f_0^+(w) = \sum_{k=0}^{\infty} \beta_k\left(f_0^+\right) w^k, \ w \in \mathbb{D}.$$

Let $\sum\limits_{k=-\infty}^{\infty} c_k\left(f_0^+\right) e^{ik\theta}$ be the Fourier series expansion of the boundary function $f_0^+ \in \mathcal{L}_{\omega_0}^{p),\theta}(\mathbb{T})$. Then by *Theorem 3.4* given in [5, p. 38] we have

$$c_k\left(f_0^+\right) = \begin{cases} \beta_k\left(f_0^+\right), & k \geq 0 \\ 0, & k < 0, \end{cases}$$

and

$$f_0^+(w) = \sum_{k=0}^{\infty} c_k\left(f_0^+\right) w^k.$$

Taking into account $f_0^+ \in \mathcal{E}_{\omega_0}^{p),\theta}(\mathbb{D})$ and $f_0^- \in \mathcal{E}_{\omega_0}^{p),\theta}(\mathbb{D}^-)$, by (4)

$$\begin{aligned}
a_k(f) \ :&= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^-(w)}{w^{k+1}} dw \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0^+(w)}{w^{k+1}} dw = \beta_k\left(f_0^+\right),
\end{aligned}$$

which shows that the $p-\varepsilon$ Faber coefficients of f are Taylor coefficients of f_0^+ at the origin, that is

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f) w^k, \ w \in \mathbb{D}.$$

Hence, we obtain

$$T_{p-\varepsilon}\left(S_n\left(f_0^+\right)\right) = T_{p-\varepsilon}\left(\sum_{k=0}^{n} c_k\left(f_0^+\right) w^k\right) = \sum_{k=0}^{n} a_k(f) F_{k,p-\varepsilon}(z) = S_n^G(f)$$

for any $n = 1, 2, \ldots$, and also

$$T_{p-\varepsilon}\left(V_{n.m}\left(f_0^+\right)\right) = V_{n,m}^G(f) \ . \tag{7}$$

By Theorem 2.3, (7) and (6) we have

$$\begin{aligned}
\left\| f - V_{n,m}^{G}(f) \right\|_{L_{\omega}^{p),\theta}(\Gamma)} &= \left\| T_{p-\varepsilon}\left(f_0^{+}\right) - T_{p-\varepsilon}\left(V_{n,m}\left(f_0^{+}\right)\right) \right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \\
&\leq \left\| T_{p-\varepsilon} \right\| \left\| f_0^{+} - V_{n,m}\left(f_0^{+}\right) \right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})} \\
&\leq c_9 \Omega_r \left(f_0^{+}, 1/n\right)_{p),\theta,\omega_0} \\
&= \mathcal{O}\left(\Omega_r (f, 1/n)_{G,p),\theta,\omega}\right).
\end{aligned}$$

□

4. Conclusions

The grand Lebesgue spaces have been considered in various fields of application, especially PDE theory. For this reason, investigations of the approximation process in these spaces have become significant. In this work, we investigate the properties of approximation aggregates as De La Vallée Poussin means of $p-\varepsilon$ Faber series in the subclass of weighted generalized Smirnov classes of analytic functions defined on simply connected domain bounded by Carleson curve. The appropriate estimation is obtained in terms of higher modulus of smoothness for a given function $f \in \mathcal{E}_{\omega}^{p),\theta}(G)$ where the ω belongs to Muckenhoupt's class. Finally, some results related to Cèsaro means constructed via $p-\varepsilon$ Faber series are given in this work.

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