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GENERALIZED CONE $b-$RANDOM METRIC SPACE WITH SOME RANDOM FIXED POINT THEOREMS

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Abstract. In this paper, we introduce generalized cone $b-$random metric space and prove some random fixed point theorems for mappings satisfying various contractive type conditions. Also some stochastic fixed point theorems for integral type contractive mappings have been proved in the above framework.

Keywords: fixed point, random fixed point, generalized cone $b-$random metric space, generalized subadditive cone integrable function.

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1. Introduction

Fixed point theory plays an important role in different branches of Mathematics for its several applications therein. Since 20th century, after the initiation of Banach contraction principle [3] several deterministic fixed point theorems have been proved by many mathematicians. The study of random operator equations employed by the power of functional analysis was first started by the Prague School of Probabilists led by A. *Š*pa*č*ek [22] and Han*š* ([6]-[7]). After that, considerable attention had been paid to the study of random fixed point theory because of its importance in probabilistic functional analysis and wide and generous applications in probabilistic models. Almost all random fixed point theorems are stochastic generalization of their classical deterministic counterparts. In search of its application one can see that random fixed point theorem was applied by W.J. Padgett [14] to prove the existence of a random solution of random nonlinear integral equation in a setting of Banach space. Following this literature Saha et al. ([15]-[19]) had succeeded to develop the applications of random fixed point theory for different types of random mappings, contractive in nature.

In 2007, Huang and Zhang [8] instigated the concept of cone metric spaces and established some fixed point theorems in the setting of normal cone metric spaces. After that several mathematicians proved various fixed point theorems in the setting of cone metric spaces. Khojasteh et al. [11] proved fixed point theorems of mappings satisfying integral

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type contractive conditions in such spaces. Recently Kadelburg et al. [10] proved some common fixed point theorems in the setting of topological vector space-valued cone metric space which is a generalization of cone metric space. In 2015, George et al. bundled together the concepts of $b-$metric space [2], rectangular metric space [4], cone metric space [8], cone-rectangular metric space [1] and cone $b-$metric space [9] and initiated the concept of generalized cone $b-$metric spaces [5] in a unified way.

Recently Mehta et al. [13] introduced cone random metric space and proved some random fixed point theorems on it. Inspired by this literature many mathematicians were attracted to prove several fixed point theorems in the setting of cone random metric space. In 2016, Saluja et al. [20] proved some common random fixed point theorems in this setting.

In this paper, we introduce generalized cone $b-$random metric space as a generalization of cone random metric space and prove some random fixed point theorems for a class of generalized mappings and integral type mappings, both are contractive in nature. Supporting example is also given in connection with this space.

2. Preliminaries

Let E be a Hausdorff topological vector space (shortly *tvs*) with null vector θ. A proper nonempty and closed subset P of E is called a cone if $P+P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \theta$. The cone P is called solid if P has a nonempty interior.

Each cone P induces a partial order $\preceq$ on E by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$. The pair (E, P) is an ordered topological vector space.

Example 2.1. [10] *Let $E = C^1_{\mathbb{R}}[0,1]$ with $||x|| = ||x||_\infty + ||x'||_\infty$ and let $P = \{x \in E : x(t) \geq 0 on\ [0,1]\}$. This cone is solid but is not normal. Consider for example, $x_n(t) = \frac{(1-\sin nt)}{(n+2)}$ and $y_n(t) = \frac{(1+\sin nt)}{(n+2)}$. Since $||x_n|| = ||y_n|| = 1$ and $||x_n + y_n|| = \frac{2}{(n+2)} \to 0$, it follows that P is a nonnormal cone.*

Now consider the space $E = C^1_{\mathbb{R}}[0,1]$ endowed with the strongest locally convex topology t^*. Then P is also t^*-solid, but not t^*-normal. Indeed, if it were normal then the space (E, t^*) would be normed, which is impossible since an infinite-dimensional space with the strongest locally convex topology cannot be metrizable.

Definition 2.1. [10] *Let X be a nonempty set and (E, P) an ordered tvs. A function $d : X^2 \to E$ is called a tvs$-$cone metric and (X, d) is called a tvs$-$cone metric space if the following conditions hold:*

(C1)$\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(C2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(C3) $d(x,z) \preceq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Definition 2.2. [10] *Let $\{x_n\} \subset X$ and $x \in X$. Then*

(i) $\{x_n\}$ tvs$-$cone converges to x if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$; we denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is a tvs$-$cone Cauchy sequence if for every $c \in E$ with $\theta \ll c$ there exists a natural number n_0 such that $d(x_m, x_n) \ll c$ for all $m, n > n_0$.

(iii) (X, d) is tvs$-$cone complete if every tvs$-$Cauchy sequence is tvs$-$convergent in X.

Now we state some basic properties of a real *tvs* E with a solid cone P and a *tvs*$-$cone metric space (X, d) over it.

Lemma 2.1. [10] *(a) Let $\theta \preceq x_n \to \theta$ in (E,P), and let $\theta \ll c$. Then there exists n_0 such that $x_n \ll c$ for each $n > n_0$.*

(b) It can happen that $\theta \preceq x_n \ll c$ for each $n > n_0$, but $x_n \nrightarrow \theta$ in (E,P).

(c) It can happen that $x_n \to x$, $y_n \to y$ in the tvs−cone metric d, but that $d(x_n, y_n) \nrightarrow d(x,y)$ in (E,P). In particular, it can happen that $x_n \to x$ in d but that $d(x_n, x) \nrightarrow \theta$ (which is impossible if the cone is normal).

(d) $\theta \preceq u \ll c$ for each $c \in intP$ implies that $u = \theta$.

(e) $x_n \to x \wedge y_n \to y$ (in the tvs−cone metric) implies that $x = y$.

(f) Each tvs−cone metric space is Hausdorff in the sense that for arbitrary distinct points x and y there exists disjoint neighbourhoods in the topology τ_c having the local base formed by the sets of the form $K_c(x) = \{z \in X : d(x,z) \ll c\}$, $c \in intP$.

Lemma 2.2. [10] *(a) If $u \preceq v$ and $v \ll w$, then $u \ll w$.*

(b) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(c) If $u \ll v$ and $v \ll w$, then $u \ll w$.

(d) Let $x \in X$, $\{x_n\}$ and $\{b_n\}$ be two sequences in X and E, respectively, $\theta \ll c$, and $\theta \preceq d(x_n, x) \preceq b_n$ for all $n \in \mathbb{N}$. If $b_n \to \theta$, then there exists a natural number n_0 such that $d(x_n, x) \ll c$ for all $n \geq n_0$.

Now let E be a real Banach space with the null vector $0(\equiv \theta)$ and $P \subset E$ be a solid cone. Also let $\preceq$ be the ordering induced by P.

Definition 2.3. [5] *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:*

(GCbM1) $0 \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(GCbM2) $d(x,y) = d(y,x)$ for all $x, y \in X$,

(GCbM3) there exists a real number $s \geq 1$ such that $d(x,y) \preceq s[d(x,u)+d(u,v)+d(v,y)]$ for all $x, y \in X$ and for all distinct points $u, v \in X \backslash \{x,y\}$.

Then d is called a generalized cone $b-$metric on X and (X,d) is called a generalized cone $b-$metric space (in short $GCbMS$) with coefficient s.

It is to be noted that every cone $b-$metric space with coefficient s is a $GCbMS$ with coefficient s^2 and every cone rectangular metric space is also $GCbMS$ but the converse is not true in general. The Examples 2.5 and 2.6 respectively in [5] support our statement.

Definition 2.4. [5] *Let (X,d) be a $GCbMS$. The sequence $\{x_n\}$ in X is said to be:*

(a) a convergent sequence if for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d(x_n, x) \ll c$ for some $x \in X$. We say that the sequence $\{x_n\}$ converges to x and we denote it by $\lim_{n\to\infty} x_n = x$;

(b) a Cauchy sequence if for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$, for all $m, n \geq n_0$.

(c) The $GCbMS$ (X,d) is said to be complete if every Cauchy sequence in X is convergent in X.

For any $x \in X$ we define the open ball with center x and radius $r > 0 (r \in E)$ in a $GCbMS$ by

$$B_r(x) = \{y \in X : d(x,y) < r\}$$

The open balls in $GCbMS$ are not necessarily open (see Remark below). Let U be the collection of all subsets A of X satisfying the condition that for each $x \in A$ there exists $r > 0$ such that $B_r(x) \subset A$. Then U defines a topology for the $GCbMS$ (X,d) which is not necessarily Hausdorff (See Remark below).

In Example 2.5 (See [5]) one can see the following:

Remark 2.1. [5] *(i)* $B_{\frac{1}{2}}(\frac{1}{2}) = \{2, 3, \frac{1}{2}\}$ *and there does not exist any open ball with center* 2 *and contained in* $B_{\frac{1}{2}}(\frac{1}{2})$. *So* $B_{\frac{1}{2}}(\frac{1}{2})$ *is not an open set.*

(ii) The sequence $\{\frac{1}{n}\}$ *converges to* 2 *and* 3 *in* $GCbMS$ *and so the limit is not unique. Also,* $d(\frac{1}{n}, \frac{1}{n+1}) = (2, 2) \nrightarrow (0, 0)$ *as* $n \to \infty$*; therefore* $\{\frac{1}{n}\}$ *is not a Cauchy sequence in* $GCbMS$*. Thus in a* $GCbMS$ *not every convergent sequence is necessarily a Cauchy sequence.*

(iii) There does not exists $r_1, r_2 > 0$ *such that* $B_{r_1}(2) \cap B_{r_2}(3) = \emptyset$ *and so* (X, d) *is not Hausdorff.*

In the following we suppose that P is a normal cone in the real Banach space E.

Example 2.2. [12] *Let* $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ *and* $d : X \times X \to E$ *defined by* $d(x, y) = (|x - y|, \alpha|x - y|)$, *where* $\alpha \geq 0$ *is a constant. Then* (X, d) *is a cone metric space with the normal constant of* P *is* $k = 1$.

Definition 2.5. [11] *Let* $a, b \in E$ *and* $a < b$. *We define*
$[a, b] = \{x \in E : x = tb + (1-t)a,$ *for some* $t \in [0, 1]\}$, $[a, b) = \{x \in E : x = tb + (1-t)a,$ *for some* $t \in [0, 1)\}$

Definition 2.6. [11] *The set* $\{a = x_0, x_1, x_2, ..., x_n = b\}$ *is called a partition for* $[a, b]$ *if and only if the sets* $\{[x_{i-1}, x_i)\}_{i=1}^n$ *are pairwise disjoint and* $[a, b] = \{\cup_{i=1}^n [x_{i-1}, x_i)\} \cup \{b\}$.

Definition 2.7. [11] *For each partition* Q *of* $[a, b]$ *and each increasing function* $\phi : [a, b] \to P$, *we define cone lower summation and cone upper summation as*
$L_n^{Con}(\phi, Q) = \sum_{i=0}^{n-1} \phi(x_i)||x_i - x_{i+1}||$, $\mathcal{U}_n^{Con}(\phi, Q) = \sum_{i=0}^{n-1} \phi(x_{i+1})||x_i - x_{i+1}||$, *respectively.*

Definition 2.8. [11] $\phi : [a, b] \to P$ *is called an integrable function on* $[a, b]$ *with respect to cone* P *or to simplicity, Cone integrable function, if and only if for all partition* Q *of* $[a, b]$
$\lim_{n\to\infty} L_n^{Con}(\phi, Q) = S^{Con} = \lim_{n\to\infty} \mathcal{U}_n^{Con}(\phi, Q)$, *where* S^{Con} *must be unique. We show the common value* S^{Con} *by*
$\int_a^b \phi(x) d_P(x)$ *or to simplicity* $\int_a^b \phi d_p$. *We denote the set of all cone integrable function* $\phi : [a, b] \to P$ *by* $\mathcal{L}^1([a, b], P)$.

Lemma 2.3. [11] *(1) If* $[a, b] \subset [a, c]$, *then* $\int_a^b f d_p \preceq \int_a^c f d_p$, *for* $f \in \mathcal{L}^1(X, P)$. *(2)* $\int_a^b (\alpha f + \beta g) d_p = \alpha \int_a^b f d_p + \beta \int_a^b g d_p$, *for* $f, g \in \mathcal{L}^1(X, P)$ *and* $\alpha, \beta \in \mathbb{R}$.

Definition 2.9. [21] *Let* X *be a nonempty set. Suppose that the mapping* $d : X \times X \to [0, +\infty)$, *satisfies:*
1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$
3. $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ *for all* $x, y \in X$ *and for all distinct points* $w, z \in X \backslash \{x, y\}$ *(rectangular property)*

Then d *is called a generalized metric and* (X, d) *is a generalized metric space (or shortly g.m.s).*

Theorem 2.1. [21] *Let* (X, d) *be a complete g.m.s,* $c \in (0, 1)$, *and let* $f : X \to X$ *be a mapping such that for each* $x, y \in X$,

$$\int_0^{d(fx,fy)} \phi(t)dt \leq c \int_0^{d(x,y)} \phi(t)dt,$$

where $\phi : [0, +\infty) \to [0, +\infty)$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* $[0, +\infty)$, *nonnegative, and such that*

$$\forall \epsilon > 0, \int_0^{\epsilon} \phi(t)dt > 0$$

Then, f *admits a unique fixed point* $a \in X$ *such that for each* $x \in X$, $\lim_{n\to\infty} f^n x = a$.

Definition 2.10. [13] *(Measurable function) Let* (Ω, Σ) *be a measurable space with* $\Sigma-$*a sigma algebra of subsets of* Ω *and* M *be a nonempty subset of a metric space* $X = (X, d)$. *Let* 2^M *be the family of nonempty subsets of* M *and* $C(M)$ *the family of all nonempty closed subsets of* M. *A mapping* $G : \Omega \to 2^M$ *is called measurable if for each open subset* U *of* M, $G^{-1}(U) \in \Sigma$, *where* $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\}$.

Definition 2.11. [13] *(Measurable selector) A mapping* $\xi : \Omega \to M$ *is called a measurable selector of a measurable mapping* $G : \Omega \to 2^M$ *if* ξ *is measurable and* $\xi(\omega) \in G(\omega)$ *for each* $\omega \in \Omega$.

Definition 2.12. [13] *(Random operator) The mapping* $T : \Omega \times M \to X$ *is said to be a random operator if and only if for each fixed* $x \in M$, *the mapping* $T(., x) : \Omega \to X$ *is measurable.*

Definition 2.13. [13] *(Continuous random operator) A random operator* $T : \Omega \times M \to X$ *is said to be continuous random operator if for each fixed* $x \in M$ *and* $\omega \in \Sigma$, *the mapping* $T(\omega, .) : M \to X$ *is continuous at* x.

Definition 2.14. [13] *(Random fixed point) A measurable mapping* $\xi : \Omega \to M$ *is a random fixed point of a random operator* $T : \Omega \times M \to X$ *if and only if* $T(\omega, \xi(\omega)) = \xi(\omega)$ *for each* $\omega \in \Omega$.

Definition 2.15. [13] *(Cone Random Metric Space) Let* M *be a nonempty set and let the mapping* $d : \Omega \times M \to P$, *where* P *is a cone,* $\omega \in \Omega$ *be a selector, satisfy the following conditions:*

(i) $d(x(\omega), y(\omega)) \succeq 0$ *and* $d(x(\omega), y(\omega)) = 0$ *if and only if* $x(\omega) = y(\omega)$ *for all* $x(\omega), y(\omega) \in \Omega \times M$,

(ii) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$ *for all* $x, y \in M$, $\omega \in \Omega$ *and* $x(\omega), y(\omega) \in \Omega \times M$,

(iii) $d(x(\omega), y(\omega)) \preceq d(x(\omega), z(\omega)) + d(z(\omega), y(\omega))$ *for all* $x, y, z \in M$ *and* $\omega \in \Omega$ *be a selector,*

(iv) for any $x, y \in M$, $\omega \in \Omega$, $d(x(\omega), y(\omega))$ *is non-increasing and left continuous.*

Then d *is called cone random metric on* M *and* (M, d) *is called a cone random metric space.*

3. Main Results

Definition 3.1. *Let* X *be a nonempty set and* (Ω, Σ) *be a measurable space with* $\Sigma-$*a sigma algebra of subsets of* Ω. *Let the mapping* $d : (\Omega \times X)^2 \to P$, *where* P *be a solid cone in topological vector space* (E, τ) *and* $\omega \in \Omega$ *be a selector, satisfies the following conditions:*

(i) $d(x(\omega), y(\omega)) \succeq \theta$ *and* $d(x(\omega), y(\omega)) = \theta$ *if and only if* $x(\omega) = y(\omega)$ *for all* $x(\omega), y(\omega) \in \Omega \times X$,

(ii) $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$ *for all* $x, y \in X$, $\omega \in \Omega$ *and* $x(\omega), y(\omega) \in \Omega \times X$,

(iii) $d(x(\omega), y(\omega)) \preceq s[d(x(\omega), u(\omega)) + d(u(\omega), v(\omega)) + d(v(\omega), y(\omega))]$ *for all* $x(\omega), y(\omega) \in \Omega \times X$ *and for all distinct points* $u(\omega), v(\omega) \in \Omega \times X \backslash \{x(\omega), y(\omega)\}$, *where the coefficient* $s \geq 1$,

(iv) for any $x, y \in X$, $\omega \in \Omega$, $d(x(\omega), y(\omega))$ is non-increasing and left continuous.
Then d is called generalized cone $b-$random metric on X and (X, d) is called a generalized cone $b-$random metric space (In short $GCbRMS$).

Definition 3.2. *Let (X, d) be a $GCbRMS$. The sequence $\{x_n(\omega)\}$ in X is said to be:*

(a) convergent and converges to some $x(\omega) \in \Omega \times X$ if for every $c \in E$ with $c \gg \theta$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d(x_n(\omega), x(\omega)) \ll c$. We denote this by $\lim_{n\to\infty} x_n(\omega) = x(\omega)$.

(b) a Cauchy sequence if for all $c \in E$ with $c \gg \theta$, there is $N \in \mathbb{N}$ such that $d(x_m(\omega), x_n(\omega)) \ll c$ for all $m, n \geq N$.

(c) The $GCbRMS$ (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Theorem 3.1. *Let (X, d) be a complete $GCbRMS$ with respect to cone $P \subset E$ and let M be a nonempty separable closed subset of X. Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega, .) : \Omega \times M \to M$ satisfying the condition:*

$$\begin{aligned} d(T(x(\omega)), T(y(\omega))) &\preceq a(\omega)d(x(\omega), y(\omega)) + b(\omega)d(x(\omega), T(x(\omega))) + \\ &\qquad c(\omega)d(y(\omega), T(y(\omega))), \end{aligned} \tag{1}$$

for all $x, y \in M$, $0 < sa(\omega) + sb(\omega) + c(\omega) < 1$, where $a(\omega), b(\omega) \geq 0$, $0 \leq c(\omega) < \frac{1}{s+1}$ and $\omega \in \Omega$. Then T has a unique random fixed point in X.

Proof. Let $x_0(\omega) \in \Sigma \times M$ and we construct a sequence $\{x_n(\omega)\}$ as follows:

$$x_n(\omega) = T(x_{n-1}(\omega)) = T^n(x_0(\omega)) \tag{2}$$

for all $n = 1, 2, \ldots$. If $x_n(\omega) = x_{n+1}(\omega)$ for some $n \in \mathbb{N} \cup \{0\}$, then clearly $x_n(\omega)$ will be a random fixed point of T. So without loss of generality we can assume that $x_n(\omega) \neq x_{n+1}(\omega)$ for all $n = 0, 1, 2, \ldots$. We can also assume that $x_n(\omega)$ is not a periodic point of T for any $n \in \mathbb{N} \cup \{0\}$. Indeed if $x_n(\omega) = x_{n+k}(\omega)$ for some $n \geq 0$ and $k \geq 2$ then

$$\begin{aligned} d(x_n(\omega), x_{n+1}(\omega)) &= d(x_n(\omega), T(x_n(\omega))) \\ &= d(x_{n+k}(\omega), T(x_{n+k}(\omega))) \\ &= d(x_{n+k}(\omega), x_{n+k+1}(\omega)) \end{aligned} \tag{3}$$

Now from the relation (1) we have,

$$\begin{aligned} d_n &= d(x_n(\omega), x_{n+1}(\omega)) \\ &= d(T(x_{n-1}(\omega)), T(x_n(\omega))) \\ &\preceq a(\omega)d(x_{n-1}(\omega), x_n(\omega)) + b(\omega)d(x_{n-1}(\omega), T(x_{n-1}(\omega))) + \\ &\qquad c(\omega)d(x_n(\omega), T(x_n(\omega))) \\ &= (a(\omega) + b(\omega))d(x_{n-1}(\omega), x_n(\omega)) + c(\omega)d(x_n(\omega), x_{n+1}(\omega)) \\ &= (a(\omega) + b(\omega))d_{n-1} + c(\omega)d_n \end{aligned} \tag{4}$$

Therefore $d_n \preceq \lambda(\omega)d_{n-1}$ that is $d_n \preceq \lambda(\omega)^n d_0$ for all $n \geq 1$, where $\lambda(\omega) = \frac{a(\omega)+b(\omega)}{1-c(\omega)}$ for all $\omega \in \Omega$.

Hence from (3) we get, $d_n = d_{n+k} \preceq \lambda(\omega)^k d_n$. Since $0 \leq \lambda(\omega) < 1$ we get, $-d_n \in P$. Therefore we must have $d_n = \theta$ and then $x_n(\omega)$ will be a random fixed point of T. Thus we can assume that $x_n(\omega) \neq x_m(\omega)$ for all distinct $n, m \in \mathbb{N}$. Again setting $d^*_n =$

$d(x_n(\omega), x_{n+2}(\omega))$ $\forall$ $n = 0, 1, 2...$ we obtain

$$\begin{aligned} d_n^* &= d(x_n(\omega), x_{n+2}(\omega)) \\ &= d(T(x_{n-1}(\omega)), T(x_{n+1}(\omega))) \\ &\preceq a(\omega)d(x_{n-1}(\omega), x_{n+1}(\omega)) + b(\omega)d(x_{n-1}(\omega), T(x_{n-1}(\omega))) + \\ &\qquad\qquad c(\omega)d(x_{n+1}(\omega), T(x_{n+1}(\omega))) \\ &= a(\omega)d(x_{n-1}(\omega), x_{n+1}(\omega)) + b(\omega)d(x_{n-1}(\omega), x_n(\omega)) + \\ &\qquad\qquad c(\omega)d(x_{n+1}(\omega), x_{n+2}(\omega)) \\ &= a(\omega)d_{n-1}^* + b(\omega)d_{n-1} + c(\omega)d_{n+1} \end{aligned} \tag{5}$$

If $b(\omega) = 0 = c(\omega)$ then $d_n^* \preceq a(\omega)^n d_0^*$ for all $n \geq 1$, otherwise for any $n \in \mathbb{N}$

$$\begin{aligned} d_n^* &\preceq a(\omega)d_{n-1}^* + b(\omega)d_{n-1} + c(\omega)d_{n+1} \\ &= a(\omega)d_{n-1}^* + b(\omega)\lambda(\omega)^{n-1}d_0 + c(\omega)\lambda(\omega)^{n+1}d_0 \\ &= a(\omega)d_{n-1}^* + (b(\omega) + c(\omega)\lambda(\omega)^2)\lambda(\omega)^{n-1}d_0 \\ &\preceq a(\omega)[a(\omega)d_{n-2}^* + (b(\omega) + c(\omega)\lambda(\omega)^2)\lambda(\omega)^{n-2}d_0] + (b(\omega) + c(\omega)\lambda(\omega)^2)\lambda(\omega)^{n-1}d_0 \\ &= a(\omega)^2 d_{n-2}^* + (b(\omega) + c(\omega)\lambda(\omega)^2)[\lambda(\omega)^{n-1} + a(\omega)\lambda(\omega)^{n-2}]d_0 \\ &\preceq a(\omega)^n d_0^* + (b(\omega) + c(\omega)\lambda(\omega)^2)[\lambda(\omega)^{n-1} + a(\omega)\lambda(\omega)^{n-2} + ... + a(\omega)^{n-1}]d_0 \\ &= a(\omega)^n d_0^* + (b(\omega) + c(\omega)\lambda(\omega)^2)\frac{\lambda(\omega)^n - a(\omega)^n}{\lambda(\omega) - a(\omega)}d_0 \\ &\preceq a(\omega)^n d_0^* + \mu(\omega)\lambda(\omega)^n d_0, \qquad \mu(\omega) = \frac{(b(\omega) + c(\omega)\lambda(\omega)^2)}{\lambda(\omega) - a(\omega)} \end{aligned} \tag{6}$$

Now here we see that $0 \leq \lambda(\omega) < \frac{1}{s}$ and $\mu(\omega) \leq 1$. We wish to show that $\{x_n(\omega)\}$ is Cauchy and so we consider two special cases for $d(x_n(\omega), x_{n+p}(\omega))$, $p \geq 1$ and for all $n \in \mathbb{N}$.

Case I: If p is odd, say $p = 2m + 1$, $m \geq 0$ then

$$\begin{aligned} d(x_n(\omega), x_{n+2m+1}(\omega)) &\preceq s[d(x_n(\omega), x_{n+1}(\omega)) + d(x_{n+1}(\omega), x_{n+2}(\omega)) + d(x_{n+2}(\omega), x_{n+2m+1}(\omega))] \\ &= s[d_n + d_{n+1}] + sd(x_{n+2}(\omega), x_{n+2m+1}(\omega)) \\ &\quad ... \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + ... + s^m[d_{n+2m-2} + d_{n+2m-1}] + \\ &\quad s^m d_{n+2m} \\ &\preceq s[\lambda(\omega)^n + \lambda(\omega)^{n+1}]d_0 + s^2[\lambda(\omega)^{n+2} + \lambda(\omega)^{n+3}]d_0 + ... + \\ &\quad s^m[\lambda(\omega)^{n+2m-2} + \lambda(\omega)^{n+2m-1}]d_0 + s^m\lambda(\omega)^{n+2m}d_0 \\ &= s(1 + \lambda(\omega))\lambda(\omega)^n[1 + s\lambda(\omega)^2 + s^2\lambda(\omega)^4 + ... + s^{m-1}\lambda(\omega)^{2m-2}]d_0 + \\ &\quad s^m\lambda(\omega)^{n+2m}d_0 \\ &\preceq s(1 + \lambda(\omega))\lambda(\omega)^n[1 + s\lambda(\omega)^2 + s^2\lambda(\omega)^4 + ... + s^m\lambda(\omega)^2 m]d_0 \\ &\preceq \frac{s(1 + \lambda(\omega))}{1 - s\lambda(\omega)^2}\lambda(\omega)^n d_0 \end{aligned} \tag{7}$$

Since $\lambda(\omega)^n \to 0$ as $n \to \infty$ so by property (a) of Lemma 2.1 and property (a) of Lemma 2.2 we can say that for any $c \gg \theta$ there exists $N_1 \in \mathbb{N}$ such that $d(x_n(\omega), x_{n+2m+1}(\omega)) \ll c$ for all $n \geq N_1$ and for all $m \geq 0$.

Case II: Also if $p = 2m$, $m \geq 1$ then

$$\begin{aligned} d(x_n(\omega), x_{n+2m}(\omega)) &\preceq s[d(x_n(\omega), x_{n+1}(\omega)) + d(x_{n+1}(\omega), x_{n+2}(\omega)) + d(x_{n+2}(\omega), x_{n+2m}(\omega))] \\ &= s[d_n + d_{n+1}] + sd(x_{n+2}(\omega), x_{n+2m}(\omega)) \\ &\dots \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + \dots + s^{m-1}[d_{n+2m-4} + d_{n+2m-3}] + \\ &\quad s^{m-1} d^*_{n+2m-2} \end{aligned} \tag{8}$$

Subcase I: If $b(\omega) = c(\omega) = 0$ then from (6) and (8) we get

$$\begin{aligned} d(x_n(\omega), x_{n+2m}(\omega)) &\preceq s[\lambda(\omega)^n + \lambda(\omega)^{n+1}]d_0 + s^2[\lambda(\omega)^{n+2} + \lambda(\omega)^{n+3}]d_0 + \dots + \\ &\quad s^{m-1}[\lambda(\omega)^{n+2m-4} + \lambda(\omega)^{n+2m-3}]d_0 + s^{m-1}a(\omega)^{n+2m-2}d_0^* \\ &= s(1+\lambda(\omega))\lambda(\omega)^n[1 + s\lambda(\omega)^2 + s^2\lambda(\omega)^4 + \dots + s^{m-2}\lambda(\omega)^{2m-4}]d_0 + \\ &\quad s^{m-1}a(\omega)^{n+2m-2}d_0^* \\ &\preceq \frac{s(1+\lambda(\omega))}{1 - s\lambda(\omega)^2}\lambda(\omega)^n d_0 + a(\omega)^n d_0^* \end{aligned} \tag{9}$$

Subcase II: If either $b(\omega) \neq 0$ or $c(\omega) \neq 0$ then from (6) and (8) we have

$$\begin{aligned} d(x_n(\omega), x_{n+2m}(\omega)) &\preceq \frac{s(1+\lambda(\omega))}{1 - s\lambda(\omega)^2}\lambda(\omega)^n d_0 + s^{m-1}[a(\omega)^{n+2m-2}d_0^* + \mu(\omega)\lambda(\omega)^{n+2m-2}d_0] \\ &\preceq \frac{s(1+\lambda(\omega))}{1 - s\lambda(\omega)^2}\lambda(\omega)^n d_0 + [a(\omega)^n d_0^* + \mu(\omega)\lambda(\omega)^n d_0] \\ &= \left[\frac{s(1+\lambda(\omega))}{1 - s\lambda(\omega)^2} + \mu(\omega)\right]\lambda(\omega)^n d_0 + a(\omega)^n d_0^* \end{aligned} \tag{10}$$

Since both $\lambda(\omega)^n \to 0$ and $a(\omega)^n \to 0$ as n tending to infinity then for any $c \gg \theta$ there exists $N_2 \in \mathbb{N}$ such that $d(x_n(\omega), x_{n+2m}(\omega)) \ll c$ for all $n \geq N_2$ and for all $m \geq 1$.
Combining Case I and Case II we see that $\{x_n(\omega)\}$ is Cauchy in $\Omega \times M$. Since X is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$ as $n \to \infty$. Since M is closed then $z(\omega) \in \Omega \times M$. Now we will show that $z(\omega)$ is a random fixed point of T.
For any $n \in \mathbb{N}$ we get

$$\begin{aligned} d(z(\omega), T(z(\omega))) &\preceq s[d(z(\omega), x_n(\omega)) + d(x_n(\omega), x_{n+1}(\omega)) + d(x_{n+1}(\omega), T(z(\omega)))] \\ &\preceq s[d(z(\omega), x_n(\omega)) + d_n + a(\omega)d(z(\omega), x_n(\omega)) + \\ &\quad b(\omega)d(x_n(\omega), x_{n+1}(\omega)) + c(\omega)d(z(\omega), T(z(\omega)))] \end{aligned} \tag{11}$$

This implies $d(z(\omega), T(z(\omega))) \preceq \frac{s(1+a(\omega))}{1-sc(\omega)}d(x_n(\omega), z(\omega)) + \frac{s(1+b(\omega))}{1-sc(\omega)}\lambda(\omega)^n d_0$. Since $x_n(\omega) \to z(\omega)$ and $\lambda(\omega)^n \to 0$ as $n \to \infty$ then for $c \gg \theta$ there exists $n_1, n_2 \in \mathbb{N}$ such that $d(x_n(\omega), z(\omega)) \ll \frac{1-sc(\omega)}{2s(1+a(\omega))}c$ if $n \geq n_1$ and $\lambda(\omega)^n d_0 \ll \frac{1-sc(\omega)}{2s(1+b(\omega))}c$ whenever $n \geq n_2$. Taking $n \geq n_0 = \max\{n_1, n_2\}$ we have $d(z(\omega), T(z(\omega)))$. Since $c \gg \theta$ is arbitrary then $T(z(\omega)) = z(\omega)$. So $z(\omega)$ is a random fixed point of T.
Now let $u(\omega)$ be another random fixed point of T then

$$\begin{aligned} d(z(\omega), u(\omega)) = d(T(z(\omega)), T(u(\omega))) &\preceq a(\omega)d(z(\omega), u(\omega)) + b(\omega)d(z(\omega), T(z(\omega))) + \\ &\quad c(\omega)d(u(\omega), T(u(\omega))) \end{aligned} \tag{12}$$

From (12) we get $d(z(\omega), u(\omega)) \preceq a(\omega)d(z(\omega), u(\omega)) \prec d(z(\omega), u(\omega))$, a contradiction. Hence $z(\omega)$ is the unique random fixed point of T. □

Corollary 3.1. *Let (X,d) be a complete GCbRMS with respect to cone $P \subset E$ and let M be a nonempty separable closed subset of X. Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega,.) : \Omega \times M \to M$ satisfying the condition:*

$$d(T(x(\omega)),T(y(\omega))) \preceq a(\omega)d(x(\omega),y(\omega)) \tag{13}$$

for all $x,y \in M$, $0 < a(\omega) < \frac{1}{s}$ and $\omega \in \Omega$. Then T has a unique random fixed point in X.

Proof. If we put $b(\omega) = 0 = c(\omega)$ in Theorem 3.1 then we get our desired result. □

Corollary 3.2. *Let (X,d) be a complete GCbRMS with respect to cone $P \subset E$ and let M be a nonempty separable closed subset of X. Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega,.) : \Omega \times M \to M$ satisfying the condition:*

$$d(T(x(\omega)),T(y(\omega))) \preceq b(\omega)[d(x(\omega),T(x(\omega))) + d(y(\omega),T(y(\omega)))] \tag{14}$$

for all $x,y \in M$, $0 < b(\omega) < \frac{1}{s+1}$ and $\omega \in \Omega$. Then T has a unique random fixed point in X.

Proof. If we set $a(\omega) = 0$ and take $b(\omega) = c(\omega)$ in Theorem 3.1 then the result follows immediately. □

Theorem 3.2. *Let (X,d) be a complete GCbRMS with respect to cone $P \subset E$ and let M be a nonempty separable closed subset of X. Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega,.) : \Omega \times M \to M$ satisfying the condition:*

$$d(T(x(\omega)),T(y(\omega))) \preceq a(\omega)d(x(\omega),y(\omega)) + L(\omega)d(y(\omega),T(x(\omega))) \tag{15}$$

where $0 < a(\omega) < \frac{1}{s}$ and $L(\omega) \geq 0$, for all $x,y \in M$ and $\omega \in \Omega$. Then T has a random fixed point in X.

Proof. Let $x_0(\omega) \in \Omega \times M$ and consider the iterative sequence $\{x_n(\omega)\}$, where $x_n(\omega) = T(x_{n-1}(\omega)) = T^n(x_0(\omega))$ for all $n \geq 0$.
Now from (15) we get

$$\begin{aligned}
d_n &= d(x_n(\omega),x_{n+1}(\omega)) \\
&= d(T(x_{n-1}(\omega)),T(x_n(\omega))) \\
&\preceq a(\omega)d(x_{n-1}(\omega),x_n(\omega)) + L(\omega)d(x_n(\omega),T(x_{n-1}(\omega))) \\
&= a(\omega)d(x_{n-1}(\omega),x_n(\omega)) = a(\omega)d_{n-1}
\end{aligned} \tag{16}$$

Therefore by routine verification we see that $x_n(\omega)$ is not a periodic point of T for any $n \in \mathbb{N} \cup \{0\}$. Without loss of generality we can also assume that for any $n \geq 0$, $x_n(\omega) \neq x_{n+1}(\omega)$.
From (15) and (16) we obtain

$$\begin{aligned}
d_n^* &= d(x_n(\omega),x_{n+2}(\omega)) \\
&= d(T(x_{n-1}(\omega)),T(x_{n+1}(\omega))) \\
&\preceq a(\omega)d(x_{n-1}(\omega),x_{n+1}(\omega)) + L(\omega)d(x_{n+1}(\omega),T(x_{n-1}(\omega))) \\
&= a(\omega)d_{n-1}^* + L(\omega)d_n \\
&\preceq a(\omega)[a(\omega)d_{n-2}^* + L(\omega)d_{n-1}] + L(\omega)d_n \\
&\preceq a(\omega)^2 d_{n-2}^* + 2L(\omega)a(\omega)^n d_0 \\
&\quad ... \\
&\preceq a(\omega)^n d_0^* + nL(\omega)a(\omega)^n d_0
\end{aligned} \tag{17}$$

Now we will show that $\{x_n(\omega)\}$ is Cauchy and for this we consider two special cases for $d(x_n, x_{n+p})$, $p \geq 1$ and for all $n \in \mathbb{N}$.

Case I: For $p = 2m+1$, $m \geq 0$ we have

$$\begin{aligned} d(x_n(\omega), x_{n+2m+1}(\omega)) &\preceq s[d(x_n(\omega), x_{n+1}(\omega)) + d(x_{n+1}(\omega), x_{n+2}(\omega)) + d(x_{n+2}(\omega), x_{n+2m+1}(\omega))] \\ &= s[d_n + d_{n+1}] + sd(x_{n+2}(\omega), x_{n+2m+1}(\omega)) \\ &\quad ... \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + ... + s^m[d_{n+2m-2} + d_{n+2m-1}] + \\ &\quad s^m d_{n+2m} \\ &\preceq s[a(\omega)^n + a(\omega)^{n+1}]d_0 + s^2[a(\omega)^{n+2} + a(\omega)^{n+3}]d_0 + ... + \\ &\quad s^m[a(\omega)^{n+2m-2} + a(\omega)^{n+2m-1}]d_0 + s^m a(\omega)^{n+2m} d_0 \\ &\preceq s(1 + a(\omega))a(\omega)^n[1 + sa(\omega)^2 + ... + s^{m-1}a(\omega)^{2m-2}]d_0 + s^m a(\omega)^{n+2m} d_0 \\ &\preceq s(1 + a(\omega))a(\omega)^n[1 + sa(\omega)^2 + ... + s^{m-1}a(\omega)^{2m-2} + s^m a(\omega)^{2m}] \\ &\preceq \frac{s(1 + a(\omega))}{1 - sa(\omega)^2} a(\omega)^n d_0 \end{aligned} \tag{18}$$

Since $0 < a(\omega) < \frac{1}{s}$ then $a(\omega)^n \to 0$ as $n \to \infty$ so for any $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $d(x_n(\omega), x_{n+2m+1}(\omega)) \ll c$ for all $n \geq n_1$ and for all $m \geq 0$.

Case II: For $p = 2m$, $m \geq 1$ we get

$$\begin{aligned} d(x_n(\omega), x_{n+2m}(\omega)) &\preceq s[d(x_n(\omega), x_{n+1}(\omega)) + d(x_{n+1}(\omega), x_{n+2}(\omega)) + d(x_{n+2}(\omega), x_{n+2m}(\omega))] \\ &= s[d_n + d_{n+1}] + sd(x_{n+2}(\omega), x_{n+2m}(\omega)) \\ &\quad ... \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + ... + s^{m-1}[d_{n+2m-4} + d_{n+2m-3}] + \\ &\quad s^{m-1} d^*_{n+2m-2} \\ &\preceq s[a(\omega)^n + a(\omega)^{n+1}]d_0 + s^2[a(\omega)^{n+2} + a(\omega)^{n+3}]d_0 + ... + \\ &\quad s^{m-1}[a(\omega)^{n+2m-4} + a(\omega)^{n+2m-3}]d_0 + s^{m-1}[a(\omega)^{n+2m-2} d^*_0 + \\ &\quad L(\omega)(n + 2m - 2)a(\omega)^{n+2m-2} d_0] \\ &\preceq \frac{s(1 + a(\omega))}{1 - sa(\omega)^2} a(\omega)^n d_0 + a(\omega)^n d^*_0 + L(\omega)(n + 2m - 2)a(\omega)^n d_0 \end{aligned} \tag{19}$$

Since $a(\omega)^n \to 0$ and $(n + 2m - 2)a(\omega)^n \to 0$ as $n \to \infty$ for any $m \geq 1$, then for $c \gg \theta$ there exists $n_2 \in \mathbb{N}$ such that $d(x_n(\omega), x_{n+2m}(\omega)) \ll c$ for all $n \geq n_2$ and for all $m \geq 1$. Therefore from Case I and Case II we see that $\{x_n(\omega)\}$ is Cauchy in $\Omega \times M$. Since X is complete then there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$ as $n \to \infty$. Since M is closed then $z(\omega) \in \Omega \times M$.

Now

$$\begin{aligned} d(z(\omega), T(z(\omega))) &\preceq s[d(z(\omega), x_n(\omega)) + d(x_n(\omega), x_{n+1}(\omega)) + d(x_{n+1}(\omega), T(z(\omega)))] \\ &\preceq s[d(z(\omega), x_n(\omega)) + d_n + a(\omega)d(z(\omega), x_n(\omega)) + \\ &\quad L(\omega)d(z(\omega), x_{n+1}(\omega))] \\ &\preceq s(1 + a(\omega))d(z(\omega), x_n(\omega)) + sa(\omega)^n d_0 + \\ &\quad sL(\omega)d(z(\omega), x_{n+1}(\omega)) \end{aligned} \tag{20}$$

Since $x_n(\omega) \to z(\omega)$ and $a(\omega)^n \to 0$ as $n \to \infty$ then for any arbitrary $c \gg \theta$ there exists $r_1, r_2, r_3 \in \mathbb{N}$ such that $d(z(\omega), x_n(\omega)) \ll \frac{c}{3s(1+a(\omega))}$ if $n \geq r_1$, $d(z(\omega), x_{n+1}(\omega)) \ll \frac{c}{3s(1+L(\omega))}$ whenever $n \geq r_2$ and $a(\omega)^n d_0 \ll \frac{c}{3s}$ if $n \geq r_3$. Therefore if $n \geq r_0 =$

$\max\{r_1, r_2, r_3\}$ then $d(z(\omega), T(z(\omega))) \ll c$ which implies $T(z(\omega)) = z(\omega)$. Hence $z(\omega)$ is a random fixed point of T in X. □

4. Result in integral setting

Let X be a nonempty set and (Ω, Σ) be a measurable space with $\Sigma-$a sigma algebra of subsets of Ω. Also let the mapping $d : (\Omega \times X)^2 \to P$, where P is a normal cone in the Banach space $(E, ||.||)$, be a generalized cone $b-$random metric on X.

Definition 4.1. *A function $\varphi : P \to E$ is called generalized subadditive cone integrable function if and only if for all $a, b \in P$ and for any $k \geq 1$*

$$\begin{aligned} (i) \int_\theta^{a+b} \varphi d_p &\preceq \int_\theta^{a} \varphi d_p + \int_\theta^{b} \varphi d_p \\ (ii) \int_\theta^{ka} \varphi d_p &\preceq k \int_\theta^{a} \varphi d_p \end{aligned} \tag{21}$$

Example 4.1. *Let $X = E = \mathbb{R}$, $P = \{x \in E : x \geq 0\}$, $d(x, y) = |x - y|$ and $\varphi(t) = \frac{1}{t+1}$ for all $t > 0$. Then for all $a, b \in P$ and for any $k \geq 1$ we have*

$$\int_0^{a+b} \frac{dt}{t+1} = ln(a+b+1), \int_0^{a} \frac{dt}{t+1} = ln(a+1),$$
$$\int_0^{b} \frac{dt}{t+1} = ln(b+1), \int_0^{ka} \frac{dt}{t+1} = ln(ka+1)$$

Since $ln(a+b+1) \leq ln(a+1) + ln(b+1)$ and $ln(ka+1) \leq k\ ln(a+1)$, it follows that φ is a generalized subadditive cone integrable function.

Theorem 4.1. *Let (X, d) be a complete GCbRMS with respect to cone $P \subset E$ and let M be a nonempty separable closed subset of X. Let T be a continuous random operator defined on M such that for $\omega \in \Omega$, $T(\omega, .) : \Omega \times M \to M$ satisfying the condition:*

$$\int_\theta^{d(T(x(\omega)),T(y(\omega)))} \varphi d_p \preceq \alpha(\omega) \int_\theta^{d(x(\omega),y(\omega))} \varphi d_p \tag{22}$$

for all $x, y \in M$, $0 < \alpha(\omega) < \frac{1}{s}$ and for any $\omega \in \Omega$, where $\varphi : P \to E$ is a nonvanishing map and generalized subadditive cone integrable function on each $[a, b] \subset P$ such that for each $\epsilon \gg \theta$, $\int_\theta^\epsilon \varphi d_p \gg \theta$. Then T has a unique random fixed point in X.

Proof. Let $x_0(\omega) \in \Omega \times M$ and we consider the sequence $\{x_n(\omega)\}$ defined by $x_n(\omega) = T(x_{n-1}(\omega))$ for all $n = 1, 2, ...$

Now

$$\begin{aligned} d_n &= \int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p = \int_\theta^{d(T(x_{n-1}(\omega)),T(x_n(\omega)))} \varphi d_p \\ &\preceq \alpha(\omega) \int_\theta^{d(x_{n-1}(\omega),x_n(\omega))} \varphi d_p = \alpha(\omega) d_{n-1}, \ \forall n \geq 1, \end{aligned} \tag{23}$$

that is

$$\begin{aligned} d_n &= \int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p \\ &\preceq \alpha(\omega)^n \int_\theta^{d(x_0(\omega),x_1(\omega))} \varphi d_p = \alpha(\omega)^n d_0, \ \forall n = 1, 2, ... \end{aligned} \tag{24}$$

First we assume that $x_n(\omega) = x_m(\omega)$ for some $m, n \in \mathbb{N}, m \neq n$. Let $m > n$, then $x_n(\omega) = x_{n+k}(\omega)$ where $k = m - n \geq 1$ and we have

$$\begin{aligned} d_n &= \int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p \\ &= \int_\theta^{d(x_{n+k}(\omega),x_{n+k+1}(\omega))} \varphi d_p \\ &\preceq \alpha(\omega)^k \int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p = \alpha(\omega)^k d_n \qquad (25) \end{aligned}$$

From (25) we see that $(1-\alpha(\omega)^k)\int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p \preceq \theta$, thus $d(x_n(\omega), x_{n+1}(\omega)) = \theta$ implying that $T(x_n(\omega)) = x_n(\omega)$ and so T has a random fixed point in X.

Therefore without loss of generality we can assume that $x_n(\omega) \neq x_m(\omega)$ for all $m, n \in \mathbb{N}, m \neq n$. We now wish to show that $\{x_n(\omega)\}$ is Cauchy and so we consider two cases for $d(x_n(\omega), x_{n+p}(\omega)), p \geq 1$ and for all $n \in \mathbb{N}$.

Case I: If p is odd say $p = 2m + 1$, $m \geq 0$ then

$$\begin{aligned} \int_\theta^{d(x_n(\omega),x_{n+2m+1}(\omega))} \varphi d_p &\preceq \int_\theta^{s[d(x_n(\omega),x_{n+1}(\omega))+d(x_{n+1}(\omega),x_{n+2}(\omega))+d(x_{n+2}(\omega),x_{n+2m+1}(\omega))]} \varphi d_p \\ &\preceq s\int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p + s\int_\theta^{d(x_{n+1}(\omega),x_{n+2}(\omega))} \varphi d_p + \\ &\quad s\int_\theta^{d(x_{n+2}(\omega),x_{n+2m+1}(\omega))} \varphi d_p \\ &= s[d_n + d_{n+1}] + s\int_\theta^{d(x_{n+2}(\omega),x_{n+2m+1}(\omega))} \varphi d_p \\ &\preceq s[d_n + d_{n+1}] + \\ &\quad s\int_\theta^{s[d(x_{n+2}(\omega),x_{n+3}(\omega))+d(x_{n+3}(\omega),x_{n+4}(\omega))+d(x_{n+4}(\omega),x_{n+2m+1}(\omega))]} \varphi d_p \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^2\int_\theta^{d(x_{n+4}(\omega),x_{n+2m+1}(\omega))} \varphi d_p \\ &\quad ... \\ &\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + ... + s^m[d_{n+2m-2} + d_{n+2m-1}] + \\ &\quad s^m d_{n+2m} \\ &\preceq \frac{s(1+\alpha(\omega))}{1-s\alpha(\omega)^2}\alpha(\omega)^n d_0 \qquad (26) \end{aligned}$$

Since $\alpha(\omega)^n \to 0$ as $n \to \infty$ so by property (a) of Lemma 2.1 and property (a) of Lemma 2.2 we can say that for any $c \gg \theta$ there exists $N_1 \in \mathbb{N}$ such that $\int_\theta^{d(x_n(\omega),x_{n+2m+1}(\omega))} \varphi d_p \ll c$ for all $n \geq N_1$ and for all $m \geq 0$.

Now

$$\begin{aligned} d_n^* = \int_\theta^{d(x_n(\omega),x_{n+2}(\omega))} \varphi d_p &= \int_\theta^{d(T(x_{n-1}(\omega)),T(x_{n+1}(\omega))} \varphi d_p \\ &\preceq \alpha(\omega)\int_\theta^{d(x_{n-1}(\omega),x_{n+1}(\omega))} \varphi d_p \\ &= \alpha(\omega)d_{n-1}^* \preceq ... \preceq \alpha(\omega)^n d_0^*, \ \ \forall n \geq 1. \qquad (27) \end{aligned}$$

Case II: If $p = 2m$ for some $m \geq 1$ then,

$$\begin{aligned}
\int_\theta^{d(x_n(\omega),x_{n+2m}(\omega))} \varphi d_p &\preceq \int_\theta^{s[d(x_n(\omega),x_{n+1}(\omega))+d(x_{n+1}(\omega),x_{n+2}(\omega))+d(x_{n+2}(\omega),x_{n+2m}(\omega))]} \varphi d_p \\
&\preceq s\int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p + s\int_\theta^{d(x_{n+1}(\omega),x_{n+2}(\omega))} \varphi d_p + \\
&\quad s\int_\theta^{d(x_{n+2}(\omega),x_{n+2m}(\omega))} \varphi d_p \\
&= s[d_n + d_{n+1}] + s\int_\theta^{d(x_{n+2}(\omega),x_{n+2m}(\omega))} \varphi d_p \\
&\preceq s[d_n + d_{n+1}] + \\
&\quad s\int_\theta^{s[d(x_{n+2}(\omega),x_{n+3}(\omega))+d(x_{n+3}(\omega),x_{n+4}(\omega))+d(x_{n+4}(\omega),x_{n+2m}(\omega))]} \varphi d_p \\
&\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^2\int_\theta^{d(x_{n+4}(\omega),x_{n+2m}(\omega))} \varphi d_p \\
&\quad \dots \\
&\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + \dots + s^{m-1}[d_{n+2m-4} + \\
&\quad d_{n+2m-3}] + s^{m-1}d^*_{n+2m-2} \\
&\preceq \frac{s(1+\alpha(\omega))}{1-s\alpha(\omega)^2}\alpha(\omega)^n d_0 + s^{m-1}\alpha(\omega)^{n+2m-2}d^*_0 \\
&\preceq \frac{s(1+\alpha(\omega))}{1-s\alpha(\omega)^2}\alpha(\omega)^n d_0 + \alpha(\omega)^n d^*_0 \qquad (28)
\end{aligned}$$

Since $\alpha(\omega)^n \to 0$ as $n \to \infty$ so for $c \gg \theta$ there exists $N_2 \in \mathbb{N}$ such that $\int_\theta^{d(x_n(\omega),x_{n+2m}(\omega))} \varphi d_p \ll c$ for all $n \geq N_2$ and for all $m \geq 1$.
Combining case I and case II we see that $\int_\theta^{d(x_n(\omega),x_{n+p}(\omega))} \varphi d_p \to \theta$ as $n \to \infty$, for any $p \geq 1$. Thus $d(x_n(\omega), x_{n+p}(\omega)) \to \theta$ as $n \to \infty$, for all $p \geq 1$ and therefore $\{x_n(\omega)\}$ is Cauchy in $\Omega \times M$. Since (X, d) is complete, there exists $z(\omega) \in \Omega \times X$ such that $x_n(\omega) \to z(\omega)$ as $n \to \infty$. Since M is closed then $z(\omega) \in \Omega \times M$.
For any $n \in \mathbb{N}$ we have

$$\begin{aligned}
\int_\theta^{d(z(\omega),T(z(\omega)))} \varphi d_p &\preceq \int_\theta^{s[d(z(\omega),x_n(\omega))+d(x_n(\omega),x_{n+1}(\omega))+d(x_{n+1}(\omega),T(z(\omega)))]} \varphi d_p \\
&\preceq s\int_\theta^{d(z(\omega),x_n(\omega))} \varphi d_p + s\int_\theta^{d(x_n(\omega),x_{n+1}(\omega))} \varphi d_p + \\
&\quad s\int_\theta^{d(x_{n+1}(\omega),T(z(\omega)))} \varphi d_p \\
&\preceq s\int_\theta^{d(z(\omega),x_n(\omega))} \varphi d_p + sd_n + s\alpha(\omega)\int_\theta^{d(z(\omega),x_n(\omega))} \varphi d_p \\
&\preceq s(1+\alpha(\omega))\int_\theta^{d(z(\omega),x_n(\omega))} \varphi d_p + s\alpha(\omega)^n d_0 \qquad (29)
\end{aligned}$$

Since $d(z(\omega), x_n(\omega)) \to \theta$ and $\alpha(\omega)^n \to 0$ as $n \to \infty$ from (29) it follows that $\int_\theta^{d(z(\omega),T(z(\omega)))} \varphi d_p = \theta$ implying that $T(z(\omega)) = z(\omega)$. So $z(\omega)$ is a random fixed point of T in X. Let $u(\omega)$ be another random fixed point of T in X then

$$\begin{aligned}\int_{\theta}^{d(z(\omega),u(\omega))} \varphi d_p &= \int_{\theta}^{d(T(z(\omega)),T(u(\omega)))} \varphi d_p \\ &\preceq \alpha(\omega) \int_{\theta}^{d(z(\omega),u(\omega))} \varphi d_p \end{aligned} \tag{30}$$

This implies $(1-\alpha(\omega)) \int_{\theta}^{d(z(\omega),u(\omega))} \varphi d_p \preceq \theta$. Therefore $d(z(\omega),u(\omega)) = \theta \Rightarrow z(\omega) = u(\omega)$, a contradiction. Hence T has a unique random fixed point in X. □

Example 4.2. *Let $E = \mathbb{R}^2$, $P = \{(x,y) : x,y \geq 0\}$, $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and B be the set of all positive integers, also $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of $[0,1]$. Let us define $d : (\Omega \times X) \times (\Omega \times X) \to E$ by*

$$d(x(\omega),y(\omega)) = d(y(\omega),x(\omega)) = \begin{cases} (0,0), & \text{if } x(\omega) = y(\omega) \\ (2,2), & \text{if } x(\omega), y(\omega) \in A \\ (\frac{1}{2n}, \frac{1}{2n}) & \text{if } x(\omega) = \frac{1}{n} \in A \text{ and } y(\omega) \in \{2,3\} \\ (1,1) & \text{otherwise} \end{cases} \tag{31}$$

Then (X,d) is a generalized cone $b-$random metric space with coefficient $s = 2$ but it is not a cone random metric space.

Example 4.3. *Let $X = \{1,2,3,4\}$ and Σ be the sigma algebra of Lebesgue's measurable subsets of $\Omega = \{1,2,3,4\}$. Also let $d : (\Omega \times X) \times (\Omega \times X) \to E$, where $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$, be defined as follows:*

$$d(x(\omega),y(\omega)) = d(y(\omega),x(\omega)) = \begin{cases} 0, & \text{if } x(\omega) = y(\omega) \\ 3, & \text{if } x(\omega) = 1, y(\omega) = 2 \\ 1, & \text{if } x(\omega) \in \{1,2\} \text{ and } y(\omega) = 3 \\ 4, & \text{if } x(\omega) \in \{1,2,3\}, y(\omega) = 4 \end{cases} \tag{32}$$

Then (X,d) is a generalized cone $b-$random metric space for any $s \geq 1$ but it is not a cone random metric space. Now let us define, $T : \Omega \times X \to X$ by

$$T(\omega,x) = \begin{cases} 3, & \text{if } x \neq 4, \omega \in \Omega \\ 1, & \text{if } x = 4, \omega \in \Omega \end{cases} \tag{33}$$

Then T satisfies (22) with $\varphi(t) = \frac{1}{t+1}$ and $\alpha(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$. Here the measurable function $\xi : \Omega \to E$ with $\xi(\omega) = 3\ \forall\ \omega \in \Omega$ is the unique fixed point of T in X.

5. Conclusion

Our manuscript is dealt with a new topological structured space namely generalized cone $b-$random metric space where we have proved random fixed point theorems for two different type contractive random operators, of which one has a unique random fixed point in the underlying space and random fixed point of the another mapping may not be unique. In this article we prove another random fixed point theorem for a random operator satisfying integral type contractive condition. To prove this theorem we use a special type integrable function known as generalized subadditive cone integrable function. Moreover we decorate our paper with several examples.

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