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NUMERICAL SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS WITH ROBIN CONDITION: GALERKIN APPROACH

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Abstract. In this paper, classical solutions of nonlinear parabolic partial differential equations with the Robin boundary condition are approximated using the Galerkin finite element method (GFEM) which is associated with the combination of the Picard iterative scheme and α-family of approximation. The uniqueness, convergence, and structural stability analysis of solutions are studied. It is proven that the iterative scheme of the numerical method is stable. To ensure the efficiency and accuracy of the method, the comparative study between the exact and approximate solutions both numerically and graphically are given by solving two nonlinear parabolic problems. A reliable error estimation also opens possibilities of acceptance of the method. The results confirmed the consistency of the method and ensured the convergence of solutions.

Keywords: Nonlinear, parabolic equations, convergence, stability, shock problem, GFEM.

AMS Subject Classification: 92D25, 35K57 (primary), 35K61, 37N25.

1. Introduction

A big class of real-life problems which model the natural systems appear in mathematics as nonlinear partial differential equations (PDEs). Parabolic partial differential equations are one of the most important PDEs and have a wide range of industrial applications [1, 2, 3, 4]. Problems of these types arise in numerous branches of science and the life demands are modeled by PDEs with applications to physics, chemistry, ecology, biology, and other important fields of science (see [5, 6, 7, 8, 9], and references therein). Some examples are (i) the approximate theory of flow through a shock wave propagation in a viscous fluid, (ii) branching Brownian motion process and circuit theory, (iii) heat transfer in a draining film, (iv) dispersion of dissolved salts in groundwater, (v) auto-catalytic chemical reaction and nuclear reactor theory, (vi) fluid mechanics, turbulence, traffic flows, gas dynamics, and (vii) logistic population growth.

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To find the analytical solutions of higher-order differential equations are not only difficult but also impossible in many cases depending on the nonlinearities, nature of the governing equation, construction of appropriate meshes, complex geometrical shapes, ill-conditioning, and singularity. So the researchers devote their concentrations to develop a robust and efficient solution methodology to simulate the problems numerically.

In the literature, many researchers introduced new innovations and solved the nonlinear parabolic PDEs using simple to robust numerical methods. As a continuation, Ahmed solved linear parabolic PDEs (advection-diffusion) with constant and variable coefficients by the finite difference method [1]. In [7], a nonlinear time-periodic solution was prepared for parabolic boundary value problems by the finite difference method. A homotopy analysis method was introduced by Fallahzadeh and Shakibi to find the numerical solutions of linear convection-diffusion parabolic PDEs [6]. A meshless localized radial basis functions collocation method is used by Siraj-ul-Islam et al. for the numerical solutions of the hyperbolic PDEs and the transient nonlinear coupled Burgers' equations [10, 11, 12].

Rashidinia and Barati studied single-space-variable nonlinear parabolic equations using the Sinc collocation method [8]. Chou and Li derived the convergence properties of the nonconforming quadrilateral Wilson element for a class of nonlinear parabolic problems in two space dimensions along with Optimal H^1 and L_2 error estimations for the continuous time Galerkin approximation [13].

Siraj-ul-Islam et al. invented two new numerically stable methods based on Haar and Legendre wavelets for the solutions of one- and two-dimensional parabolic PDEs [14].

The multi-domain bivariate spectral collocation method was introduced by Sydney for solving nonlinear parabolic PDEs [9]. Tadmor provided a brief description on the development of the finite element method which is based on the Rayleigh-Ritz principle [15]. The FitzHugh-Nagumo equation, recently solved using Galerkin finite element method which was limited only for Neumann boundary conditions [5]. Chawla et al. had described new time-integration schemes for the linear convection-diffusion equation with Dirichlet and Neumann boundary conditions [16]. The nonlinear parabolic PDEs with Robin boundary conditions were solved by Sapa by introducing finite difference methods [17]. Chen and Zhang analyzed convection–diffusion equations and solved Burgers' equations by weak Galerkin finite element method which was limited within Dirichlet boundary condition only [18, 19]. Qi and Song also solved a parabolic equation using the Galerkin approach with an implicit θ scheme and this was also limited in Dirichlet boundary conditions only [20].

In this study, the nonlinear parabolic PDEs with Robin boundary conditions by Galerkin finite element method are solved; whose solution and the current approach are not available yet in the literature to the best of authors' knowledge. The main novelty of this paper is that we solved the nonlinear parabolic PDEs with Robin boundary conditions by Galerkin finite element method in an easy and efficient way. To apply GFEM, the important significance herein is that it is not necessary to convert the boundary value problems into initial ones.

The paper is organized as follows. In Section 2, the detailed formulation of GFEM for nonlinear parabolic PDEs with Robin boundary conditions is described. In Section 3, the convergence of this method is presented. The stability of this method along with the iterative schemes is narrated in Section 4. The numerical solutions of two nonlinear parabolic PDEs with Robin boundary conditions are presented in Section 5 and at the end of this research article, the conclusion is drawn along with the implementation, application, and the efficiency of the proposed scheme in Section 6.

In the following section, the formulation of GFEM for a class of second-order nonlinear parabolic PDEs will be discussed.

2. Mathematical Formulation

Let us consider a non-linear parabolic partial differential equation of the following form defined on an open bounded domain $\Lambda = [0, T] \times \Omega$, where $\Omega \in \mathbb{R}$,

$$\varepsilon \frac{\partial}{\partial x}\left(\delta(x)\frac{\partial Z(t,x)}{\partial x}\right) = \sigma(x)\frac{\partial Z(t,x)}{\partial t} + \mathcal{L}\left(x, t, Z(t,x), \frac{\partial Z(t,x)}{\partial x}\right). \tag{1}$$

The initial and the boundary conditions of the Robin type are

$$Z(x,0) = \Xi(x), \qquad x \in \Omega, \tag{2}$$

$$\alpha_1(t,x)Z(t,x) + \beta_1(t,x)\frac{\partial Z(t,x)}{\partial x} = \gamma_1(t,x), \qquad (t,x) \in \partial\Lambda, \tag{3}$$

$$\alpha_2(t,x)Z(t,x) + \beta_2(t,x)\frac{\partial Z(t,x)}{\partial x} = \gamma_2(t,x), \qquad (t,x) \in \partial\Lambda. \tag{4}$$

Where both $\alpha_1(t,x)$ and $\alpha_2(t,x)$ or $\beta_1(t,x)$ and $\beta_2(t,x)$ are not equal to zero simultaneously. To derive the mathematical formulation by Galerkin finite element method, first of all discretize the domain of x into a finite number of subdomains for a particular value of $t \geq 0$. Each subdomain is called an *element*. The length of the elements need not to be equal. The elements are numbered from left to right with parenthesis $[e]$. If the domain of x is discretize into n elements and each element contains m nodes, then the total number of degrees of freedom will be $N = (m-1) \times n + 1$.

Let the trial solution for a particular element $[e]$ be given by

$$\tilde{Z}(t,x) = \sum_{j=1}^{m} z_j(t)\varphi_j(x). \tag{5}$$

Then the weighted residual equation for the element $[e]$ becomes

$$\begin{aligned}
&\int_{[e]}\left[\sigma(x)\frac{\partial \tilde{Z}}{\partial t} - \varepsilon\frac{\partial}{\partial x}\left(\delta(x)\frac{\partial \tilde{Z}}{\partial x}\right) + \mathcal{L}\left(x, t, \tilde{Z}, \frac{\partial \tilde{Z}}{\partial x}\right)\right]\varphi_i dx = 0 \\
\Rightarrow &\int_{[e]}\sigma(x)\frac{\partial \tilde{Z}}{\partial t}\varphi_i dx - \int_{[e]}\varepsilon\frac{\partial}{\partial x}\left(\delta(x)\frac{\partial \tilde{Z}}{\partial x}\right)\varphi_i dx + \int_{[e]}\mathcal{L}\left(x, t, \tilde{Z}, \frac{\partial \tilde{Z}}{\partial x}\right)\varphi_i dx = 0 \\
\Rightarrow &\int_{[e]}\sigma(x)\frac{\partial \tilde{Z}}{\partial t}\varphi_i dx + \int_{[e]}\frac{\partial \varphi_i}{\partial x}\varepsilon\delta(x)\frac{\partial \tilde{Z}}{\partial x}dx + \int_{[e]}\mathcal{L}\left(x, t, \tilde{Z}, \frac{\partial \tilde{Z}}{\partial x}\right)\varphi_i dx = \left[\varepsilon\delta(x)\frac{\partial Z}{\partial x}\varphi_i\right]_{[e]}.
\end{aligned} \tag{6}$$

Use the equation (5) into the equation (6) and then simplify it, we obtain

$$\begin{aligned}
&\sum_{j=1}^{m}\frac{dz_j(t)}{dt}\int_{[e]}\sigma(x)\varphi_i\varphi_j dx + \sum_{j=1}^{m} z_j(t)\int_{[e]}\left[\varepsilon\delta(x)\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + \mathcal{L}\left(x, t, \varphi_j, \left(\sum_{k=1}^{m} z_k(t)\frac{d\varphi_k(x)}{dx}\right)\right)\varphi_i\right]dx \\
&= \left[\varepsilon\delta(x)\left(\sum_{k=1}^{m} z_k(t)\frac{d\varphi_k}{dx}\right)\varphi_i\right]_{[e]}.
\end{aligned} \tag{7}$$

Which can be written in the matrix form

$$\left[\mathcal{C}^{[e]}\right]\left\{\frac{dz(t)}{dt}^{[e]}\right\}+\left[\mathcal{K}^{[e]}\right]\left\{z(t)^{[e]}\right\}=\left\{\mathcal{F}^{[e]}\right\}, \tag{8}$$

where $\left[\mathcal{K}^{[e]}\right]$ and $\left[\mathcal{C}^{[e]}\right]$ are called the stiffness and forced matrices and the matrix $\left[\mathcal{F}^{[e]}\right]$ is called the Load vector. Since $\left[\mathcal{C}^{[e]}\right]$ is symmetric so it is also referred to as the damping matrix. This represents a non-linear system of ordinary differential equations. Here $\left\{z(t)^{[e]}\right\}, \left\{\frac{dz(t)}{dt}^{[e]}\right\}$, and $\left\{\mathcal{F}^{[e]}\right\}$ represents an $m \times 1$ vectors and $\left[\mathcal{C}^{[e]}\right], \left[\mathcal{K}^{[e]}\right]$ represents an $m \times m$ matrices, where m is the number of nodes in each element. By using time approximation discussed in later Section 2.1, equation (8) can be reduced to a set of the system of non-linear equations of the form [4]

$$\left[\mathbb{K}^{[e]}\right]_{\tau+1}\left\{z(t)^{[e]}\right\}_{\tau+1}=\left\{\mathbb{F}^{[e]}\right\}_{\tau,\tau+1}, \tag{9}$$

where $\left[\mathbb{K}^{[e]}\right], \left\{\mathbb{F}^{[e]}\right\}$ are known in terms of $\left[\mathcal{C}^{[e]}\right], \left[\mathcal{K}^{[e]}\right], \left\{\mathcal{F}^{[e]}\right\}, \left\{\frac{dz(t)}{dt}^{[e]}\right\}$, and $\left\{z(t)^{[e]}\right\}$. The subscript $\tau+1$ refers to the time $t_{\tau+1}$ at which the solution is sought.

2.1. **Time Approximations.** For time approximation purpose, use two type schemes as implicit and explicit. In the explicit type, z_j is found at time $t_{\tau+1}$ using the value of z_j at time t_τ which is known. In the explicit schemes, the time step size is limited approximately to the time taken for an elastic wave to cross the smallest element dimension in the mesh. So it is conditionally stable.

In the implicit type, z_j is found at time $t_{\tau+1}$ using both the known value of z_j and the unknown values of z_{j+1} at time t_τ and $t_{\tau+1}$, respectively. Implicit schemes have no limitation like explicit schemes and the time steps size can be greater than the explicit scheme's time step size. But at that time, the accuracy of the solution will decrease. So the adaptiveness between the time step size and accuracy depend on the followings [4]

(i) explicit scheme's stability,
(ii) the computational cost of the implicit scheme,
(iii) the relative size of the time step between the implicit and the explicit scheme that gives acceptable accuracy, and
(iv) the size of the computational model.

Using this time approximation in equation (8), there arises a system of nonlinear equations which can be solved by iterative procedure on their recurrent equations, that is discussed elaborately below.

2.2. **Recurrent Equations.** By using time approximation, transform the system of the ordinary differential equations (8) into the system of nonlinear algebraic equations (9). For this, finite difference schemes can be used such as *forward difference, backward difference, central difference schemes.* But in reality, no particular scheme works best for all non-linear problems. In this regard, the α-family of approximation is widely used that interpolates the weighted average of the time derivative at two consecutive time steps

$$(1-\alpha)\left\{\frac{dz(t)}{dt}\right\}_\tau+\alpha\left\{\frac{dz(t)}{dt}\right\}_{\tau+1}\approx\frac{\{z(t)\}_{\tau+1}-\{z(t)\}_\tau}{\Delta t}\qquad \text{for } 0\le\alpha\le 1, \tag{10}$$

where $\{\}_\tau$ refers to the value of the enclosed quantity at time $t=t_\tau$, and $\Delta t=t_{\tau+1}-t_\tau$. For different values of α, the following well-known numerical integration schemes arise for the equation (10) [4]

$$\alpha = \begin{cases} 0, & \text{the forward difference scheme (conditionally stable); order of accuracy} = O(\Delta t), \\ \frac{1}{2}, & \text{the Crank-Nicolson scheme (stable); order of accuracy} = O((\Delta t)^2), \\ \frac{2}{3}, & \text{the Galerkin method (stable); order of accuracy} = O((\Delta t)^2), \\ 1, & \text{the backward difference scheme (stable); order of accuracy} = O(\Delta t). \end{cases}$$

Then the equation (8) becomes

$$\left(\frac{\mathcal{C}}{\Delta t} + \alpha\mathcal{K}\right) z_{\tau+1} = \left(\frac{\mathcal{C}}{\Delta t} - (1-\alpha)\mathcal{K}\right) z_{\tau} + \mathcal{F}_{\tau}. \tag{11}$$

This is the recurrent formula that starts computing by taking an initial guess from the initial condition. At each iteration, an approximate solution is found by using *Picard iterative scheme* [23] for a particular time level.
Next Section 3 contains one important branch of this study, the convergence analysis which ensures that the iterative procedure is convergent.

3. Convergence Analysis

Let the shape functions are $\varphi_j(x) \in H_2^1,\ j = 1, 2, ..., m,$ where H_2^1 is the Hilbert space. Since $\delta(x), \frac{d\delta(x)}{dx}, \sigma(x)$, and $\mathcal{L}\left(x, t, Z(t,x), \frac{\partial Z(t,x)}{\partial x}\right)$ are continuous functions, so the solution of equation (1) uniquely exist [21]. Now substitute the trial function from the equation (5) into the equation (8), then the equation (8) can be written as

$$\left\{\frac{dz(t)^{[e]}}{dt}\right\} + \left[\mathcal{C}^{[e]}\right]^{-1}\left[\mathcal{K}^{[e]}\right]\left\{z(t)^{[e]}\right\} = \left[\mathcal{C}^{[e]}\right]^{-1}\left\{\mathcal{F}^{[e]}\right\}. \tag{12}$$

Equation (12) is an initial value problem

$$\begin{aligned} &\frac{dz_j}{dt} + \zeta_j z_j = \Im_j, \\ &z_j(0) = \Xi(x). \end{aligned} \tag{13}$$

The integrating factor of the equation (13) is

$$e^{\int \zeta_j dt} = e^{\zeta_j t}.$$

Multiply this on both sides of the equation (13) and then integrate over $[0, T]$

$$\begin{aligned} e^{\zeta_j t} z_j &= \int_0^T e^{\zeta_j \xi} \Im_j(\xi) d\xi \\ \Rightarrow z_j &= \int_0^T e^{-\zeta_j (t-\xi)} \Im_j(\xi) d\xi, \end{aligned} \tag{14}$$

and

$$|z_j|^2 \leq \frac{1}{2\zeta_j} \int_0^T |\Im_j(\xi)|^2 d\xi. \tag{15}$$

So in energy norm for a fixed t, the following is found

$$\begin{aligned}\left\|\tilde{Z}(t,x)\right\|_E^2 &= \sum_{j=1}^{m} |z_j(t)|^2 \zeta_j \\ &\leq \sum_{j=1}^{m} \left(\frac{1}{2\zeta_j}\int_0^T |\Im_j(\xi)|^2 d\xi\right)\zeta_j \\ &= \sum_{j=1}^{m} \left(\frac{1}{2}\int_0^T |\Im_j(\xi)|^2 d\xi\right).\end{aligned}$$

So the series in equation (5) converges for a particular value of t.
Let $\mathfrak{s}^{p,m}$ be the space of $(p-1)$ times continuously differentiable functions on $\bar{\Omega}$, the closure of Ω, for which the restriction to $\Omega_j,\ j = 0, 1, 2, ..., \mathcal{N} - 1$ is a polynomial of degree at most $m - 1$, where m is the number of local nodes in each element. Let $\tilde{Z} \in \mathfrak{s}^{p,m}$ be the finite element solution for a particular value of t, then for all $v \in \mathfrak{s}^{p,m}$

$$\left(\frac{\partial \tilde{Z}}{\partial t}, v\right)_0 + \mathcal{B}\left(\tilde{Z}, v\right) = (\mathcal{L}, v)_0, \tag{16}$$

where $(.,.)_0$ is the inner product, and $\mathcal{B}(.,.)$ is the bilinear transformation defined in [21, 24].
Equation(16) provides an initial value problem instead of the system of ordinary differential equations and is well known that for decreasing the increments in x, this result will converge to the exact solution of equation (1) for a particular t.
Now the iterative schemes can be applied in a relaxed way if it is numerically stable.
In the following Section 4, the stability of the iterative procedure will be discussed that demonstrate the stability of the proposed method also.

4. Stability Analysis

Equation (11) can be rewritten as

$$\left[\bar{\mathbb{K}}\right]\{z\}_{\tau+1} = [\hat{\mathbb{K}}]\{z\}_\tau + \left\{\bar{\mathbb{F}}_\tau\right\}, \tag{17}$$

$$\{z\}_{\tau+1} = [\mathbb{A}]\{z\}_\tau + \left[\bar{\mathbb{K}}\right]^{-1}\left\{\bar{\mathbb{F}}_\tau\right\}, \tag{18}$$

where $[\mathbb{A}] = \left[\bar{\mathbb{K}}\right]^{-1}[\hat{\mathbb{K}}]$ is the amplification matrix. Here $\{z\}_{\tau+1}, \{z\}_\tau$ are the solution vector at time $t + 1$ and t respectively.
The solution $\{z\}_{\tau+1}$ at time $t + 1$ depends on the solution $\{z\}_\tau$ at time t. So error can grow with iteration. An iterative method is said to be stable if the error does not grow boundlessly with iteration. The necessary and sufficient conditions to bound the error within a borderline, the eigenvalue λ_{max} of the amplification matrix $[A]$ must be less than or equal to unity such that

$$([A] - \lambda_{max}[I])\{z\} = 0. \tag{19}$$

Equation (19) will be an *unconditionally stable* eigenvalue problem if λ_{max} is less than or equal to unity for any time steps (Δt) [4]. If λ_{max} depends on the time step size (Δt) to be less than or equal to unity, then the procedure will be called *conditionally stable*.

Let us explain the case of *conditional stability*. Since the time step size can be related to the value of α in α-family of approximation as follows

$$\Delta t < \frac{2}{(1-2\alpha)\lambda_{max}}. \tag{20}$$

And it is known that if $\alpha < \frac{1}{2}$, all numerical schemes are stable for the time increment that satisfies the above relation. And for $\alpha \geq \frac{1}{2}$, the largest eigenvalue of the amplification matrix satisfies the following inequality

$$\lambda_{max} = \|\frac{1-(1-\alpha)\Delta t\lambda_{max}}{1+\alpha\Delta t\lambda_{max}}\| \leq 1, \tag{21}$$

which reveals that α-family of approximation is unconditionally stable. So it is concluded that for all values of α as well as Δt, the numerical schemes are unconditionally stable. In the following Section 5, the solutions of two nonlinear parabolic PDEs under Robin boundary conditions are given to demonstrate the efficiency of this method. Also, the role played among different parametric values, exact-approximate solution, and error terms are studied while time varies.

5. Computational Results and Discussion

In this section, the developed algorithm as discussed in the previous sections is applied to two well-known non-linear physical problems; parabolic partial differential equations with Robin boundary conditions. The results are presented both graphically in the diagram and numerically via the tabular form. All computations are done by MATLAB. In each format, the comparison between the approximate and the exact solution is presented. To compute the L_∞ norm and the mean of errors, the following two formulae are introduced

$$L_\infty = {}^{max}_{\;j}|Z(t,x)_j^{exact} - Z(t,x)_j^{GFEM}|,$$
$$mean = \frac{\sum_{j=1}^{n}|Z(t,x)_j^{exact} - Z(t,x)_j^{GFEM}|}{n}.$$

5.1. **Uniformly Propagating Shock Problem.** Let us consider the uniformly propagating shock problem [17, 26, 27]

$$\frac{\partial Z}{\partial t} = \frac{1}{Re}\frac{\partial^2 Z(t,x)}{\partial x^2} - Z(t,x)\frac{\partial Z(t,x)}{\partial x}, \tag{22}$$

subject to the initial-boundary conditions

$$\begin{aligned} Z(0,x) &= \frac{x-4}{x-2}, \\ Z(t,-1) + \frac{\partial Z(t,-1)}{\partial x} &= \frac{t^2+8T+13}{(t+3)^2}, \\ Z(t,1) - \frac{\partial Z(t,1)}{\partial x} &= \frac{t^2+4T+5}{(t+1)^2}. \end{aligned} \tag{23}$$

Here Re is the Reynolds number in the range $1 \leq Re \leq 10^5$ and for $(t,x) \in [0,1]\times[-1,1]$, $Z(t,x) \in (1.5, 3)$. The exact solution to this problem is $Z(t,x) = 1 - \frac{2}{x-t-2}$. For the computational purpose, 40 linear elements are taken. The iterative scheme that has been used here is the Crank-Nicolson iterative scheme with increments $(\Delta x, \Delta t) = (0.05, 0.05)$. The graphs (both 2D and 3D) of exact and approximate solutions for different time levels are depicted in Figure 1 with $Re = 1$ and the error graph is displayed in Figure 2. From Figure 3, it is found that for high Reynolds number, the computed results are stable and give a good agreement with the exact solution.

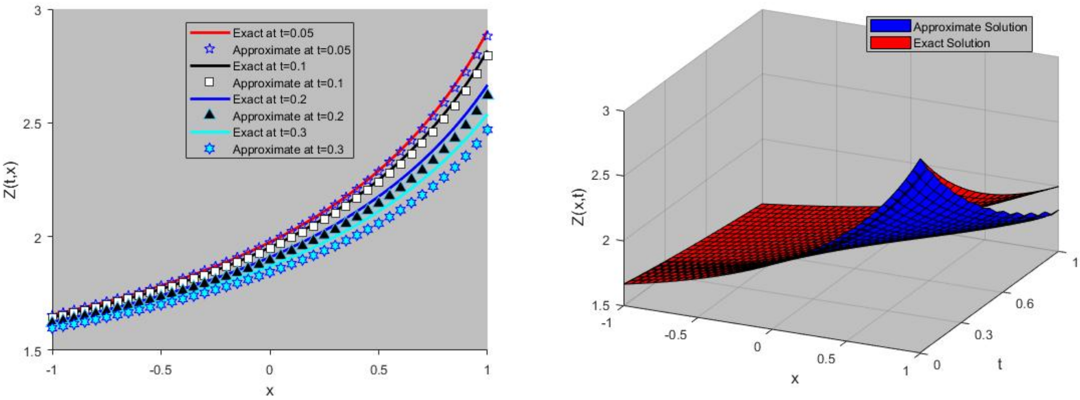

FIGURE 1. Comparison between the exact and approximate solutions at different time levels of equation (22) over the domain $[-1, 1]$.

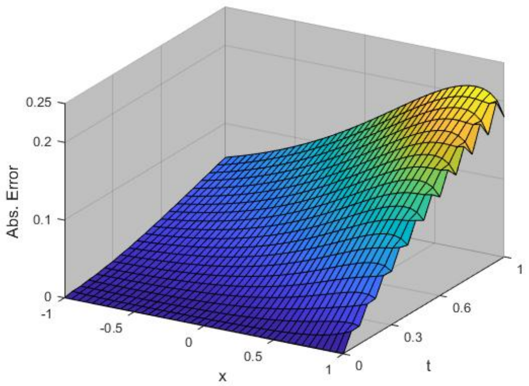

FIGURE 2. A plot of absolute error for the approximate solutions of equation (22).

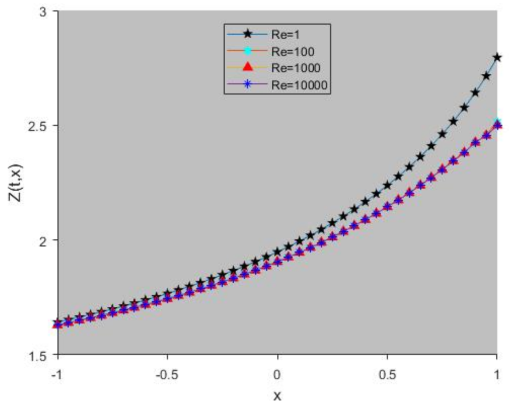

FIGURE 3. Behaviour of the approximate solutions of equation (22) at different Reynold number.

Comparisons between approximate and exact solutions at different nodes of x and different time levels are reported in Table 1. The L_∞ norm and the mean of error are reported in Table 2 which gives significantly better accuracy than the existing results in the literature [22] where the finite difference explicit method was being used and revealed L_∞ norm and the mean of error that exceeded 10^{46} and 10^{44}, respectively [17].

TABLE 1. Comparison between exact and approximate solutions of equation (22)

x	$t=0.1$			$t=0.2$			$t=0.3$		
	GFEM	Exact	Error	GFEM	Exact	Error	GFEM	Exact	Error
-1.00	1.6427	1.6452	2.4126×10^{-3}	1.6198	1.6250	5.2247×10^{-3}	1.5970	1.6061	9.0996×10^{-3}
-0.80	1.6873	1.6897	2.3255×10^{-3}	1.6610	1.6667	5.6517×10^{-3}	1.6348	1.6452	1.0381×10^{-2}
-0.60	1.7387	1.7407	2.0661×10^{-3}	1.7080	1.7143	6.2450×10^{-3}	1.6774	1.6897	1.2238×10^{-2}
-0.40	1.7977	1.8000	2.2673×10^{-3}	1.7617	1.7692	7.5615×10^{-3}	1.7255	1.7407	1.5211×10^{-2}
-0.20	1.8666	1.8696	2.9217×10^{-3}	1.8235	1.8333	9.8118×10^{-3}	1.7803	1.8000	1.9679×10^{-2}
0.00	1.9483	1.9524	4.1077×10^{-3}	1.8957	1.9091	1.3397×10^{-2}	1.8434	1.8696	2.6127×10^{-2}
0.20	2.0465	2.0526	6.1079×10^{-3}	1.9812	2.0000	1.8804×10^{-2}	1.9174	1.9524	3.5020×10^{-2}
0.40	2.167	2.1765	9.4681×10^{-3}	2.0846	2.1111	2.6479×10^{-2}	2.0061	2.0526	4.6565×10^{-2}
0.60	2.3184	2.3333	1.4910×10^{-2}	2.2135	2.2500	3.6456×10^{-2}	2.1164	2.1765	6.0098×10^{-2}
0.80	2.5163	2.5385	2.2159×10^{-2}	2.3815	2.4286	4.7071×10^{-2}	2.2607	2.3333	7.2607×10^{-2}
1.00	2.7949	2.8182	2.3302×10^{-2}	2.6203	2.6667	4.6317×10^{-2}	2.4696	2.5385	6.8858×10^{-2}

TABLE 2. The L_∞ norm and mean of absolute error at different times for equation (22)

t	L_∞ norm	Error mean
0.00	7.35×10^{-40}	8.27×10^{-41}
0.10	2.49×10^{-02}	7.72×10^{-03}
0.20	5.01×10^{-02}	1.94×10^{-02}
0.30	7.53×10^{-02}	3.32×10^{-02}
0.40	9.92×10^{-02}	4.81×10^{-02}
0.50	1.22×10^{-01}	6.34×10^{-02}
0.60	1.43×10^{-01}	7.88×10^{-02}
0.70	1.62×10^{-01}	9.39×10^{-02}
0.80	1.80×10^{-01}	1.09×10^{-01}
0.90	1.97×10^{-01}	1.23×10^{-01}
1.00	2.12×10^{-01}	1.36×10^{-01}

5.2. **Strongly Nonlinear Reaction-Diffusion Equation.** As a second problem, consider the following nonlinear reaction-diffusion equation [25]

$$\frac{\partial Z}{\partial t}=(\kappa_0+m)\left(\frac{\partial Z(t,x)}{\partial x}\right)^2-3\kappa_0\frac{\partial Z(t,x)}{\partial x}+\kappa_0\left(\frac{m+2}{m}\right)x\frac{\partial^2 Z(t,x)}{\partial x^2}+\frac{1}{\kappa_0\,(1+\mu t)^{\frac{m}{m+1}}}. \tag{24}$$

Subject to the initial condition

$$Z(x,0) = \left[A - \frac{1}{2\kappa_0}\left(\frac{m}{m+2}\right)\mu x^2\right]^{\frac{1}{m}},$$

and the boundary conditions

$$Z(0,t) + \frac{\partial Z(0,t)}{\partial x} = A^{\frac{1}{m}}\left(1+\mu t\right)^{\frac{1}{m+1}},$$
$$Z(1,t) + \frac{\partial Z(1,t)}{\partial x} = \Theta^{\frac{1}{m}}\left[1 - \frac{\mu}{\kappa_0(m+2)(1+\mu t)}\Theta^{-1}\right],$$

where

$$\Theta = A(1+\mu t)^{\frac{m}{m+1}} - \frac{m\mu}{2\kappa_0(m+2)(1+\mu t)}.$$

The explicit analytical solution of equation (24) is [25]

$$Z(t,x) = \left[A(1+\mu t)^{\frac{m}{m+1}} - \frac{1}{2\kappa_0}\left(\frac{m}{m+2}\right)\frac{\mu x^2}{(1+\mu t)}\right]^{\frac{1}{m}}.$$

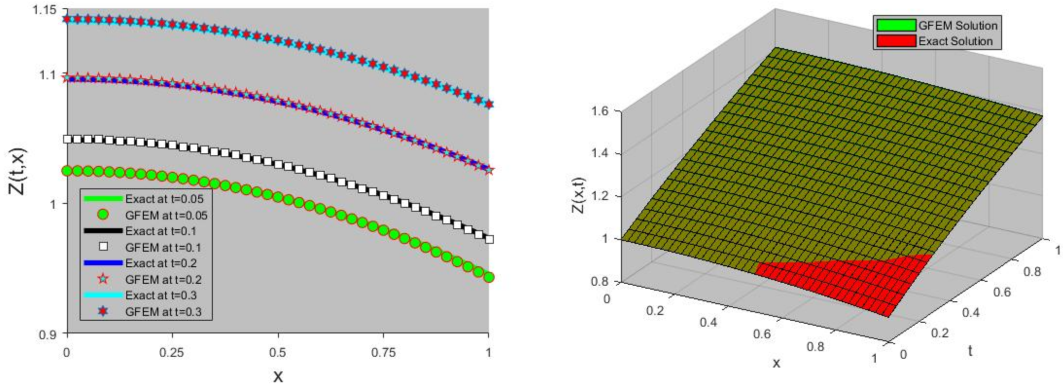

FIGURE 4. Exact and approximate solutions of equation (24) for $Z(t,x)$.

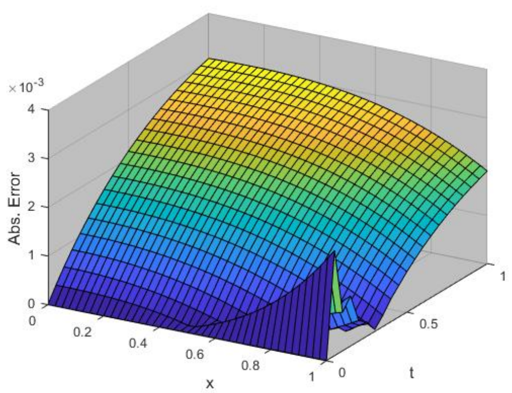

FIGURE 5. A plot of absolute error for the approximate solution of equation (24).

Here $Z(t,x) \in (0.9, 1.5)$ for the domain $(t,x) \in [0,1] \times [0,1]$. For computation 40 quadratic elements with the time increment 0.05 is taken. In α-family of approximation, consider $\alpha = 0.5$. For numerical approximation, the values $A = \mu = 1$, and $\kappa_0 = 2$ are being chosen.
Both curve and surface plots of the results are presented graphically in Figure 4 along with the three-dimensional error graph in Figure 5. The numerical results are reported in Table 3. The L_∞ norm and the mean of error are presented in Table 4. The L_∞ norm shows the higher accuracy and error mean ensures the stability of this algorithm.

TABLE 3. Comparison of exact and approximate solution of equation (24)

x	$t = 0.1$			$t = 0.2$			$t = 0.3$		
	GFEM	Exact	Error	GFEM	Exact	Error	GFEM	Exact	Error
0.00	1.0494	1.0488	5.7664×10^{-04}	1.0965	1.0954	1.0652×10^{-03}	1.1417	1.1402	1.4853×10^{-03}
0.10	1.0486	1.0481	5.6630×10^{-04}	1.0958	1.0948	1.0567×10^{-03}	1.1410	1.1395	1.4806×10^{-03}
0.20	1.0463	1.0458	5.2648×10^{-04}	1.0937	1.0927	1.0212×10^{-03}	1.1391	1.1376	1.4532×10^{-03}
0.30	1.0424	1.0420	4.5370×10^{-04}	1.0902	1.0892	9.5650×10^{-04}	1.1358	1.1344	1.3999×10^{-03}
0.40	1.0370	1.0367	3.4575×10^{-04}	1.0852	1.0843	8.6046×10^{-04}	1.1312	1.1299	1.3177×10^{-03}
0.50	1.0301	1.0299	2.0250×10^{-04}	1.0788	1.0781	7.3107×10^{-04}	1.1254	1.1241	1.2034×10^{-03}
0.60	1.0216	1.0215	2.6604×10^{-05}	1.0710	1.0704	5.6675×10^{-04}	1.1182	1.1171	1.0535×10^{-03}
0.70	1.0115	1.0117	1.7561×10^{-04}	1.0618	1.0614	3.6684×10^{-04}	1.1096	1.1088	8.6508×10^{-04}
0.80	0.9999	1.0003	3.9313×10^{-04}	1.0511	1.0510	1.3213×10^{-04}	1.0998	1.0991	6.3530×10^{-04}
0.90	0.9868	0.9875	6.0908×10^{-04}	1.0391	1.0392	1.3420×10^{-04}	1.0886	1.0883	3.6201×10^{-04}
1.00	0.9723	0.9731	7.9935×10^{-04}	1.0256	1.0260	4.2553×10^{-04}	1.0761	1.0761	4.4167×10^{-05}

TABLE 4. The L_∞ norm and mean of absolute error at different times for equation (24)

t	L_∞ norm	Error mean
0.00	5.74×10^{-11}	2.72×10^{-11}
0.10	7.99×10^{-04}	3.96×10^{-04}
0.20	1.07×10^{-03}	6.60×10^{-04}
0.30	1.49×10^{-03}	1.07×10^{-03}
0.40	1.85×10^{-03}	1.47×10^{-03}
0.50	2.16×10^{-03}	1.81×10^{-03}
0.60	2.42×10^{-03}	2.09×10^{-03}
0.70	2.64×10^{-03}	2.32×10^{-03}
0.80	2.82×10^{-03}	2.52×10^{-03}
0.90	2.97×10^{-03}	2.68×10^{-03}
1.00	3.08×10^{-03}	2.80×10^{-03}

6. CONCLUSION

In this paper, the complete formulation of the Galerkin Finite Element Method (FEM) is derived for nonlinear parabolic PDEs. The method was applied successfully to solve the nonlinear PDEs with the Robin boundary condition. In this case, the convergence and stability analysis were presented to ensure the validity and reliability of the proposed method. The results of particular nonlinear problems were depicted graphically and numerically which proved that the proposed method is very accurate, mathematically efficient, unconditionally stable, and computationally faster which converges rapidly to the exact solution. The absolute error map provided a very small error which is negligible. Data-structured tables and graphical maps of approximate and exact solutions ensured a cool agreement for a wide range of time and space steps. Afterward, it was clear from both numerical and graphical presentations that the characteristic of all solutions was harmonic due to its higher-order accuracy and low cost. This method can be applied for solving nonlinear parabolic PDEs with any other boundary conditions effortlessly.

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Conflict of interest

The authors declare no conflict of interest.

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