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ALMOST PERIODIC POSITIVE SOLUTIONS FOR A DELAYED NONLINEAR DENSITY DEPENDENT MORTALITY NICHOLSON'S BLOWFLIES MODEL ON TIME SCALES

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Abstract. In this paper we discuss a nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time varying delays. By contraction mapping theorem, we derived the necessary conditions for the existence of almost periodic positive solutions and by selecting suitable Lyapunov functionnal we study global asymptotic stability of the addressed model. Finally, some numerical simulations are listed to show the validity of our methods.

Keywords: Time scale, Nicholson's blowflies model; almost periodic positive solution, global asymptotic stability.

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1. Introduction

The delay differential equation

$$\vartheta'(t) = \alpha\vartheta(t) + \beta\vartheta(t-\tau)e^{-\gamma\vartheta(t-\tau)},\ t \in \mathbb{R}$$

describes a population of the Australian sheep blowfly proposed by Gurney [10] in 1980 and is agreed with the experimental data obtained by Nicholson [18] in 1954. Since this equation explains Nicholson blowfly more accurately, the model and it's modifications have been now refereed to as the Nicholson's blowflies model. The theory of Nicholson's blowflies model has made remarkable progress (see[6, 12, 17, 21] and references therein). Recently, Qian and Wang [22], studied a nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time-varying delays

$$\vartheta'(t) = a(t) + b(t)e^{-\vartheta(t)} + \sum_{j=1}^{m} \beta_j(t)\vartheta(t-h_j(t))e^{-\gamma_j(t)\vartheta(t-g_j(t))},\ t \in \mathbb{R}, \tag{1}$$

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and by utilising differential inequality techniques and the fluctuation lemma, a delay-independent criterion was determined to ensure the global asymptotic stability of the model.

Many authors [1, 9] believe that the discrete time model governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Recently, Zhu [25] considered the following discrete delayed Ricker model with survival rate,

$$\vartheta(t+1) = \gamma(t)\vartheta(t) + \vartheta(\tau(t))e^{r(t)\left[1-\frac{\vartheta(\tau(t))}{k(t)}\right]}, \; t \in \mathbb{Z}^+, \tag{2}$$

and established global attractivity, extreme stability, and the periodicity of the solution of the model.

The differential, difference and dynamic equations on time scales are three theories which play important role for modeling in the environment. Among them, the theory of dynamic equations on time scales is the most recent and was introduced by Stefan Hilger in his PhD thesis in 1988 with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set, so we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviors [4, 5, 23]. Moreover, basic results on this issue have been well documented in the articles [2, 3] and monographs of Bohner and Peterson [7, 8]. In the real world phenomena, since the almost periodic variation of the environment plays a crucial role in many biological and ecological dynamical systems and is more frequent and general than the periodic variation of the environment. In this paper we systematically unify the existence of almost periodic solutions for nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time varying delays modelled by ordinary differential equations and their discrete analogues in the form of difference equations and to extend these results to more general time scales. The concept of almost periodic time scales was proposed by Li and Wang [13]. Based on this concept, some works have been done (see [14, 15, 16, 19, 20]).

Motivated by aforementioned works, in this paper we study almost periodic positive solutions of a nonlinear density dependent mortality Nicholson's blowflies equation with multiple pairs of time-varying delays,

$$\vartheta^{\Delta}(t) = -a(t)\vartheta(t) + b(t)e^{-\vartheta(t)} + \sum_{\ell=1}^{n} \beta_\ell(t)\vartheta(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))}, \tag{3}$$

where $t \in \mathbb{T}$ ($\mathbb{T}$ is an arbitrary almost periodic time scale), $a(t)\vartheta(t) - b(t)e^{-\vartheta(t)}$ represents the death rate of the population, $\beta_\ell(t)\vartheta(t-h_\ell(t))\ e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))}$ describes the time dependent birth function which involves maturation delay $h_\ell(t)$ and incubation delay $g_\ell(t)$ and gains the reproduces at its maximum rate $\frac{1}{\gamma_\ell(t)}$, all parameter functions of (3) are nonnegative, bounded positive almost periodic functions, and $\ell \in \mathfrak{J} := \{1, 2, ..., n\}$. When $\mathbb{T} = \mathbb{Z}^+$, the model (3) is similar to the model (2).

For any bounded function $f(t)$, we denote $f^+ = \sup_{t\in\mathbb{T}} f(t)$, $f^- = \inf_{t\in\mathbb{T}} f(t)$.

We assume the following conditions are true throughout the paper:

(H_1) We assume that the bounded almost periodic functions $a(t)$, $b(t)$, $\beta_\ell(t)$, $g_\ell(t)$, $h_\ell(t)$ satisfy $0 < a^- \leq a(t) \leq a^+$, $0 < b^- \leq b(t) \leq b^+$, $0 < \beta_\ell^- \leq \beta_\ell(t) \leq \beta_\ell^+$, $0 < g_\ell^- \leq g_\ell(t) \leq g_\ell^+$, $0 < h_\ell^- \leq h_\ell(t) \leq h_\ell^+$ for $\ell = 1, 2, 3, \cdots, n$.

(H_2) The initial functions associated with equation (3) is given by

$$\vartheta(t;\varphi) = \varphi(t) \text{ for } t \in [-\varrho^*, 0]_{\mathbb{T}},\ \varrho^* = \max\left\{\max_{\ell\in\mathfrak{J}} g_\ell^+,\ \max_{\ell\in\mathfrak{J}} h_\ell^+\right\}$$

where $\varphi(\cdot)$ denotes a real-valued bounded and continuous functions defined on $[-\varrho^*, 0]_{\mathbb{T}}$.

Due to biological reasons of the model (3), positive solutions are only meaningful. So, we restrict our attention to positive solutions of equation (3).

2. Preliminaries

In this section, we introduce some definitions and state some preliminary results which are useful in the sequel.

Definition 2.1. [7] *A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to [0, \infty)$ are defined by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$ and $\mu(t) = \rho(t) - t$, respectively.*

- *In this definition we put* $\inf \emptyset = \sup \mathbb{T}$ *and* $\sup \emptyset = \inf \mathbb{T}$.
- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$.*
- *By an interval time scale, we mean the intersection of a real interval with a given time scale. i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$ other intervals can be defined similarly.*

Definition 2.2. [7] *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$; $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive provided $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}^k$ The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ and the set of all positively regressive functions and rd-continuous functions will be denoted by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$.*

Definition 2.3. [7] *If p is regressive function, then the generalized exponential function e_p is defined by*

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(x)}(p(x))\Delta x\right\}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{Log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Lemma 2.1. [7] *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions; then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$; *(ii)* $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$;

(iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$; *(iv)* $(e_p(\cdot, s))^\Delta = p(t)e_p(t, s)$.

Lemma 2.2. [7] *Suppose that* $p \in \mathcal{R}^+$, *then*

(i) $e_p(t,s) > 0$ *for all* $t, s \in \mathbb{T}$;
(ii) *if* $p(t) \le q(t)$ *for all* $t \ge s$, $t, s \in \mathbb{T}$, *then* $e_p(t,s) \le e_q(t,s)$ *for all* $t \ge s$.

Lemma 2.3. [7] *If* $p \in \mathcal{R}$ *and* $a, b, c \in \mathbb{T}$, *then* $[e_p(c,\cdot)]^\Delta = -p[e_p(c,\cdot)]^\sigma$, *and*

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

Lemma 2.4. [7] *Let* $p : \mathbb{T} \to \mathbb{R}$ *be right-dense continuous and regressive,* $a \in \mathbb{T}$ *and* $u_a \in \mathbb{R}$. *Then the unique solution of the initial value problem*

$$u^\Delta(t) = p(t)u(t) + f(t), \quad u(a) = u_a$$

is given by

$$u(t) = e_r(t,a)u_a + \int_a^t e_r(t,\sigma(s))f(s)\Delta s.$$

Lemma 2.5 ([7], Corollary 6.7, pp 257). *Let* $p \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$, $p(t) \ge 0$, $u(t) \in \mathcal{R}$ *and* $\alpha \in \mathbb{R}$. *Then*

$$u(t) \le \alpha + \int_{t_0}^t u(s)p(s)\Delta(s), \ \forall t \in \mathbb{T},$$

implies

$$u(t) \le \alpha e_p(t,t_0), \ \forall t \in \mathbb{T}.$$

Definition 2.4. [13] *A time scale* $\mathbb{T}$ *is called an almost periodic time scale if*

$$\Pi := \{\tau \in \mathbb{R} : t + \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \ne \{0\}.$$

Definition 2.5. [13] *Let* $\mathbb{T}$ *be an almost periodic time scale. A function* $f \in C(\mathbb{T},\mathbb{R})$ *is said to be almost periodic on* $\mathbb{T}$, *if, for any* $\varepsilon > 0$, *the set*

$$E(\varepsilon, f) = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}$; *that is, for any* $\varepsilon > 0$, *there exists a constant* $l(\varepsilon) > 0$ *such that each interval of length* $l(\varepsilon)$ *contains at least one* $\tau \in E(\varepsilon, f)$ *such that*

$$|f(t+\tau) - f(t)| < \varepsilon, \ \forall t \in \mathbb{T}.$$

The set $E(\varepsilon, f)$ *is called the* $\varepsilon-$*translation number of* $f(t)$. *We denote the set of all such functions by* $AP(\mathbb{T})$.

Lemma 2.6. [13] *If* $f \in C(\mathbb{T},\mathbb{R})$ *is an almost periodic function, then* f *is bounded on* $\mathbb{T}$.

Lemma 2.7. [13] *If* $f, g \in C(\mathbb{T},\mathbb{R})$ *are almost periodic functions, then* $f + g$, fg *are also almost periodic.*

Definition 2.6. [24] *Let* $\vartheta \in \mathbb{R}^m$ *and* $\mathcal{A}(t)$ *be an* $m \times m$ *rd-continuous matrix on* $\mathbb{T}$; *the linear system*

$$\vartheta^\Delta(t) = \mathcal{A}(t)\vartheta(t), \ t \in \mathbb{T}, \tag{4}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ *if there exist positive constants* k, α, *projection* P, *and the fundamental solution matrix* $\vartheta(t)$ *of* (4) *satisfying*

$$|\vartheta(t)\mathcal{P}\vartheta^{-1}(\sigma(\tau))|_0 \le ke_{\ominus\alpha}(t,\sigma(\tau)), \ \tau, t \in \mathbb{T}, \ t \ge \tau,$$

$$|\vartheta(t)(\mathcal{I} - \mathcal{P})\vartheta^{-1}(\sigma(s))|_0 \le ke_{\ominus\alpha}(\sigma(\tau), t), \ \tau, t \in \mathbb{T}, \ t \le \tau,$$

where $|\cdot|_0$ *is a matrix norm on* $\mathbb{T}$; *that is, if* $\mathcal{A} = (a_{ij})_{m\times m}$, *then we can take* $|\mathcal{A}|_0 = (\sum_{i=1}^m \sum_{j=1}^m |a_{ij}|^2)^{1/2}$.

Lemma 2.8. [13] *If the linear system* (4) *admits an exponential dichotomy, then the following system* $\vartheta^{\Delta}(t) = \mathcal{A}(t)\vartheta(t) + f(t),\ t \in \mathbb{T}$, *has a solution as follows:*

$$\vartheta(t) = \int_{-\infty}^{t} \vartheta(t)\mathcal{P}\vartheta^{-1}(\sigma(\tau))f(\tau)\Delta\tau - \int_{t}^{+\infty} \vartheta(t)(\mathcal{I} - \mathcal{P})\vartheta^{-1}(\sigma(\tau))f(\tau)\Delta\tau,$$

where $\vartheta(t)$ *is the fundamental solution matrix of* (4).

Lemma 2.9. [13] *Let* $\mathcal{A}(t)$ *be a regressive* $n \times n$ *matrix-valued function on* $\mathbb{T}$. *Let* $t_0 \in \mathbb{T}$ *and* $\vartheta_0 \in \mathbb{R}^n$, *then the initial value problem* $\vartheta^{\Delta}(t) = \mathcal{A}(t)\vartheta(t),\ \vartheta(t_0) = \vartheta_0$ *has a unique solution* $\vartheta(t) = e_{\mathcal{A}}(t, t_0)\vartheta_0$.

Lemma 2.10. [13] *Let* $d_i(t) > 0$ *be a function on* $\mathbb{T}$ *such that* $-d_i(t) \in \mathcal{R}^+$ *for all* $t \in \mathbb{T}$ *and* $\min\limits_{1\le i\le m} \left\{ \inf\limits_{t\in\mathbb{T}} d_i(t) \right\} > 0$. *Then the linear system*

$$\vartheta^{\Delta}(t) = diag\big(- d_1(t), -d_2(t), \cdots, -d_m(t)\big)\vartheta(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

3. Existence of the unique positive almost periodic solution

Let $\mathcal{B} = \{\vartheta(t) : \vartheta \in \mathcal{C}(\mathbb{T}, \mathbb{R}),\ \vartheta(t)$ is almost periodic function$\}$ with norm

$$\|\vartheta\|_{\mathcal{B}} = \sup_{t\in\mathbb{T}} |\vartheta(t)|.$$

Then $\mathcal{B}$ is a Banach space.

Theorem 3.1. *Assume that* (H_1) *and* (H_2) *hold. Let* $\mathfrak{M} > \mathfrak{m}$ *be two positive constants satisfy*

(i) $\mathfrak{M} = (\|\varphi\|_{\mathcal{B}} + b^*)e^+,\ b^* = \max\limits_{t\in[t_0,+\infty)_{\mathbb{T}}} \int_{t_0}^{t} b(\tau)\Delta\tau,\ e^+ = \max\limits_{t\in[t_0,+\infty)_{\mathbb{T}}} e^{\int_{t_0}^{t} \sum_{\ell=1}^{n} \beta_\ell(s)\Delta s}$.

(ii) $\dfrac{1}{a^+}\left[b^- e^{-\mathfrak{M}} + \sum\limits_{\ell=1}^{n} \beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}} \right] \ge \mathfrak{m} \ge \dfrac{1}{a^+}\sum\limits_{\ell=1}^{n} \beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}}$.

Then the solution $\vartheta(t) = \vartheta(t, t_0, \varphi) \ge 0$ *for all* $t \in [t_0, \eta(\varphi))_{\mathbb{T}}$, *of* (3) *satisfies*

$$\mathfrak{m} \le \vartheta(t) \le \mathfrak{M},\ t \in [t_0, +\infty)_{\mathbb{T}}.$$

Proof. Let $\vartheta(t) = \vartheta(t, t_0, \varphi)$ is a solution of (3) with the initial condition $\vartheta(t_0) = \varphi$, where $\varphi(\cdot)$ denotes a real-valued bounded and continuous functions defined on $[-\varrho^*, 0]_{\mathbb{T}}$. At first, we prove that $\vartheta(t) \le \mathfrak{M},\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}$, where $[t_0, \eta(\varphi))_{\mathbb{T}}$ is the maximal right interval of existence of $\vartheta(t, t_0, \varphi)$. For all $t \in [t_0, \eta(\varphi))_{\mathbb{T}}$, let $\varpi(t) = \max\limits_{t_0-\varrho\le\tau\le t} \vartheta(\tau)$, we get

$$\begin{aligned}
\vartheta^{\Delta}(t) =& - a(t)\vartheta(t) + b(t)e^{-\vartheta(t)} + \sum_{\ell=1}^{n} \beta_\ell(t)\vartheta(t - h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))} \\
&\le b(t) + \sum_{\ell=1}^{n} \beta_\ell(t)\vartheta(t - h_\ell(t)) \le b(t) + \sum_{\ell=1}^{n} \beta_\ell(t)\varpi(t).
\end{aligned}$$

So,

$$\begin{aligned}
\vartheta(t) &\le \vartheta(t_0) + b^* + \int_{t_0}^{t} \left[\sum_{\ell=1}^{n} \beta_\ell(\tau)\varpi(\tau) \right] \Delta\tau \\
&\le \|\varphi\|_{\mathcal{B}} + b^* + \int_{t_0}^{t} \left[\sum_{\ell=1}^{n} \beta_\ell(\tau) \right] \varpi(\tau)\Delta\tau,\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}.
\end{aligned} \tag{5}$$

Since (5) is true for every $t \in [t_0, \eta(\varphi))_{\mathbb{T}}$ and $\varpi(t) = \max_{t_0-\varrho\leq\tau\leq t} \vartheta(\tau)$, it follows that

$$\varpi(t) \leq \|\varphi\|_{\mathcal{B}} + b^* + \int_{t_0}^{t} \left[\sum_{\ell=1}^{n} \beta_\ell(\tau)\right] \varpi(\tau)\Delta\tau,\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}.$$

Now by Lemma 2.5, we get

$$\vartheta(t) \leq \varpi(t) \leq (\|\varphi\|_{\mathcal{B}} + b^*) \exp\left\{\int_{t_0}^{t} \left[\sum_{\ell=1}^{n} \beta_\ell(\tau)\right] \Delta\tau\right\},\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}.$$

Thus,

$$\vartheta(t) \leq \mathfrak{M},\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}.$$

Next, we show that

$$\mathfrak{m} \leq \vartheta(t),\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}. \tag{6}$$

To prove this claim, we show that for any $\lambda < 1$, the following inequality holds

$$\vartheta(t) > \lambda\mathfrak{m},\ t \in [t_0, \eta(\varphi))_{\mathbb{T}}. \tag{7}$$

By way of contradiction, assume that (7) does not hold. Then, there exists $t^* \in [t_0, \eta(\varphi))_{\mathbb{T}}$ such that

$$\vartheta(t^*) \leq \lambda\mathfrak{m},\ \vartheta(t) > \lambda,\ t \in [t_0 - \varrho, t^*)_{\mathbb{T}}.$$

Therefore, there must be a positive constant $\mu \leq 1$ such that

$$\vartheta(t^*) = \lambda\mu\mathfrak{m},\ \vartheta(t) > \lambda\mu,\ t \in [t_0 - \varrho, t^*)_{\mathbb{T}}.$$

Since $\lambda\mu < 1$, it follows that

$$\begin{aligned}
0 \geq \vartheta^{\Delta}(t^*) &= -a(t^*)\vartheta(t^*) + b(t^*)e^{-\vartheta(t^*)} + \sum_{\ell=1}^{n} \beta_\ell(t^*)\vartheta(t^* - h_\ell(t^*))e^{-\gamma_\ell(t^*)\vartheta(t^* - g_\ell(t^*))} \\
&\geq -a^+\lambda\mu\mathfrak{m} + b^-e^{-\mathfrak{M}} + \sum_{\ell=1}^{n} \beta_\ell^-\lambda\mu e^{-\gamma_\ell^+\mathfrak{M}} \geq b^-e^{-\mathfrak{M}} - \lambda\mu\left[a^+\mathfrak{m} - \sum_{\ell=1}^{n} \beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}}\right] \\
&\geq b^-e^{-\mathfrak{M}} - \left[a^+\mathfrak{m} - \sum_{\ell=1}^{n} \beta_\ell^- e^{-\gamma_\ell^+\mathfrak{M}}\right] > 0.
\end{aligned}$$

Which is a contradiction and hence (7) holds. Letting $\lambda \to 1$, we get (6). Similar to the proof of Theorem 2.3.1 in [11], we can obtain that $\eta(\varphi) = +\infty$. Therefore,

$$\mathfrak{m} \leq \vartheta(t) \leq \mathfrak{M},\ t \in [t_0, +\infty)_{\mathbb{T}}.$$

□

For $\varpi \in \mathcal{B}$, consider the equation

$$\vartheta^{\Delta}(t) = -a(t)\vartheta(t) + b(t)e^{-\varpi(t)} + \sum_{\ell=1}^{n} \beta_\ell(t)\varpi(t - h_\ell(t))e^{-\gamma_\ell(t)\varpi(t - g_\ell(t))}. \tag{8}$$

Since $\inf_{t\in\mathbb{T}} a(t) = a^- > 0$, then from Lemma 2.10 the linear equation $\vartheta^{\Delta}(t) = -a(t)\vartheta(t)$ admits exponential dichotomy on $\mathbb{T}$. Hence, by Lemma 2.8, the equation (8) has exactly one almost periodic solution,

$$\vartheta_\varpi(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(\tau))\left[b(\tau)e^{-\varpi(\tau)} + \sum_{\ell=1}^{n} \beta_\ell(\tau)\varpi(\tau - h_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau - g_\ell(\tau))}\right]\Delta\tau.$$

Define the operator $\aleph : \mathcal{B} \to \mathcal{B}$,

$$(\aleph\varpi)(t) = \int_{-\infty}^{t} e_{-a}(t, \sigma(\tau))\Big[b(\tau)e^{-\varpi(\tau)} + \sum_{\ell=1}^{n} \beta_\ell(\tau)\varpi(\tau - h_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Big]\Delta\tau.$$

It is clear that, $\varpi(t)$ is the almost periodic solution of equation (3) if and only if ϖ is the fixed point of the operator $\aleph$.

For convenience, we take $M = \max\Big\{\sum_{\ell=1}^{n} \beta_\ell^+\|\varphi\|_{\mathcal{B}}e^+,\, a^+\mathfrak{M}\Big\}$,

Theorem 3.2. *Suppose that the hypothesis of Theorem 3.1 satisfied. Then equation* (3) *has a unique almost periodic positive solution.*

Proof. It is clear from the Theorem 3.1 that $\aleph$ is self mapping on Ξ, where

$$\Xi = \big\{\varpi(t) \in \mathcal{B} : \mathfrak{m} \le \varpi(t) \le \mathfrak{M},\, t \in \mathbb{T}\big\}.$$

Next, we prove that $\aleph$ is a contraction mapping on Ξ. For $\vartheta, \varpi \in \Xi$, consider

$$\begin{aligned}
&\|\aleph\vartheta - \aleph\varpi\|_{\mathcal{B}} = \sup_{t\in\mathbb{T}} |(\aleph\vartheta)(t) - (\aleph\varpi)(t)| \\
&\le \sup_{t\in\mathbb{T}}\Bigg\{\int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[-b(\tau)\Big(e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\Big) \\
&\quad + \sum_{\ell=1}^{n}\beta_\ell(\tau)\Big(\big[\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\big] \\
&\quad + \int_{\tau-g_\ell(\tau)}^{\tau-h_\ell(\tau)} \vartheta^\Delta(s)e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))}\Delta s - \int_{\tau-g_\ell(\tau)}^{\tau-h_\ell(\tau)} \varpi^\Delta(s)e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Delta s\Big)\Big]\Delta\tau\Bigg\}
\end{aligned}$$

From Theorem 3.1, we note that

$$\vartheta^\Delta(t) \le \sum_{\ell=1}^{n}\beta_\ell(t)\varpi(t) \le \sum_{\ell=1}^{n}\beta_\ell(t)\|\varphi\|_{\mathcal{B}}\exp\Bigg\{\int_{t_0}^{t}\sum_{\ell=1}^{n}\beta_\ell(s)\Delta s\Bigg\} \le \sum_{\ell=1}^{n}\beta_\ell^+\|\varphi\|_{\mathcal{B}}e^+,$$

and $\varpi^\Delta(t) \ge -a(t)\varpi(t) \ge -a^+\mathfrak{M}$. Therefore,

$$\begin{aligned}
&\|\aleph\vartheta - \aleph\varpi\|_{\mathcal{B}} \le \sup_{t\in\mathbb{T}}\int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[b^+\Big|e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\Big| \\
&\quad + \sum_{\ell=1}^{n}\beta_\ell^+\Big|\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Big| \\
&+ \sum_{\ell=1}^{n}\beta_\ell^+\big[g_\ell(\tau) - h_\ell(\tau)\big]\Big|\sum_{\ell=1}^{n}\beta_\ell^+\|\varphi\|_{\mathcal{B}}e^+e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - (d^+ + a^+\mathfrak{M})e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Big|\Big]\Delta\tau \\
&\le \sup_{t\in\mathbb{T}}\int_{-\infty}^{t} e_{-a}(t,\sigma(\tau))\Big[b^+\Big|e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\Big| \\
&\quad + \sum_{\ell=1}^{n}\beta_\ell^+\Big|\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Big| \\
&\quad + \sum_{\ell=1}^{n}\beta_\ell^+\big[g_\ell(\tau) - h_\ell(\tau)\big]M\Big|e^{-\gamma_\ell(\tau)\vartheta(\tau-g_\ell(\tau))} - e^{-\gamma_\ell(\tau)\varpi(\tau-g_\ell(\tau))}\Big|\Big]\Delta\tau.
\end{aligned}$$

By mean value theorem, we have $\left|e^{-\vartheta(\tau)} - e^{-\varpi(\tau)}\right| \leq e^{-\xi_1}\left|\vartheta(\tau) - \varpi(\tau)\right|$ where ξ_1 lies between $\vartheta(\tau)$ and $\varpi(\tau)$,

$$\begin{aligned}&\left|\vartheta(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\vartheta(\tau - g_\ell(\tau))} - \varpi(\tau - g_\ell(\tau))e^{-\gamma_\ell(\tau)\varpi(\tau - g_\ell(\tau))}\right|\\ &\qquad\qquad \leq (1 - \gamma_\ell\xi_2)e^{-\gamma_\ell^-\xi_2}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right|,\end{aligned}$$

where ξ_2 lies between $\vartheta(\tau - g_\ell(\tau))$ and $\varpi(\tau - g_\ell(\tau))$, and

$$\left|e^{-\gamma_\ell(\tau)\vartheta(\tau - g_\ell(\tau))} - e^{-\gamma_\ell(\tau)\varpi(\tau - g_\ell(\tau))}\right| \leq \gamma_\ell^+ e^{-\gamma_\ell^-\xi_3}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right|,$$

where ξ_3 lies between $\vartheta(\tau - g_\ell(\tau))$ and $\varpi(\tau - g_\ell(\tau))$.
Hence,

$$\begin{aligned}\|\aleph\vartheta - \aleph\varpi\|_{\mathcal{B}} &\leq \sup_{t\in\mathbb{T}} \int_{-\infty}^t e_{-a}(t, \sigma(\tau))\Big[b^+ e^{-\xi_1}\left|\vartheta(\tau) - \varpi(\tau)\right|\\ &+ \sum_{\ell=1}^n \beta_\ell^+(1 - \gamma_\ell\xi_2)e^{-\gamma_\ell^-\xi_2}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right|\\ &+ M\sum_{\ell=1}^n \beta_\ell^+\left[g_\ell(\tau) - h_\ell(\tau)\right]\gamma_\ell^+ e^{-\gamma_\ell^-\xi_3}\left|\vartheta(\tau - g_\ell(\tau)) - \varpi(\tau - g_\ell(\tau))\right|\Big]\Delta\tau\\ &\leq \sup_{t\in\mathbb{T}} \int_{-\infty}^t e_{-a}(t, \sigma(\tau))\Big[b^+ e^{-\xi_1} + \sum_{\ell=1}^n \beta_\ell^+(1 - \gamma_\ell^-\xi_2)e^{-\gamma_\ell^-\xi_2}\\ &\qquad + M\sum_{\ell=1}^n \beta_\ell^+ g_\ell^+\gamma_\ell^+ e^{-\gamma_\ell^-\xi_3}\Big]\Delta\tau\, \|\vartheta - \varpi\|_{\mathcal{B}}\\ &\leq \frac{1}{a^-}\Big[b^+ e^{-\xi_1} + \sum_{\ell=1}^n \beta_\ell^+(1 - \gamma_\ell^-\xi_2)e^{-\gamma_\ell^-\xi_2} + M\sum_{\ell=1}^n \beta_\ell^+ g_\ell^+\gamma_\ell^+ e^{-\gamma_\ell^-\xi_3}\Big]\|\vartheta - \varpi\|_{\mathcal{B}}\end{aligned}$$

Since $\frac{1}{a^-}\Big[b^+ e^{-\xi_1} + \sum_{\ell=1}^n \beta_\ell^+(1 - \gamma_\ell^-\xi_2)e^{-\gamma_\ell^-\xi_2} + M\sum_{\ell=1}^n \beta_\ell^+ g_\ell^+\gamma_\ell^+ e^{-\gamma_\ell^-\xi_3}\Big] < 1$, it follows that $\aleph$ is a contraction mapping. Thus, by the contraction mapping fixed point theorem, the operator $\aleph$ has a unique fixed point ϑ^* in Ξ. This implies that the equation (3) has a unique almost periodic positive solution $\vartheta^*(t)$ and $\mathfrak{m} \leq \vartheta^*(t) \leq \mathfrak{M}$. □

For convenience, we take

$$\begin{aligned}\Gamma = 2\Big[b^- - \sum_{\ell=1}^n \beta_\ell^+ - \mathfrak{M}\sum_{\ell=1}^n \beta_\ell^+\gamma_\ell^- - M\sum_{\ell=1}^n \beta_\ell^+ g_\ell^+\gamma_\ell^+\Big]&\\ \times\Big[b^+ + \sum_{\ell=1}^n \beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^n \beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^n \beta_\ell^+ g_\ell^+\gamma_\ell^+\Big]^{-2}&.\end{aligned}$$

Theorem 3.3. *Suppose that the hypothesis of the Theorem 3.2 is satisfied and for any* $t_0 \in [-\varrho^*, +\infty)_{\mathbb{T}}$,

$$\int_{t_0}^t (\Gamma - \mu(\tau))\Delta\tau \to +\infty \text{ as } t \to +\infty.$$

Then equation (3) *has unique globally asymptotically stable almost periodic positive solution.*

Proof. By Theorem 3.2, we know that (3) has a unique almost periodic positive solution $\vartheta^*(t)$, and $\mathfrak{m} \leq \vartheta^*(t) \leq \mathfrak{M}$. Suppose $\vartheta(t)$ is any arbitrary solution of (8) with initial function $\varphi(t) > 0$, $t \in [\varrho^*, 0]_{\mathbb{T}}$. Now we prove that $\vartheta^*(t)$ is globally asymptotically stable.

Let $\varpi(t) = \vartheta(t) - \vartheta^*(t)$ and define $\mathcal{V}(\varpi) = \varpi^2$. Then, we have

$$\begin{aligned}
\mathcal{V}^{\Delta}(\varpi) =& 2\varpi(t)\varpi^{\Delta}(t) + \mu(t)(\varpi^{\Delta}(t))^2 \\
=& 2\varpi\Big[- a(t)\big(\vartheta(t) - \vartheta^*(t)\big) + b(t)\left(e^{-\vartheta(t)} - e^{-\vartheta^*(t)}\right) \\
&+ \sum_{\ell=1}^{n} \beta_\ell(t)\left(\vartheta(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))} - \vartheta^*(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta^*(t-g_\ell(t))}\right)\Big] \\
&+ \mu(t)\Big[- a(t)\big(\vartheta(t) - \vartheta^*(t)\big) + b(t)\left(e^{-\vartheta(t)} - e^{-\vartheta^*(t)}\right) \\
&+ \sum_{\ell=1}^{n} \beta_\ell(t)\left(\vartheta(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta(t-g_\ell(t))} - \vartheta^*(t-h_\ell(t))e^{-\gamma_\ell(t)\vartheta^*(t-g_\ell(t))}\right)\Big]^2
\end{aligned}$$

Similar argument employed in Theorem 3.2 yields,

$$\begin{aligned}
\mathcal{V}^{\Delta}(\varpi) =& 2\left[-b^- + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]\|\varpi\|_{\mathcal{B}} \\
&+ \mu(t)\left[b^+ + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]^2\|\varpi\|_{\mathcal{B}} \\
=& -\left[b^+ + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]^2\Big[\Gamma - \mu(t)\Big]\|\varpi\|_{\mathcal{B}}.
\end{aligned}$$

Let $\Omega(\vartheta) = \left[b^+ + \sum_{\ell=1}^{n}\beta_\ell^+ + \mathfrak{M}\sum_{\ell=1}^{n}\beta_\ell^+\gamma_\ell^- + M\sum_{\ell=1}^{n}\beta_\ell^+ g_\ell^+\gamma_\ell^+\right]^2\vartheta^2$, then

$$\mathcal{V}^{\Delta}(\varpi(t)) \leq -\Big[\Gamma - \mu(t)\Big]\Omega(\|\vartheta\|_{\mathcal{B}}).$$

Integrating from t_0 to t, we obtain

$$\mathcal{V}(\varpi(t)) \leq \mathcal{V}(\varpi(t_0)) - \int_{t_0}^{t}\Big[\Gamma - \mu(\tau)\Big]\Omega(\|\vartheta\|_{\mathcal{B}})\Delta\tau.$$

So, we get

$$\int_{t_0}^{t}\Big[\Gamma - \mu(\tau)\Big]\Omega(\|\vartheta\|_{\mathcal{B}})\Delta\tau \leq \mathcal{V}(\varpi(t_0)) - \mathcal{V}(\varpi(t)) < \mathcal{V}(\varpi(t_0)) < +\infty.$$

Since

$$\int_{t_0}^{t}(\Gamma - \mu(\tau))\Delta\tau \to +\infty \text{ as } t \to +\infty,$$

it follows that

$$\Omega(\|\vartheta\|_{\mathcal{B}}) \to 0, \ i.e., \ \|\vartheta(t) - \vartheta^*(t)\|_{\mathcal{B}} \to 0.$$

Hence, $\vartheta^*(t)$ is globally asymptotically stable. $\square$

4. Examples

Example 4.1. *Consider following nonlinear density dependent mortality Nicholson's blowflies model for* $\mathbb{T} = \mathbb{R}$.

$$\left.\begin{aligned} \vartheta^{\Delta}(t) =& -(2+\sin(2t))\vartheta(t) + |\cos(t)|e^{-\vartheta(t)} \\ & + |\sin(t)|\vartheta\left(t - 2e^{\cos(\sqrt{2}t)}\right) e^{-(2+\sin(t))\vartheta\left(t-(4+2e^{\sin(\sqrt{2}t)})\right)}, \\ \vartheta(0) =& \, 0.1. \end{aligned}\right\} \tag{9}$$

It is clear that (9) satisfies all the assumptions of Theorem 3.3. Therefore, equation (9) has a unique almost periodic positive solution $\vartheta^*(t)$ which is globally asymptotically stable. The numerical simulations in Fig. 1 strongly support the conclusion.

Example 4.2. *Consider following nonlinear density dependent mortality Nicholson's blowflies model for* $\mathbb{T} = \mathbb{Z}^+$.

$$\left.\begin{aligned} \vartheta(t+1) =& \, \vartheta(t) - (1+\cos t)\vartheta(t) + |\sin(t)|e^{-\vartheta(t)} \\ & + |\sin(t)|\vartheta\left(t - e^{\sin(\sqrt{2}t)}\right) e^{-(2+\sin(t))\vartheta\left(t-(4+2e^{\cos(\sqrt{2}t)})\right)}, \\ \vartheta(0) =& \, 0.05. \end{aligned}\right\} \tag{10}$$

It is clear that (10) satisfies all the assumptions of Theorem 3.3. Therefore, equation (10) has a unique almost periodic positive solution $\vartheta^*(t)$ which is globally asymptotically stable. The numerical simulations in Fig. 2 strongly support the conclusion.

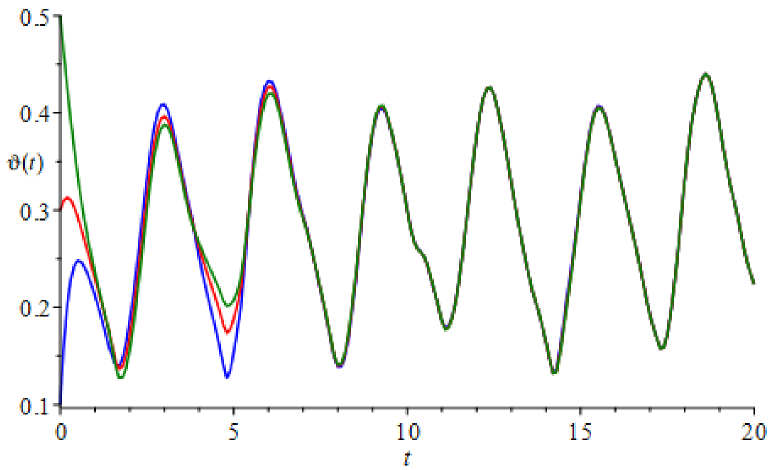

Figure 1. Numerical solution $\vartheta(t)$ of equation (9) for initial value $\varphi(\tau) = 0.1, 0.3, 0.5 \, \tau \in [-(4+2e), 0]$.

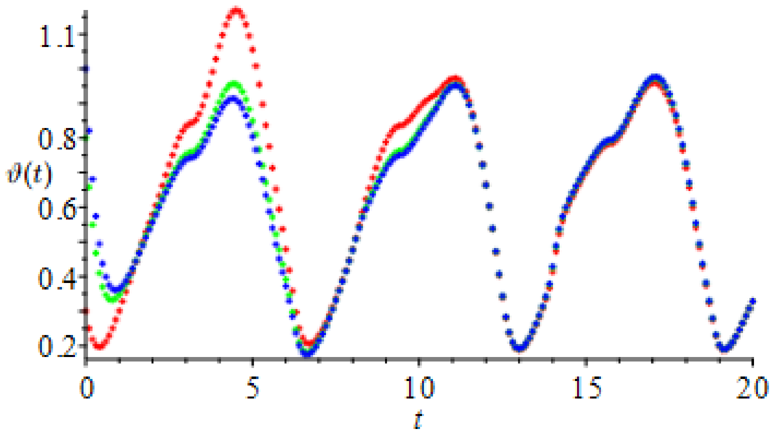

FIGURE 2. Numerical solution $\vartheta(t)$ of equation (9) for initial value $\varphi(\tau) = 0.3, 0.8, 1\,\tau \in [-(4+2e), 0]$.

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