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A PRODUCT TOPOLOGY AND STRONG CONVERGENCE SCHEME FOR FINDING COMMON FIXED POINTS OF A FAMILY OF NONEXPANSIVE SEMIGROUP

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ABSTRACT. In this paper, we consider the product space E^I with the product topology generated by the strong topologies on E for each $i \in I$ and using a family of nonexpansive semigroup in a product spaces E^I, where E is a real strictly convex and reflexive smooth Banach space and I is a nonempty set. And also, we introduce an algorithm in the product space E^I consisting of all functions from I to E and prove the convergence theorem of the proposed algorithms.

Keywords: Product topology, Fixed point, Nonexpansive mapping, Representation, Sunny nonexpansive retraction.

AMS Subject Classification: 47H10, 47H09.

1. Introduction

Let C be a nonempty, closed and convex subset of a Banach space E and let E^* be the dual space of E. Throughout $\langle .,. \rangle$ denotes the pairing between E and E^*. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. In the sequel, we use j to denote the single-valued normalized duality mapping. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth or said to have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for all $x \in U$. E is said to be uniformly smooth

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or said to have a uniformly Fr$\acute{e}$chet differentiable norm if the limit is attained uniformly for $x, y \in U$, it is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E. A Banach space E is smooth if the duality mapping J of E is single valued. We know that if E is smooth, then J is norm to weak* continuous; for more details, see [17].

Throughout this paper, unless otherwise stated, S will denote a semigroup, E a Banach space, C a nonempty, closed convex subset of E, and E^* the dual space of E. Let C be a nonempty, closed and convex subset of a Banach space E. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$ and a mapping f is an α_i-contraction on E if $\|f(x) - f(y)\| \leq \alpha_i \|x - y\|$, $x, y \in E$ such that $0 \leq \alpha_i < 1$.

Suppose that I is a nonempty set. We introduce the following general algorithm in the product space E^I for finding an element of E^I such that it's values are the common fixed points of the representations $\mathcal{S}_i = \{T_{t,i} : t \in S\}$ of a semigroup S as nonexpansive mappings from C_i into itself with respect to a left regular sequence of means defined on an appropriate subspace of bounded real-valued functions of a semigroup. In fact , our goal is to prove that there exists a unique sunny nonexpansive retraction P_i of C_i onto $\mathrm{Fix}(\mathcal{S}_\mathrm{i})$ and $x_i \in C_i$ for each $i \in I$ such that the sequence $\{g_n : I \to E\}$ in E^I generated by

$$\begin{cases} g_n(i) = z_{n,i}, \qquad i \in I, \\ z_{n,i} = \epsilon_n f_i(z_{n,i}) + (1 - \epsilon_n) T_{\mu_{n,i}} z_{n,i} \qquad i \in I, \end{cases}$$

converges to a function $g : I \to E$ in E^I defined by $g(i) = P_i x_i$ in the product topology on E^I. To see more related works, the readers can refer [3, 5, 8, 9, 10, 11, 15, 16, 18].

2. Preliminaries

Let S be a semigroup. We denote by $B(S)$ or $l^\infty(S)$ the Banach space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$ and $f \in B(S)$ we define l_s and r_s in $B(S)$ by $(l_s f)(t) = f(st)$, $(r_s f)(t) = f(ts)$, $(t \in S)$. Let X be a subspace of $B(S)$ containing 1. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e. $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup (see page 29 of [17]). A net $\{\mu_{\alpha_i}\}$ of means on X is said to be left regular if $\lim_{\alpha_i} \|l_s^* \mu_{\alpha_i} - \mu_{\alpha_i}\| = 0$, for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let f be a function from semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $B(S)$ containing all the functions $t \to \langle f(t), x^* \rangle$ with $x^* \in E^*$. We know from [6] that for any $\mu \in X^*$, there exists a unique element f_μ in E such that $\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle$ for all $x^* \in E^*$. We denote such f_μ by $\int f(t) \, \mathrm{d}\mu(t)$. Moreover, if μ is a mean on X, then from [7], $\int f(t) \, \mathrm{d}\mu(t) \in \overline{\mathrm{co}} \{f(t) : t \in S\}$.

Let C be a nonempty, closed and convex subset of E. Then, a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from C into itself is said to be a representation of S as nonexpansive mapping on C into itself if $\mathcal{S}$ satisfies the following :
(1) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;
(2) for every $s \in S$ the mapping $T_s : C \to C$ is nonexpansive.

We denote by Fix($\mathcal{S}$) the set of common fixed points of T_s, that is
Fix($\mathcal{S}$)=$\bigcap_{s\in S}\{x \in C : T_s x = x\}$.

Let $\{X_\alpha\}_\alpha$ be a family of topological spaces. If the spaces X_α are all equal to some fixed space X, the product $\Pi_{\alpha\in A}X_\alpha$ is just the set X^A of mappings from A to X, and the product topology is just the topology of pointwise convergence. More precisely:

Proposition 2.1. [4, Proposition 4. 12] *If X is a topological space, A is a nonempty set, and $\{f_n\}$ is a sequence in X^A then $f_n \to f$ in the product topology iff $f_n \to f$ pointwise.*

Lemma 2.2. ([13, Lemma 3.2]). *Let S be a semigroup and let C be a closed convex subset of a reflexive Banach space E. Suppose that $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as nonexpansive mapping from C into itself such that weak closure of $\{T_t x : t \in S\}$ is weakly compact for each $x \in C$ and X is a subspace of $B(S)$ such that $1 \in X$. Also suppose that the mapping $t \to \langle T_t x, x^*\rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Let μ be a mean on X. If we write $T_\mu x$ instead of $\int T_t x\, d\mu(t)$, then the followings hold.*
(i) *T_μ is a nonexpansive mapping from C into C.*
(ii) *$T_\mu x = x$ for each $x \in Fix(\mathcal{S})$.*
(iii) *$T_\mu x \in \overline{co}\,\{T_t x : t \in S\}$ for each $x \in C$.*
(iv) *If X is r_s-invariant for each $s \in S$ and μ is right invariant, then $T_\mu T_t = T_\mu$ for each $t \in S$.*

Definition 2.3. [1, Definition 5.2.8] *Let C be a nonempty subset of a Banach space X and $T : C \to X$ be a mapping. Then T is said to be demiclosed at $v \in X$ if for any sequence $\{x_n\}$ in C the following implication holds:*

$$x_n \rightharpoonup u \in C \quad \textit{and} \quad Tx_n \to v \quad \textit{imply} \quad Tu = v.$$

Note that $S_X = \{x \in X : \|x\| = 1\}$ shows the unit sphere of X.

Definition 2.4. [1, Definition 2.1.1] *A Banach space X is said to be strictly convex if*

$$x, y \in S_X \quad \textit{with} \quad x = y \Rightarrow \|(1-\lambda)x + \lambda y\| < 1 \quad \textit{for all} \quad \lambda \in (0, 1).$$

This says that the midpoint $\frac{x+y}{2}$ of two distinct points x and y in the unit sphere S_X of X does not lie on S_X. In other words, if $x, y \in S_X$ with $\|x\| = \|y\| = \|\frac{x+y}{2}\|$, then $x = y$.

Definition 2.5. [1, Definition 2.2.1] *A Banach space X is said to be uniformly convex if for any ϵ, $0 < \epsilon \leq 2$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq 1$ imply that there exists a $\delta = \delta(\epsilon) > 0$ such that $\|\frac{x+y}{2}\| \leq 1 - \delta$.*

Definition 2.6. [1, Definition 2.6.1] *A Banach space X is said to be smooth if for each $x \in S_X$, there exists a unique functional $j_x \in X^*$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$.*

Remark 2.1. *From, Theorem 4.1.6 in* [17], *every uniformly convex Banach space is strictly convex and reflexive.*

Remark 2.2. *To see retraction and sunny nonexpansive retract concepts, refer to* [1, 17]. *For example, we know from* [1, Proposition 2.10.20] *that in the case that E is a smooth Banach space and R is a retraction from C onto D where C is a nonempty convex subset of E and D a nonempty subset of C, then we have R is sunny and nonexpansive, if and only if for each $x \in C$ and $z \in D$,*

$$\langle x - Rx\,,\, J(z - Rx)\rangle \leq 0.$$

Lemma 2.7. [12, Lemma 1] *Let S be a semigroup and E be a real uniformly convex and smooth Banach space. Suppose that C is a nonempty compact convex subset of E. Also suppose that $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as nonexpansive mappings from*

C into itself such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. *Let X be a left invariant subspace of $l^\infty(S)$ such that $1 \in X$, and $t \mapsto \langle T_t x, x^* \rangle$ belongs to X for each $x \in C$ and $x^* \in E^*$. If μ is a left invariant mean on X and if J is weakly sequentially continuous, then* $\mathrm{Fix(T_\mu)} = \mathrm{T_\mu(C)} = \mathrm{Fix}(\mathcal{S})$ *and there exists a unique sunny nonexpansive retraction from C onto* $\mathrm{Fix}(\mathcal{S})$.

Let E be a Banach space and $B \subseteq E$. Suppose that $D \subseteq E$. Let P be a retraction of B onto D, that is, $Px = x$ for each $x \in D$. Then P is said to be sunny, if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$, $P(Px + t(x - Px)) = Px$. A subset D of B is said to be a sunny nonexpansive retract of B if there exists a sunny nonexpansive retraction P of B onto D. We know that if E is smooth and P is a retraction of B onto D, then P is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$, $\langle x - Px\,,\, J(z - Px)\rangle \leq 0$. For more details, see [17].

In this paper, we denote B_r for an open ball of radius r centered at 0. Also for $\epsilon > 0$ and a mapping $T : C \to C$, we denote $F_\epsilon(T; G)$ for the set of ϵ-approximate fixed points of T for a subset G of C, *i.e.*, $F_\epsilon(T; G) = \{x \in G : \|x - Tx\| \leq \epsilon\}$.

3. Main results

In this section, we deal with a product topology convergence approximation scheme for finding an element of E^I such that it's values are the common fixed points of the representations of a family of the representations of nonexpansive mappings.

Theorem 3.1. *Let I be a nonempty set and S be a semigroup. Let C_i be a nonempty compact convex subset of a real strictly convex and reflexive smooth Banach space E for each $i \in I$. Consider the product space E^I with the product topology generated by the strong topologies on E for each $i \in I$. Suppose that $\mathcal{S}_i = \{T_{s,i} : s \in S\}$ be a representation of S as nonexpansive mapping from C_i into itself such that* $\mathrm{Fix}(\mathcal{S}_\mathrm{i}) \neq \emptyset$ *for each $i \in I$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, x^* \rangle$ is an element of X for each $x \in C_i$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a left regular sequence of means on X. Suppose that f_i is an α_i-contraction on C_i for each $i \in I$. Let $\{\epsilon_n\}$ be a sequence in $(0, 1)$ such that $\lim_n \epsilon_n = 0$. Then there exists a unique sunny nonexpansive retraction P_i of C_i onto* $\mathrm{Fix}(\mathcal{S}_\mathrm{i})$ *and $x_i \in C_i$ for each $i \in I$ such that the sequence $\{g_n : I \to E\}$ in E^I generated by*

$$\begin{cases} g_n(i) = z_{n,i}, & i \in I, \\ z_{n,i} = \epsilon_n f_i(z_{n,i}) + (1 - \epsilon_n) T_{\mu_{n,i}} z_{n,i} & i \in I, \end{cases} \tag{1}$$

converges to the function $g : I \to E$ defined by $g(i) = P_i x_i$ in the product topology on E^I.

Proof. Since every Banach space is a complete metric space and E is reflexive, from Proposition 1.7.3 and [1, Theorem 1.9.21], every compact subset C_i of a reflexive Banach space E, is weakly compact. By [1, Proposition 1.9.18], we know that every closed convex subset of a weakly compact subset C_i of a Banach space E is weakly compact. So [1, Proposition 1.9.13] implies that each convex subset C_i of a normed space E is weakly closed if and only if C_i is closed. Hence, weak closure of $\{T_{t,i} x : t \in S\}$ is weakly compact for each $x \in C_i$.

The proof is divided into six steps.

Step 1. The existence of $z_{n,i}$ which satisfies (1).
This concludes from the fact that the following mapping $N_{n,i}$ is a contraction on C_i for every $n \in \mathbb{N}$ and $i \in I$,

$$N_{n,i} x_i := \epsilon_n f_i(x_i) + (1 - \epsilon_n) T_{\mu_{n,i}} x_i \quad (x_i \in C_i).$$

Indeed, put $\beta_n = (1 + \epsilon_n(\alpha_i - 1))$, then $0 \leq \beta_n < 1$ $(n \in \mathbb{N})$. Hence, we have,

$$\begin{aligned}\|N_{n,i}x_i - N_{n,i}y_i\| \leq& \epsilon_n\|f_i(x_i) - f_i(y_i)\| + (1 - \epsilon_n)\,\|T_{\mu_{n,i}}x_i - T_{\mu_{n,i}}y_i\|\\ \leq& \epsilon_n\alpha_i\|x_i - y_i\| + (1 - \epsilon_n)\|x_i - y_i\|\\ =&(1 + \epsilon_n(\alpha_i - 1))\|x_i - y_i\| = \beta_n\|x_i - y_i\|.\end{aligned}$$

Hence, by Banach Contraction Principle [1, Theorem 4.1.5], there exists a unique point $z_{n,i} \in C_i$ that $N_{n,i}z_{n,i} = z_{n,i}$.

Step 2. $\lim_{n\to\infty} \|z_{n,i} - T_{t,i}z_{n,i}\| = 0$, for all $i \in I$ and $t \in S$.

Consider $t \in S$, $i \in I$ and let $\epsilon > 0$. By [14, Lemma 1], there exists $\delta > 0$ such that $\overline{\text{co}}F_\delta(T_{t,i}) + 2B_\delta \subseteq F_\epsilon(T_{t,i})$. From [2, Corollary 2.8], there also exists a natural number N such that

$$\Big\|\frac{1}{N+1}\sum_{j=0}^{N} T_{t^js,i}y - T_{t,i}\Big(\frac{1}{N+1}\sum_{j=0}^{N} T_{t^js,i}y\Big)\Big\| \leq \delta, \tag{2}$$

for all $s \in S$ and $y \in C_i$. Let $p_i \in \text{Fix}(\mathcal{S}_\text{i})$ and $M_{0,i}$ be a positive number such that, $\sup\limits_{y\in C_i}\|y\| \leq M_{0,i}$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$ such that $\|\mu_n - l^*_{t^j}\mu_n\| \leq \frac{\delta}{(3M_{0,i})}$ for $n \geq N_0$ and $j = 1, 2, \cdots, N$. Therefore, we conclude

$$\begin{aligned}&\sup_{y\in C_i}\Big\|T_{\mu_{n,i}}y - \int\frac{1}{N+1}\sum_{j=0}^{N} T_{t^js,i}y\,\mathrm{d}\mu_n(s)\Big\|\\ &= \sup_{y\in C_i}\sup_{\|x^*\|=1}\Big|\langle T_{\mu_{n,i}}y, x^*\rangle - \Big\langle\int\frac{1}{N+1}\sum_{j=0}^{N} T_{t^js,i}y\,\mathrm{d}\mu_n(s), x^*\Big\rangle\Big|\\ &= \sup_{y\in C_i}\sup_{\|x^*\|=1}\Big|\frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_{s,i}y, x^*\rangle - \frac{1}{N+1}\sum_{j=0}^{N}(\mu_n)_s\langle T_{t^js,i}y, x^*\rangle\Big|\\ &\leq \frac{1}{N+1}\sum_{j=0}^{N}\sup_{y\in C_i}\sup_{\|x^*\|=1}\Big|(\mu_n)_s\langle T_{s,i}y, x^*\rangle - (l^*_{tj}\mu_n)_s\langle T_{s,i}y, x^*\rangle\Big|\\ &\leq \max_{j=1,2,\cdots,N}\|\mu_n - l^*_{tj}\mu_n\|(M_{0,i} + 2\|p_i\|)\\ &\leq \max_{j=1,2,\cdots,N}\|\mu_n - l^*_{tj}\mu_n\|(3M_{0,i})\\ &\leq \delta \quad (\mathrm{n} \geq \mathrm{N}_0).\end{aligned} \tag{3}$$

Applying Lemma 2.2, we have

$$\int\frac{1}{N+1}\sum_{j=0}^{N} T_{t^js,i}y\,\mathrm{d}\mu_n(s) \in \overline{\text{co}}\left\{\frac{1}{N+1}\sum_{j=0}^{N} T_{t^j,i}(T_{s,i}y) : s \in S\right\}. \tag{4}$$

By (2)-(4) we deduce

$$\begin{aligned}T_{\mu_{n,i}}y \in& \overline{\text{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^js,i}y : s \in S\right\} + B_\delta\\ &\subset \overline{\text{co}}F_\delta(T_{t,i}) + 2B_\delta \subset F_\epsilon(T_{t,i}),\end{aligned}$$

for all $y \in C_i$ and $n \geq N_0$. Hence, $\limsup_{n\to\infty} \sup_{y\in C_i} \|T_{t,i}(T_{\mu_{n,i}}y) - T_{\mu_{n,i}}y\| \leq \epsilon$. Because $\epsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty} \sup_{y\in C_i} \|T_{t,i}(T_{\mu_{n,i}}y) - T_{\mu_{n,i}}y\| = 0. \tag{5}$$

Let $t \in S$ and $\epsilon > 0$. Then there exists $\delta > 0$, which satisfies (2). Put $L_{0,i} = (1 + \alpha_i)2M_{0,i} + \|f_i(p_i) - p_i\|$. Now, by $\lim_n \epsilon_n = 0$ and using (5) there exists a natural number N_1 such that $T_{\mu_{n,i}}y \in F_\delta(T_{t,i})$ for each $y \in C_i$ and $\epsilon_n < \frac{\delta}{2L_{0,i}}$ for each $n \geq N_1$. Since $p_i \in \mathrm{Fix}(\mathcal{S}_\mathrm{i})$, we conclude

$$\begin{aligned}
\epsilon_n\|f_i(z_{n,i})-&T_{\mu_{n,i}}z_{n,i}\| \\
&\leq \epsilon_n\Big(\|f_i(z_{n,i}) - f_i(p_i)\| + \|f_i(p_i) - p_i\| \\
&\quad + \|T_{\mu_{n,i}}p_i - T_{\mu_{n,i}}z_{n,i}\|\Big) \\
&\leq \epsilon_n\Big(\alpha_i\|z_{n,i} - p_i\| + \|f_i(p_i) - p_i\| + \|z_{n,i} - p_i\|\Big) \\
&\leq \epsilon_n\left(\alpha_i\|z_{n,i} - p_i\| + \|f_i(p_i) - p_i\| + \|z_{n,i} - p_i\|\right) \\
&\leq \epsilon_n\left((1 + \alpha_i)\|z_{n,i} - p_i\| + \|f_i(p_i) - p_i\|\right) \\
&\leq \epsilon_n\left((1 + \alpha_i)2M_{0,i} + \|f_i(p_i) - p_i\|\right) \\
&= \epsilon_n L_{0,i} \leq \frac{\delta}{2},
\end{aligned}$$

for all $n \geq N_1$. Observe that

$$\begin{aligned}
z_{n,i} &= \epsilon_n f_i(z_{n,i}) + (1 - \epsilon_n)T_{\mu_{n,i}}z_{n,i} \\
&= T_{\mu_{n,i}}z_{n,i} + \epsilon_n\left(f_i(z_{n,i}) - T_{\mu_{n,i}}z_{n,i}\right) \\
&\in F_\delta(T_{t,i}) + B_{\frac{\delta}{2}} \\
&\subseteq F_\delta(T_{t,i}) + 2B_\delta \\
&\subseteq F_\epsilon(T_{t,i}).
\end{aligned}$$

for each $n \geq N_1$. Then we conclude that
$\|z_{n,i} - T_{t,i}z_{n,i}\| \leq \epsilon \quad (\mathrm{n} \geq \mathrm{N}_1)$.
Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{n\to\infty} \|z_{n,i} - T_{t,i}z_{n,i}\| = 0$.

Step 3. For each $i \in I$, $\mathfrak{S}\{z_{n,i}\} \subset \mathrm{Fix}(\mathcal{S}_\mathrm{i})$, where $\mathfrak{S}\{z_{n,i}\}$ denotes the set of strongly limit points (i.e., $z \in \mathfrak{S}\{z_{n,i}\}$ means that there exists a subsequence $\{z_{n_j,i}\}$ of $\{z_{n,i}\}$ such that $z_{n_j,i} \to z$) of $\{z_{n,i}\}$.
Let $i \in I$. Consider $z_i \in \mathfrak{S}\{z_{n,i}\}$ and let $\{z_{n_j,i}\}$ be a subsequence of $\{z_{n,i}\}$ such that $z_{n_j,i} \to z_i$.

$$\begin{aligned}
\|T_{t,i}z_i - z_i\| &\leq \|T_{t,i}z_i - T_{t,i}z_{n_j,i}\| + \|T_{t,i}z_{n_j,i} - z_{n_j}\| + \|z_{n_j,i} - z_i\| \\
&\leq 2\|z_{n_j,i} - z_i\| + \|T_{t,i}z_{n_j,i} - z_{n_j,i}\|,
\end{aligned}$$

applying step 2 we have, $\|T_{t,i}z_i - z_i\| \leq 2\lim_j \|z_{n_j,i} - z_i\| + \lim_j \|T_{t,i}z_{n_j,i} - z_{n_j}\| = 0$, hence, $z_i \in \mathrm{Fix}(\mathcal{S}_\mathrm{i})$.

Step 4. For each $i \in I$, there exists a unique sunny nonexpansive retraction P_i of C_i onto $\mathrm{Fix}(\mathcal{S}_\mathrm{i})$ and $x_i \in C_i$ such that

$$\Gamma_i := \limsup_n \langle x_i - P_i x_i\,,\ J(z_{n,i} - P_i x_i)\rangle \leq 0. \tag{6}$$

Applying Lemma 2.7, there exists a unique sunny nonexpansive retraction P_i of C_i onto $\mathrm{Fix}(\mathcal{S}_\mathrm{i})$. Using Banach Contraction Principle, we have that f_iP_i has a unique fixed point $x_i \in C_i$. We prove that $\Gamma_i := \limsup_n \langle x_i - P_ix_i\,,\, J(z_{n,i} - Px_i)\rangle \le 0$. Observe that, by the definition of Γ_i and by the fact that C_i is a compact subset of E_i, we can select a subsequence $\{z_{n_j,i}\}$ of $\{z_{n,i}\}$ with the following properties:
(i) $\lim_j \langle x_i - P_ix_i\,,\, J(z_{n_j,i} - P_ix_i)\rangle = \Gamma_i$;
(ii) $\{z_{n_j,i}\}$ converges strongly to a point z_i.
Applying Step 3, we have $z_i \in \mathrm{Fix}(\mathcal{S}_\mathrm{i})$. From the fact that E_i is smooth, we conclude

$$\Gamma_i = \lim_j \langle x_i - P_ix_i\,,\, J(z_{n_j,i} - P_ix_i)\rangle = \langle x_i - P_ix_i\,,\, J(z_i - P_ix_i)\rangle \le 0.$$

Since $f_iP_ix_i = x_i$, we have $(f_i - I)P_ix_i = x_i - P_ix_i$. From [17, page 99], we have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\epsilon_n(\alpha_i - 1)\|z_{n,i} - P_ix_i\|^2\\
&\ge\Big[\epsilon_n\alpha_i\big\|z_{n,i} - P_ix_i\big\| + (1-\epsilon_n)\|z_{n,i} - P_ix_i\|\Big]^2 - \|z_{n,i} - P_ix_i\|^2\\
&\ge\Big[\epsilon_n\big\|f(z_{n,i}) - f(P_ix_i)\big\| + (1-\epsilon_n)\|T_{\mu_{n,i}}z_{n,i} - P_ix_i\|\Big]^2 - \|z_{n,i} - P_ix_i\|^2\\
&\ge 2\Big\langle \epsilon_n\Big(f(z_{n,i}) - f(P_ix_i)\Big)\\
&\quad + (1-\epsilon_n)(T_{\mu_{n,i}}z_{n,i} - P_ix_i) - (z_{n,i} - P_ix_i)\,,\, J(z_{n,i} - P_ix_i)\Big\rangle\\
&= -2\epsilon_n\langle (f - I)P_ix_i\,,\, J(z_{n,i} - P_ix_i)\rangle\\
&= -2\epsilon_n\langle x_i - P_ix_i\,,\, J(z_{n,i} - P_ix_i)\rangle.
\end{aligned}$$

Therefore,

$$\|z_{n,i} - P_ix_i\|^2 \le \frac{2}{1-\alpha_i}\langle x_i - P_ix_i\,,\, J(z_{n,i} - P_ix_i)\rangle. \tag{7}$$

Step 5. $\{z_{n,i}\}$ strongly converges to P_ix_i.
(6), (7) and the fact that $P_ix_i \in \mathrm{Fix}(\mathcal{S}_\mathrm{i})$, imply that

$$\limsup_n \|z_{n,i} - P_ix_i\|^2 \le \frac{2}{1-\alpha_i}\limsup_n \langle x_i - P_ix_i\,,\, J(z_{n,i} - P_ix_i)\rangle \le 0.$$

Hence, $z_{n,i} \to P_ix_i$.
Step 6. $\{g_n\}$ converges to g in the product topology on E^I.
As we know, the topology on E^I is that of strongly pointwise convergence. Hence from (1) and step 5, we conclude the desired results. □

4. **Examples and Corollaries**

In this section, we deal with some examples and corollaries.

Corollary 4.1. *Let I be a nonempty set. Let C_i be a nonempty compact convex subset of a real Hilbert space H for each $i \in I$. Consider the product space H^I with the product topology generated by the strong topologies on H. Suppose that $\{T_i\}_{i\in I}$ is a family of nonexpansive mappings from C_i into itself such that* $\mathrm{Fix}(\mathrm{T_i}) \neq \emptyset$ *for each $i \in I$. Suppose that f_i is an α_i-contraction on C_i for each $i \in I$. Let ϵ_n be a sequence in $(0,1)$ such that $\lim_n \epsilon_n = 0$. Then there exists a unique sunny nonexpansive retraction P_i of C_i onto*

$\mathrm{Fix(T_i)}$ *and* $x_i \in C_i$ *for each* $i \in I$ *such that the sequence* $\{g_n : I \to H\}$ *in* H^I *generated by*

$$\begin{cases} g_n(i) = z_{n,i}, & i \in I, \\ z_{n,i} = \epsilon_n f_i(z_{n,i}) + (1-\epsilon_n)\frac{1}{n}\sum_{k=1}^{n} T_i^k z_{n,i} & i \in I, \end{cases}$$

converges to the function $g : I \to E$ *defined by* $g(i) = P_i x_i$ *in the product topology on* E^I.

Proof. Let $\mathcal{S}_i = \{T_i^j : j \in S\}$ where $S = \{1, 2, ...\}$. For a function $f = (x_1, x_2, ...) \in B(S)$, define

$$\mu_n(f) = \frac{1}{n}\sum_{k=1}^{n} x_k \qquad (n \in \mathbb{N}),$$

then $\{\mu_n\}$ is a left regular sequence of means on $B(S)$ [17]. Hence, we have

$$T_{\mu_{n,i}} x = \frac{1}{n}\sum_{k=1}^{n} T_i^k x \qquad (n \in \mathbb{N}).$$

Then from Theorem 3.1, we get the results. □

Corollary 4.2. *Let* I *be a nonempty set. Let* C_i *be a nonempty compact convex subset of a real Hilbert space* H *for each* $i \in I$. *Consider the product space* H^I *with the product topology generated by the strong topologies on* H. *Suppose that* $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \le t < +\infty\}$ *and* $\mathcal{S}_i = \{T_{t,i} : t \in \mathbb{R}^+\}$ *be a representation of* $\mathbb{R}^+$ *as nonexpansive mapping from* C_i *into itself such that* $\mathrm{Fix}(\mathcal{S}_\mathrm{i}) \neq \emptyset$ *for each* $i \in I$. *Let* X *be a left invariant subspace of* $C(\mathbb{R}^+)$ *such that* $1 \in X$. *Suppose that* f_i *is an* α_i*-contraction on* C_i *for each* $i \in I$. *Let* $\{\epsilon_n\} \subseteq (0,1)$ *such that* $\lim_n \epsilon_n = 0$ *and* $\{a_n\} \subseteq (0,\infty)$ *such that* $\lim_n a_n = \infty$. *Then there exists a unique sunny nonexpansive retraction* P_i *of* C_i *onto* $\mathrm{Fix(T_i)}$ *and* $x_i \in C_i$ *for each* $i \in I$ *such that the sequence* $\{g_n : I \to H\}$ *in* H^I *generated by*

$$\begin{cases} g_n(i) = z_{n,i}, & i \in I, \\ z_{n,i} = \epsilon_n f_i(z_{n,i}) + (1-\epsilon_n)\frac{1}{a_n}\int_0^{a_n} T_{t,i} z_{n,i}\,\mathrm{d}t & (n \in \mathbb{N}), \qquad i \in I, \end{cases}$$

converges to the function $g : I \to E$ *defined by* $g(i) = P_i x_i$ *in the product topology on* E^I.

Proof. For a function $f \in C(\mathbb{R}^+)$, define

$$\mu_n(f) = \frac{1}{a_n}\int_0^{a_n} f(t)\,\mathrm{d}t \qquad (n \in \mathbb{N}),$$

then $\{\mu_n\}$ is a left regular sequence of means on $B(S)$ [17]. Hence, we have

$$T_{\mu_{n,i}} x = \frac{1}{a_n}\int_0^{a_n} T_{t,i} x\,\mathrm{d}t \qquad (n \in \mathbb{N}).$$

Then from Theorem 3.1, we get the results. □

5. Conclusion

In this paper, we introduced an algorithm in a product space that is new in the literature. Then we proved, the proposed scheme is convergent with respect to the product topology. Also, using a family of representations, a mean μ and the mapping T_μ as a nonexpansive mapping, we constructed the algorithm.

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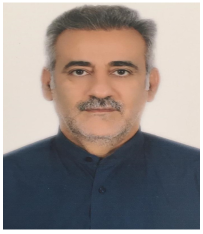

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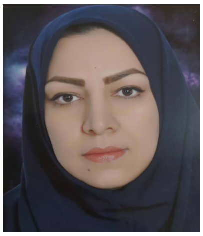

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