

BOUNDARY VALUE PROBLEM SOLVING FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL AND INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we will study a boundary value problem for semilinear fractional differential equations of order $q \in (1, 2]$ with nonlocal and integral boundary conditions. Some existence and uniqueness results with illustrative examples will be presented by applying some fixed point theorems.

Keywords: Fractional differential equations, Banach space, Banach's fixed point theorem, Krasnoselskii's fixed point theorem, nonlocal condition.

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1. INTRODUCTION

Fractional derivatives have an excellent tool for describing memory and hereditary properties of various materials and processes. these characteristics of fractional derivatives make fractional order models more realistic and practical than the standar integer models. Recently, boundary value problems for nonlinear fractional differential equations have been studied by several authors. In fact, fractional differential equations arise in many scientific disciplines such as, Biology, Chemistry, Physics, Economics, Control theory, Signal processing, etc, [14, 19, 20]. For more details of developments on the subject, one can see for example [2, 3, 5, 6, 7, 8, 9, 10, 16, 17, 21, 22] and the references therein..

Fractional differential equations with integral boundary conditions constitute a very interesting and important class of problems. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, cellular systems etc. see [1, 13]. For a detailed description of the integral boundary conditions, we refer the reader to the papers [4, 6, 10] and references cited therein.

In the present paper, we consider a boundary value problem for semilinear fractional differential equations of order $q \in (1, 2]$ with nonlocal and integral boundary conditions given by:

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$$\begin{cases} {}^C D^q x(t) = f(t, x(t)), & 1 < q \leq 2, \quad t \in J = [0, 1], \\ x(0) = g(x) + \alpha \int_0^\xi x(s) ds, & 0 < \xi < 1, \\ x(1) = h(x) + \beta \int_0^\eta x(s) ds, & 0 < \eta < 1. \end{cases} \quad (1)$$

where ${}^C D^q$ denotes the Caputo fractional derivative of order q , $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function, $g, h : [0, 1] \rightarrow \mathbb{R}$ are two continuous functions satisfying some hypotheses that will be specified later.

Nonlocal conditions were initiated by Bitsadze [12]. Byszewski to notice that the nonlocal condition may be more useful than the standard initial condition to describe some physical phenomenas. For example, $g(x)$ and $h(x)$ can be given in the form $\sum_{i=1}^p c_i x(t_i)$, où $(c_i = 1, 2, \dots, p)$ are given constants and $0 < t_1 < t_2 < \dots < T$. As examples for recent papers on nonlocal fractional boundary value problems the interested reader is referred to [8, 10, 11, 23] and the references therein.

The first aim of this paper is to study the existence and uniqueness result for the problem (1), where we apply the Banach contraction principle. For the second result, we use Krasnoselskii's fixed point theorem to establish the existence of the solution to the boundary value problem (1), finally, the last result is based on a lemma of D. O'Regan.

2. PRELIMINARY NOTIONS OF FRACTIONAL CALCULUS

Firstly, before starting our work, we need some basic definitions and lemmas of fractional calculus that we can find in [14, 19, 20].

Definition 2.1. [19, 20] *If $g \in C((a, b); \mathbb{R})$ and $q \in \mathbb{R}_+$, then the fractional integral of order q is defined by:*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds,$$

where Γ denotes the Gamma function.

Definition 2.2. [19, 20] *For a continuous function $g : [0, +\infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^C D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q] + 1,$$

where $[q]$ is the integer part of the real number q .

Lemma 2.1. [14] *For $q > 0$, the homogenous fractional differential equation ${}^C D^q g(t) = 0$, has a solution*

$$g(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $(i=0, 1, \dots, n-1)$ and $n = [q] + 1$.

Lemma 2.2. [14] *Let $g \in C([0, 1], \mathbb{R})$ such that $D^q g \in C([0, 1], \mathbb{R})$. Then*

$$I^q {}^C D^q g(t) = g(t) - c_0 - c_1 t - c_2 t^2 - \dots - c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $(i=0, 1, \dots, n-1)$ and $n = [q] + 1$.

Lemma 2.3. [14] *Let $p, q \geq 0$ and $g \in L^1([0, 1], \mathbb{R})$. Then,*

$$I^p I^q g(t) = I^{p+q} g(t) = I^q I^p g(t),$$

and

$${}^C D^p I^p g(t) = g(t), \quad \forall t \in [0, 1].$$

Lemma 2.4. [14] *Let $q > p > 0$ and $g \in L^1([0, 1], \mathbb{R})$. Then for all $t \in [0, 1]$, we have*

$${}^C D^p I^q g(t) = I^{q-p} g(t).$$

$L^1([0, 1], \mathbb{R})$ is the Banach space of Lebesgue integrable functions from $[0, 1]$ into \mathbb{R} .

3. MAIN RESULTS

In the context of studying our existence and uniqueness results, we need the following auxiliary lemma.

Lemma 3.1. *Let $\sigma : [0, 1] \rightarrow \mathbb{R}$ be a given continuous function. The solution $x(t)$ of the boundary value problem:*

$$\begin{cases} {}^C D^q x(t) = \sigma(t), & 1 < q \leq 2, \quad t \in J = [0, 1], \\ x(0) = g(x) + \alpha \int_0^\xi x(s) ds, & 0 < \xi < 1, \\ x(1) = h(x) + \beta \int_0^\eta x(s) ds, & 0 < \eta < 1. \end{cases} \quad (2)$$

is given by

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sigma(s) ds \\ & - \frac{\alpha}{\gamma \Gamma(q)} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \int_0^\xi \left(\int_0^s (s-m)^{q-1} \sigma(m) dm \right) ds \\ & + \frac{\beta}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \int_0^\eta \left(\int_0^s (s-m)^{q-1} \sigma(m) dm \right) ds \\ & - \frac{1}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \int_0^1 (1-s)^{q-1} \sigma(s) ds \\ & - \frac{1}{\gamma} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] g(x) + \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] h(x), \end{aligned} \quad (3)$$

where

$$\gamma = \frac{1}{2} [(1-\alpha\xi)(2-\beta\eta^2) + \alpha\xi^2(1-\beta\eta)] \neq 0.$$

Proof. We have:

$${}^C D^q x(t) = \sigma(t),$$

i.e.,

$$I^q {}^C D^q x(t) = I^q \sigma(t).$$

Then, in view of lemma 2.1, it follows that:

$$x(t) = c_0 + c_1 t + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sigma(s) ds. \quad (4)$$

By integrating the expression (4) on $[0, \xi]$, and adding $g(x)$ to two sides after multiplied it by α , we find that:

$$g(x) + \alpha \int_0^\xi x(s)ds = g(x) + \alpha c_0 \xi + \alpha c_1 \frac{\xi^2}{2} + \frac{\alpha}{\Gamma(q)} \int_0^\xi \left(\int_0^s (s-m)^{q-1} \sigma(m) dm \right) ds.$$

In a similar way, we get:

$$h(x) + \beta \int_0^\eta x(s)ds = h(x) + \beta c_0 \eta + \beta c_1 \frac{\eta^2}{2} + \frac{\beta}{\Gamma(q)} \int_0^\eta \left(\int_0^s (s-m)^{q-1} \sigma(m) dm \right) ds.$$

Using the boundary conditions for (2), we obtain:

$$(1 - \alpha\xi)c_0 - \alpha \frac{\xi^2}{2} c_1 = g(x) + \alpha A \quad (5)$$

$$(1 - \beta\eta)c_0 + \left(1 - \beta \frac{\eta^2}{2}\right) c_1 = h(x) + \beta B - C, \quad (6)$$

where,

$$A = \frac{1}{\Gamma(q)} \int_0^\xi \left(\int_0^s (s-m)^{q-1} \sigma(m) dm \right) ds, \quad (7)$$

$$B = \frac{1}{\Gamma(q)} \int_0^\eta \left(\int_0^s (s-m)^{q-1} \sigma(m) dm \right) ds,$$

$$C = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} \sigma(s) ds.$$

The resolution of the system (5)-(6), gives:

$$c_0 = \frac{1}{\gamma} \left[\alpha \beta \frac{\xi^2}{2} B - \alpha \frac{\xi^2}{2} C - \alpha \left(1 - \beta \frac{\eta^2}{2}\right) A + \alpha \frac{\xi^2}{2} h(x) - \left(1 - \beta \frac{\eta}{2}\right) g(x) \right],$$

and

$$c_1 = \frac{1}{\gamma} \left[\beta(1 - \alpha\xi)B - (1 - \alpha\xi)C - \alpha(1 - \beta\eta)A + (1 - \alpha\xi)h(x) - (1 - \beta\eta)g(x) \right].$$

By substituting the values of c_0 and c_1 in (4), we obtain (3). \square

We equip the Banach space $C([0, 1], \mathbb{R})$ of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0, 1]\}$, and, before announcing our theorems, we need the following assumptions:

$$(H1) : |f(t, x) - f(t, y)| \leq L|x - y|, \quad \forall t \in [0, 1], \quad L > 0, \quad x, y \in \mathbb{R},$$

$$(H2) : |g(x) - g(y)| \leq l_1|x - y|, \quad l_1 > 0, \quad \forall x, y \in \mathbb{R},$$

$$(H3) : |h(x) - h(y)| \leq l_2|x - y|, \quad l_2 > 0, \quad \forall x, y \in \mathbb{R},$$

$$(H4) : |f(t, x)| \leq \mu(t), \quad \forall (t, x) \in [0, 1] \times \mathbb{R} \text{ and } \mu \in C([0, 1], \mathbb{R}_+).$$

Now for convenience, let use set:

$$\begin{aligned}
 A_0 &= \frac{1}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \tag{8} \\
 A_1 &= |\alpha| (|2 - \beta\eta^2| + 2|1 - \beta\eta|) \xi^{q+1}, \\
 A_2 &= (|\alpha|\xi^2 + 2|1 - \alpha\xi|) (|\beta|\eta^{q+1} + q + 1), \\
 A_3 &= |2 - \beta\eta^2| + 2|1 - \beta\eta|, \\
 A_4 &= |\alpha|\xi^2 + 2|1 - \alpha\xi|.
 \end{aligned}$$

Theorem 3.1. *Assume that $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which satisfies the assumption (H1), g and h are two real functions bounded on the real line that satisfy respectively the assumptions (H2) and (H3). If $L^*A_0 < 1$, where $L^* = \max\{L, l_1, l_2\}$ and A_0 is given by (8) Then the boundary value problem (1) has a unique solution.*

Proof. For the proof of the theorem 3.1 and in view of lemma 3.1, we define the operator: $N : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ by:

$$\begin{aligned}
 (Nx)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \\
 &\quad - \frac{\alpha}{\gamma\Gamma(q)} \left[\frac{2 - \beta\eta^2}{2} + (1 - \beta\eta)t \right] \int_0^\xi \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\
 &\quad + \frac{\beta}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] \int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\
 &\quad - \frac{1}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] \int_0^1 (1-s)^{q-1} f(s, x(s)) ds \\
 &\quad - \frac{1}{\gamma} \left[\frac{2 - \beta\eta^2}{2} + (1 - \beta\eta)t \right] g(x) + \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] h(x)., \tag{9}
 \end{aligned}$$

Letting

$$M = \sup_{t \in [0,1]} |f(t, 0)|, \quad M_1 = \sup_{x \in \mathbb{R}} |g(x)|, \quad M_2 = \sup_{x \in \mathbb{R}} |h(x)|,$$

and choosing $\rho \geq \frac{A_0 M^*}{1 - L^* A_0}$, where $M^* = \max\{M, M_1, M_2\}$.

Firstly, we show that $NB_\rho \subset B_\rho$, where $B_\rho = \{x \in C([0, 1], \mathbb{R}) : \|x\| \leq \rho\}$. For $x \in B_\rho$, we

have:

$$\begin{aligned}
& |(Nx)(t)| \\
\leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds \\
& + \left| \frac{\alpha}{\gamma \Gamma(q)} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \right| \int_0^\xi \left(\int_0^s (s-m)^{q-1} |f(m, x(m))| dm \right) ds \\
& + \left| \frac{\beta}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| \int_0^\eta \left(\int_0^s (s-m)^{q-1} |f(m, x(m))| dm \right) ds \\
& + \left| \frac{1}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| \int_0^1 (1-s)^{q-1} |f(s, x(s))| ds \\
& + \left| \frac{1}{\gamma} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \right| |g(x)| + \left| \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| |h(x)| \\
\leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
& + \left| \frac{\alpha}{\gamma \Gamma(q)} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \right| \int_0^\xi \left(\int_0^s (s-m)^{q-1} (|f(m, x(m)) - f(m, 0)| \right. \\
& \left. + |f(m, 0)|) dm \right) ds \\
& + \left| \frac{\beta}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| \int_0^\eta \left(\int_0^s (s-m)^{q-1} (|f(m, x(m)) - f(m, 0)| \right. \\
& \left. + |f(m, 0)|) dm \right) ds \\
& + \left| \frac{1}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| \int_0^1 (1-s)^{q-1} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) ds \\
& + \left| \frac{1}{\gamma} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \right| (|g(x) - g(0)| + |g(0)|) \\
& + \left| \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| (|h(x) - h(0)| + |h(0)|) \\
\leq & (L\rho + M) \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \right. \\
& + \frac{|\alpha|}{|\gamma \Gamma(q)} \left(\frac{|2-\beta\eta^2|}{2} + |1-\beta\eta| \right) \int_0^\xi \left(\int_0^s (s-m)^{q-1} dm \right) ds \\
& + \frac{|\beta|}{|\gamma \Gamma(q)} \left(\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right) \int_0^\eta \left(\int_0^s (s-m)^{q-1} dm \right) ds \\
& + \frac{1}{|\gamma \Gamma(q)} \left(\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right) \int_0^1 (1-s)^{q-1} ds \Big] \\
& + \frac{1}{|\gamma|} \left(\frac{|2-\beta\eta^2|}{2} + |1-\beta\eta| \right) (l_1\rho + M_1) \\
& + \frac{1}{|\gamma|} \left(\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right) (l_2\rho + M_2) \\
\leq & (L^*\rho + M^*) \left[\frac{1}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \right] \\
= & (L^*\rho + M^*) A_0 \leq \rho.
\end{aligned}$$

Thus, $\|Nx\| \leq \rho$.

For each $t \in [0, 1]$ and $x, y \in C([0, 1], \mathbb{R})$, we have:

$$\begin{aligned}
 & |(Nx)(t) - (Ny)(t)| \\
 \leq & \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s)) - f(s, y(s))| ds \\
 & + \left| \frac{\alpha}{\gamma \Gamma(q)} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \right| \int_0^\xi \left(\int_0^s (s-m)^{q-1} |f(m, x(m)) - f(m, y(m))| dm \right) ds \\
 & + \left| \frac{\beta}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| \int_0^\eta \left(\int_0^s (s-m)^{q-1} |f(m, x(m)) - f(m, y(m))| dm \right) ds \\
 & + \left| \frac{1}{\gamma \Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| \int_0^1 (1-s)^{q-1} |f(s, x(s)) - f(s, y(s))| ds \\
 & + \left| \frac{1}{\gamma} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \right| |g(x) - g(y)| + \left| \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \right| |h(x) - h(y)| \\
 \leq & L|x-y| \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \right. \\
 & + \frac{|\alpha|}{|\gamma| \Gamma(q)} \left(\frac{|2-\beta\eta^2|}{2} + |1-\beta\eta| \right) \int_0^\xi \left(\int_0^s (s-m)^{q-1} dm \right) ds \\
 & + \frac{|\beta|}{|\gamma| \Gamma(q)} \left(\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right) \int_0^\eta \left(\int_0^s (s-m)^{q-1} dm \right) ds \\
 & \left. + \frac{1}{|\gamma| \Gamma(q)} \left(\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right) \int_0^1 (1-s)^{q-1} ds \right] \\
 & + l_1|x-y| \frac{1}{|\gamma|} \left(\frac{|2-\beta\eta^2|}{2} + |1-\beta\eta| \right) \\
 & + l_2|x-y| \frac{1}{|\gamma|} \left(\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right) \\
 \leq & L^*|x-y| \left[\frac{1}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \right] \\
 = & L^*A_0|x-y|.
 \end{aligned}$$

Hence, $\|Nx - Ny\| \leq L^*A_0\|x - y\|$. The number L^* depends only on the parameters indicated in our problem. Since $L^*A_0 < 1$, then N is a contraction. Thus, by Banach's fixed point theorem, it follows that our boundary value problem (1) has a unique solution. \square

Example 3.1. Consider the following fractional boundary value problem:

$$\begin{cases}
 {}^C D^{\frac{3}{2}} x(t) = \frac{1}{(t+4)^2} \frac{|x|}{1+|x|} + 1 + \cos^2 t, & t \in J = [0, 1], \\
 x(0) = \frac{1}{16} x(\mu) + \frac{1}{2} \int_0^{\frac{1}{4}} x(s) ds, & \mu \in [0, 1], \\
 x(1) = \frac{1}{12} x(\nu) + \int_0^{\frac{3}{4}} x(s) ds, & \nu \in [0, 1].
 \end{cases} \tag{10}$$

In this example, we have:

$$q = \frac{3}{2}, \quad \alpha = \frac{1}{2}, \quad \beta = 1, \quad \xi = \frac{1}{4}, \quad \eta = \frac{3}{4},$$

and

$$f(t, x) = \frac{1}{(t+4)^2} \frac{|x|}{1+|x|} + 1 + \cos^2 t, \quad g(x) = \frac{1}{16}x(\mu), \quad h(x) = \frac{1}{12}x(\nu).$$

Then,

$$\begin{aligned} |f(t, x) - f(t, y)| &\leq \frac{1}{16}|x - y|, & L &= \frac{1}{16}, \\ |g(x) - g(y)| &= \frac{1}{16}|x - y|, & l_1 &= \frac{1}{16}, \\ |h(x) - h(y)| &= \frac{1}{12}|x - y|, & l_2 &= \frac{1}{12}, \end{aligned}$$

Furthermore,

$$\begin{aligned} L^* &= \frac{1}{12}, \quad \gamma = \frac{81}{128}, \quad A_1 = \frac{31}{1024}, \quad A_2 = \frac{513\sqrt{3} + 4560}{1024}, \\ A_3 &= \frac{31}{16}, \quad A_4 = \frac{57}{32}, \quad 2|\gamma|(q+1) = \frac{405}{128}, \quad 2|\gamma| = \frac{162}{128}, \end{aligned}$$

Therefore,

$$L^*A_0 = \frac{1}{9\sqrt{\pi}} \left(\frac{19\sqrt{3}}{120} + \frac{7831}{3240} \right) + \frac{119}{1944} \approx 0.337498344 < 1.$$

So, all the hypotheses of the theorem 3.1 are satisfied and consequently the boundary value problem (10) has a unique solution.

Our second result is based on the fixed point theorem of Krasnoselskii .

Theorem 3.2. [15] (*Krasnoselskii's fixed point theorem*) Let M be a closed convex and nonempty subset of a Banach space X . Let A, B be two operators such that:

- (i) $Ax + By \in M$ whenever $x, y \in M$,
- (ii) A is compact and continuous,
- (iii) B is a contraction mapping.

Then, there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.3. Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a jointly continuous function mapping bounded subsets of $[0, 1] \times \mathbb{R}$ into relatively compact subsets of \mathbb{R} , and the assumptions (H1) – (H4) hold. If

$$L^* \left[\frac{1}{\Gamma(q+1)} \left(\frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \right] < 1, \quad (11)$$

then, the boundary value problem (1) has at least one solution on $[0, 1]$.

Proof. Letting $M_3 = \sup_{t \in [0,1]} |\mu(t)|$. We fix

$$\bar{\rho} \geq \widetilde{M} \left[\frac{1}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \right]$$

where $\widetilde{M} = \max\{M_1, M_2, M_3\}$ and consider $B_{\bar{\rho}} = \{x \in C([0, 1], \mathbb{R}) : \|x\| \leq \bar{\rho}\}$. To apply Theorem 3.2, we define two operators \mathcal{P} and \mathcal{Q} by:

$$\begin{aligned}
 (\mathcal{P}x)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \\
 (\mathcal{Q}x)(t) &= -\frac{\alpha}{\gamma\Gamma(q)} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] \int_0^\xi \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\
 &\quad + \frac{\beta}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\
 &\quad - \frac{1}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] \int_0^1 (1-s)^{q-1} f(s, x(s)) ds \\
 &\quad - \frac{1}{\gamma} \left[\frac{2-\beta\eta^2}{2} + (1-\beta\eta)t \right] g(x) + \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1-\alpha\xi)t \right] h(x),
 \end{aligned}$$

- For $x, y \in B_{\bar{\rho}}$, we find that:

$$\begin{aligned}
 \|\mathcal{P}x + \mathcal{Q}y\| &\leq \frac{M_3}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{M_1A_3 + M_2A_4}{2|\gamma|} \\
 &\leq \widetilde{M} \left[\frac{1}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \right] \\
 &\leq \bar{\rho}.
 \end{aligned}$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\bar{\rho}}$.

- For $x, y \in C([0, 1], \mathbb{R})$ and each $t \in [0, 1]$, we have:

$$|(\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)| \leq L^* \left[\frac{1}{\Gamma(q+1)} \left(\frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \right] |x - y|.$$

which implies that:

$$\|(\mathcal{Q}x) - (\mathcal{Q}y)\| \leq L^* \left[\frac{1}{\Gamma(q+1)} \left(\frac{A_1 + A_2}{2|\gamma|(q+1)} \right) + \frac{A_3 + A_4}{2|\gamma|} \right] \|x - y\|.$$

So, it follows by the condition (11) that \mathcal{Q} is a contraction mapping.

- The continuity of f implies that the operator \mathcal{P} is continuous. In addition we have:

$$\|\mathcal{P}x\| \leq \frac{M_3}{\Gamma(q+1)},$$

which means that \mathcal{P} is uniformly bounded on $B_{\bar{\rho}}$.

- Now, we prove that the operator \mathcal{P} is compact.

Taking into account the condition (H1), we define $f^* = \sup_{(t,x) \in [0,1] \times B_{\bar{\rho}}} |f(t, x)|$.

Then for $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, we have:

$$\begin{aligned}
 |(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)| &= \frac{1}{\Gamma(q)} \left| \int_0^{t_1} (t_1 - s)^{q-1} f(s, x(s)) ds - \int_0^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds \right| \\
 &= \frac{1}{\Gamma(q)} \left| \int_0^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right] f(s, x(s)) ds \right. \\
 &\quad \left. - \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds \right| \\
 &\leq \frac{1}{\Gamma(q)} \left[\int_0^{t_1} \left((t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right) |f(s, x(s))| ds \right. \\
 &\quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{q-1} |f(s, x(s))| ds \right] \\
 &\leq \frac{f^*}{\Gamma(q)} \left[\int_0^{t_1} \left((t_1 - s)^{q-1} - (t_2 - s)^{q-1} \right) ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \right] \\
 &\leq \frac{f^*}{\Gamma(q+1)} \left[2(t_2 - t_1)^q + t_2^q - t_1^q \right]. \tag{12}
 \end{aligned}$$

The second member in (12) is independent of x and tends to zero when $t_2 - t_1 \rightarrow 0$, so \mathcal{P} is equicontinuous. Using the fact that f maps bounded subsets into relatively compact subsets, we obtain that $\mathcal{P}(\mathbf{B})(t)$ is relatively compact in \mathbb{R} for every t , (where \mathbf{B} is a bounded subset of $C([0, 1] \times \mathbb{R})$). Then \mathcal{P} is relatively compact on $B_{\bar{\rho}}$. Therefore, by the Ascoli-Arzelà theorem, we conclude that \mathcal{P} is compact on $B_{\bar{\rho}}$. Thus, all the assumptions of Theorem 3.2 are satisfied. Then the boundary value problem (1) has at least one solution on $[0, 1]$. □

Our next main result is based on the following lemma established by D. O'Regan in [18].

Lemma 3.2. *Denote by U an open set in a closed convex set C of a Banach space E . Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \rightarrow C$ is a given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \rightarrow E$ is continuous and completely continuous and $F_2 : \bar{U} \rightarrow E$ is a nonlinear contraction (i.e. there exists a nonnegative nondecreasing function $\phi : [0, +\infty) \rightarrow (0, +\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in \bar{U}$). Then either*

- (C₁) *F has a fixed point $x \in \bar{U}$; or*
- (C₂) *there exist a point $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where \bar{U} and ∂U , respectively represent the closure and boundary of U .*

Now, for convenience we define:

$$\Omega_r = \{x \in C([0, 1], \mathbb{R}) : \|x\| < r\},$$

and

$$M_r = \max\{|f(t, x)| : (t, x) \in [0, 1] \times [-r, r]\}.$$

Theorem 3.4. *Let $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Suppose that (H1) hold. In addition, we assume that:*

- (H5) *there exist two positive constants ρ_1, ρ_2 and two continuous functions $\phi_1, \phi_2 : [0, +\infty) \rightarrow$*

$(0, +\infty)$ such that:

- $\phi_1(z) \leq \rho_1 z$, and $|g(u) - g(v)| \leq \phi_1(|u - v|)$, for all $u, v \in \mathbb{R}$.
- $\phi_2(z) \leq \rho_2 z$, and $|h(u) - h(v)| \leq \phi_2(|u - v|)$, for all $u, v \in \mathbb{R}$.

(H6) $g(0) = 0$ and $h(0) = 0$.

(H7) There exists a nonnegative function $p \in C([0, 1], \mathbb{R}_+)$ and a nondecreasing function $\psi : [0, +\infty) \rightarrow (0, +\infty)$ such that:

$$|f(t, u)| \leq p(t)\psi(|u|), \text{ for all } (t, u) \in [0, 1] \times \mathbb{R}.$$

(H8) $\sup_{r \in \mathbb{R}_+} \frac{r}{p_0 \psi(r)} > \frac{|\gamma|}{|\gamma| - \rho_1 A_3 - \rho_2 A_4}$, where

$$p_0 = \frac{1}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} p(s) ds + \frac{A_1}{2|\gamma|} \int_0^\xi \left(\int_0^s (s-m)^{q-1} p(m) dm \right) ds \right. \\ \left. + \frac{|\beta|(|\alpha|\xi^2 + 2|1 - \alpha\xi|)}{2|\gamma|} \int_0^\eta \left(\int_0^s (s-m)^{q-1} p(m) dm \right) ds \right. \\ \left. + \frac{|\alpha|\xi^2 + 2|1 - \alpha\xi|}{2|\gamma|} \int_0^1 (1-s)^{q-1} p(s) ds \right].$$

Then, the boundary value problem (1) has at least one solution on $[0, 1]$.

Proof. In the first place, we consider the operator $N : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$ as that defined by (9) and we put:

$$(Nx)(t) = (N_1x)(t) + (N_2x)(t), \quad t \in [0, 1],$$

where,

$$(N_1x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \\ - \frac{\alpha}{\gamma\Gamma(q)} \left[\frac{2 - \beta\eta^2}{2} + (1 - \beta\eta)t \right] \int_0^\xi \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\ + \frac{\beta}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] \int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\ - \frac{1}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] \int_0^1 (1-s)^{q-1} f(s, x(s)) ds$$

and

$$(N_2x)(t) = -\frac{1}{\gamma} \left[\frac{2 - \beta\eta^2}{2} + (1 - \beta\eta)t \right] g(x) + \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] h(x),$$

From (H8) there exists a strictly positive number r_0 ($r_0 > 0$) such that:

$$\frac{r_0}{p_0 \psi(r_0)} > \frac{|\gamma|}{|\gamma| - \rho_1 A_3 - \rho_2 A_4}. \tag{13}$$

Now, for the proof of our theorem, we shall prove that the operators N_1 and N_2 satisfy all the hypotheses of lemma 3.2. So, the proof is done in four steps.

Step 1: The operator N_1 is continuous and completely continuous.

We show that $N_1(\overline{\Omega}_{r_0})$ is bounded. For all $x \in \overline{\Omega}_{r_0}$, we have:

$$\begin{aligned}
 |(N_1x)(t)| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds \\
 &+ \frac{|\alpha|}{|\gamma|\Gamma(q)} \left[\frac{|2-\beta\eta^2|}{2} + |1-\beta\eta| \right] \int_0^\xi \left(\int_0^s (s-m)^{q-1} |f(m, x(m))| dm \right) ds \\
 &+ \frac{|\beta|}{|\gamma|\Gamma(q)} \left[\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right] \int_0^\eta \left(\int_0^s (s-m)^{q-1} |f(m, x(m))| dm \right) ds \\
 &+ \frac{1}{|\gamma|\Gamma(q)} \left[\frac{|\alpha|\xi^2}{2} + |1-\alpha\xi| \right] \int_0^1 (1-s)^{q-1} |f(s, x(s))| ds \\
 &\leq \frac{M_r}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right). \tag{14}
 \end{aligned}$$

Then,

$$\|N_1x\| \leq \frac{M_r}{\Gamma(q+1)} \left(1 + \frac{A_1 + A_2}{2|\gamma|(q+1)} \right).$$

This means that $N_1(\overline{\Omega}_{r_0})$ is uniformly bounded. Furthermore, for each $t_1, t_2 \in [0, 1], t_1 < t_2$, we have:

$$\begin{aligned}
 &|(N_1x)(t_1) - (N_1x)(t_2)| \\
 &\leq \frac{1}{\Gamma(q)} \int_0^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1} \right] |f(s, x(s))| ds \\
 &+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} |f(s, x(s))| ds \\
 &+ \frac{|\alpha|}{|\gamma|\Gamma(q)} |1-\beta\eta|(t_2-t_1) \int_0^\xi \left(\int_0^s (s-m)^{q-1} |f(m, x(m))| dm \right) ds \\
 &+ \frac{|\beta|}{|\gamma|\Gamma(q)} |1-\alpha\xi|(t_2-t_1) \int_0^\eta \left(\int_0^s (s-m)^{q-1} |f(m, x(m))| dm \right) ds \\
 &+ \frac{|1|}{|\gamma|\Gamma(q)} |1-\alpha\xi|(t_2-t_1) \int_0^1 (1-s)^{q-1} |f(s, x(s))| ds \\
 &\leq \frac{M_r}{\Gamma(q)} \left\{ \int_0^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1} \right] ds + \int_{t_1}^{t_2} (t_2-s)^{q-1} ds \right. \\
 &+ \frac{|\alpha||1-\beta\eta|(t_2-t_1)}{|\gamma|} \int_0^\xi \left(\int_0^s (s-m)^{q-1} dm \right) ds \\
 &+ \frac{|\beta||1-\alpha\xi|(t_2-t_1)}{|\gamma|} \int_0^\eta \left(\int_0^s (s-m)^{q-1} dm \right) ds \\
 &\left. + \frac{|1-\alpha\xi|(t_2-t_1)}{|\gamma|} \int_0^1 (1-s)^{q-1} ds \right\},
 \end{aligned}$$

which is independent of x and tends to zero as $t_2 - t_1 \rightarrow 0$. Then N_1 is equicontinuous. Hence, by Ascoli-Arzelà Theorem, we conclude that $N_1(\overline{\Omega}_{r_0})$ is a relatively compact set.

Let $\{x_n\} \subset \overline{\Omega}_{r_0}$ which $\|x_n - x\| \rightarrow 0$. Then $|x_n(t) - x(t)| \rightarrow 0$ on $[0, 1]$. From the uniform continuity of $(t, x) \mapsto f(t, x)$ on the compact set $[0, 1] \times \overline{\Omega}_{r_0}$ it follows that $|f(t, x_n(t)) - f(t, x(t))| \rightarrow 0$ uniformly on $[0, 1]$. Hence $\|N_1x_n - N_1x\| \rightarrow 0$ when $n \rightarrow +\infty$ which means that N_1 is completely continuous.

Step 2: The operator $N_2 : \overline{\Omega}_{r_0} \rightarrow C([0, 1], \mathbb{R})$ is a contraction. This is deduced directly from the condition (H5).

Step 3: The set $N(\overline{\Omega}_{r_0})$ is bounded.

From the assumption (H5), we obtain that:

$$\begin{aligned} \|N_2x\| &\leq \frac{2}{|\gamma|} \left[\frac{|2 - \beta\eta^2|}{2} + |1 - \beta\eta| \right] \rho_1 r_0 + \frac{2}{|\gamma|} \left[\frac{|\alpha|\xi^2}{2} + |1 - \alpha\xi| \right] \rho_2 r_0 \\ &= \frac{\rho_1 A_3 + \rho_2 A_4}{|\gamma|} r_0 \end{aligned} \tag{15}$$

for any $x \in \overline{\Omega}_{r_0}$. Since the set $N_1(\overline{\Omega}_{r_0})$ is bounded, then the set $N(\overline{\Omega}_{r_0})$ is also bounded.

Step 4: Finally, Just show that the condition (C₂) in lemma 3.2, does not occur. To this end, we proceed by contradiction. We suppose that (C₂) holds. Then, there exists $\lambda \in (0, 1)$ and $x \in \partial\overline{\Omega}_{r_0}$ such that $x = \lambda N x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned} x(t) = \lambda \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \right. \\ - \frac{\alpha}{\gamma\Gamma(q)} \left[\frac{2 - \beta\eta^2}{2} + (1 - \beta\eta)t \right] \int_0^\xi \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\ + \frac{\beta}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] \int_0^\eta \left(\int_0^s (s-m)^{q-1} f(m, x(m)) dm \right) ds \\ - \frac{1}{\gamma\Gamma(q)} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] \int_0^1 (1-s)^{q-1} f(s, x(s)) ds \\ \left. - \frac{1}{\gamma} \left[\frac{2 - \beta\eta^2}{2} + (1 - \beta\eta)t \right] g(x) + \frac{1}{\gamma} \left[\frac{\alpha\xi^2}{2} + (1 - \alpha\xi)t \right] h(x) \right\}, t \in [0, 1]. \end{aligned}$$

With the hypotheses (H6) – (H8), we have:

$$\begin{aligned} r_0 \leq \frac{\psi(r_0)}{\Gamma(q)} \left[\int_0^t (t-s)^{q-1} p(s) ds + \frac{A_1}{2|\gamma|} \int_0^\xi \left(\int_0^s (s-m)^{q-1} p(m) dm \right) ds \right. \\ + \frac{|\beta|(|\alpha|\xi^2 + 2|1 - \alpha\xi|)}{2|\gamma|} \int_0^\eta \left(\int_0^s (s-m)^{q-1} p(m) dm \right) ds \\ \left. + \frac{|\alpha|\xi^2 + 2|1 - \alpha\xi|}{2|\gamma|} \int_0^1 (1-s)^{q-1} p(s) ds \right] \\ + \frac{\rho_1 A_3 + \rho_2 A_4}{|\gamma|} r_0. \end{aligned}$$

This means that:

$$r_0 \leq p_0 \psi(r_0) + \frac{\rho_1 A_3 + \rho_2 A_4}{|\gamma|} r_0.$$

Thus,

$$\frac{r_0}{p_0 \psi(r_0)} \leq \frac{|\gamma|}{|\gamma| - \rho_1 A_3 - \rho_2 A_4},$$

this is a contradiction with (13). Consequently, the operators N_1 and N_2 satisfy all the assumptions of the lemma 3.2 . Hence the operator N has at least one fixed point x in $\overline{\Omega}_{r_0}$, which is solution of the boundary value problem (1). So, this completes the proof \square

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