TWMS J. App. and Eng. Math. V.12, N.3, 2022, pp. 1083-1094

A CERTAIN SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY LAMBDA OPERATOR

B. VENKATESWARLU[1], P. THIRUPATHI REDDY[2] , G. SWAPNA[1], §

Abstract. In this paper, we introduce a new subclass of uniformly convex functions with negative coefficients defined by lambda operator. We obtain the coefficient bounds, growth distortion properties, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $TS(\upsilon, \varrho, \mu, s)$. Furthermore, we obtained modified Hadamard product, convolution and integral operators for this class.

Keywords: analytic, coefficient bounds, extreme points, convolution.

AMS Subject Classification: 30C45.

1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{\eta=2}^{\infty} a_\eta z^\eta \tag{1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in E. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \le \upsilon < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E). \tag{2}$$

We denote this class with $S^*(\upsilon)$.

A function $u \in A$ is a convex function of the order $\upsilon, 0 \le \upsilon < 1$, if it satisfy

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E). \tag{3}$$

[1] Department of Mathematics- GITAM University-Doddaballapur- 562163- Bengaluru Rural- India.
e-mail: bvlmaths@gmail.com; ORCID: https://orcid.org/0000-0003-3669-350X.
e-mail: swapna.priya38@gmail.com; ORCID: https://orcid.org/0000-0002-8869-8647.
[2] Department of Mathematics, Kakatiya Univeristy, Warangal, 506 009, Telangana, India.
e-mail: reddypt2@gmail.com; ORCID: https://orcid.org/0000-0002-0034-444X.
§ Manuscript received: May 03, 2020; accepted: July 11, 2020.

We denote this class with $K(\upsilon)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in E respectively.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta, \;\; a_\eta \geq 0 \; (z \in E) \tag{4}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon), \; C(\upsilon) = T \cap K(\upsilon)$. The class $T^*(\upsilon)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [13]. Recently, some subclasses of T have investigated by [1, 3] and others.

For $u \in A$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{\eta=2}^{\infty} b_\eta z^\eta$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{\eta=2}^{\infty} a_\eta b_\eta z^\eta = (g * u)(z) \; (z \in E).$$

Note that $u * g \in A$.

For following Goodman [4, 5] and Ronning [9, 10] introduced and studied the following subclasses:

(1). A function $u \in A$ is said to be in the class $UCV(\varrho, \gamma)$, uniformly $\gamma-$ convex function if is satisfies the condition

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)} - \varrho\right\} > \gamma \left|\frac{zu''(z)}{u'(z)}\right|, \tag{5}$$

where $\gamma \geq 0, -1 < \varrho \leq 1$ and $\varrho + \gamma \geq 0$.

(2). A function $u \in A$ is said to be in the class $SP(\varrho, \gamma)$, uniformly $\gamma-$ starlike function if is satisfies the condition

$$\Re\left\{\frac{zu'(z)}{u(z)} - \gamma\right\} > \varrho \left|\frac{zu'(z)}{u(z)} - 1\right|, \tag{6}$$

where $\gamma \geq 0, -1 < \varrho \leq 1$ and $\varrho + \gamma \geq 0$.

Indeed it follows from (5) and (6) that

$$u \in UCV(\varrho, \gamma) \Leftrightarrow zu' \in SP(\varrho, \gamma). \tag{7}$$

For $\gamma = 0$, we get respectively, the classes $K(0) = K$ and $S^*(0) = S^*$. The function of the class $UCV(0, 1) \equiv UCV$ is called uniformly convex functions were introduced by Goodman with geometric interpretation in [4]. The class $SP(0, 1) \equiv SP$ is defined by Ronning in [9]. For $\varrho = 0$, the class $UCV(0, \gamma) \equiv \gamma - UCV$ and $SP(0, \gamma) \equiv \gamma - SP$ are defined respectively, by Kanas and Wisniowska in [6, 7].

Further, Murugusundarmoorthy and Magesh [8], Santosh et al. [11], and Thirupathi Reddy and Venkateswarlu [15] have studied and investigated interesting properties for the classes $UCV(\varrho, \gamma)$ and $SP(\varrho, \gamma)$.

In [14], Spanier and Oldham defined lambda function as

$$\lambda(z, s) = \sum_{\eta=2}^{\infty} \frac{z^\eta}{(2\eta + 1)^s},$$

where $z \in E, s \in \mathbb{C}$, when $|z| < 1, \Re(s) > 1$, when $|z| = 1$ and let $\lambda^{(-1)}(z,s)$ be defined such that

$$\lambda(z,s) * \lambda^{(-1)}(z,s) = \frac{1}{(1-z)^{\mu+1}},\ \mu > -1.$$

We now define $\left(z\lambda^{(-1)}(z,s)\right)$ as the following

$$\begin{aligned}(z\lambda(z,s)) * \left(z\lambda^{(-1)}(z,s)\right) &= \frac{z}{(1-z)^{\mu+1}} \\ &= z + \sum_{\eta=2}^{\infty} \frac{(\mu+1)_{\eta-1}}{(\eta-1)!} z^{\eta}, \mu > -1\end{aligned}$$

and obtain the following linear operator

$$\mathcal{I}_{\mu,s}u(z) = \left(z\lambda^{(-1)}(z,s)\right) * u(z)$$

where $u \in A, z \in E$ and

$$\left(z\lambda^{(-1)}(z,s)\right) = z + \sum_{\eta=2}^{\infty} \frac{(\mu+1)_{\eta-1}(2\eta-1)^s}{(\eta-1)!} z^{\eta}.$$

A simple computation gives us

$$\mathcal{I}_{\mu,s}u(z) = z + \sum_{\eta=2}^{\infty} \phi(\mu,s,\eta) a_{\eta} z^{\eta} \tag{8}$$

$$\text{where } \phi(\mu,s,\eta) = \frac{(\mu+1)_{\eta-1}(2\eta-1)^s}{(\eta-1)!}. \tag{9}$$

where $(\mu)_\eta$ is the Pochhammer symbol defined in terms of the Gamma function by $(\mu)_\eta = \frac{\Gamma(\mu+\eta)}{\Gamma(\mu)} = \begin{cases} 1, & \text{if } \eta = 0; \\ \mu(\mu+1)\cdots(\mu+\eta-1), & \text{if } \eta \in \mathbb{N} \end{cases}.$

Now, by making use of the linear operator $\mathcal{I}_{\mu,s}u$, we define a new subclass of functions belonging to the class A.

Definition 1.1. *For $-1 \leq \upsilon < 1$ and $\varrho \geq 0$, we let $TS(\upsilon,\varrho,\mu,s)$ be the subclass of A consisting of functions of the form (4) and satisfying the analytic criterion*

$$\Re\left\{\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)} - \upsilon\right\} \geq \varrho\left|\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)} - 1\right|, \tag{10}$$

for $z \in E$.

By suitably specializing the values of μ and s, the class $TS(\upsilon,\varrho,\mu,s)$ can be reduces to the class studied earlier by Ronning [9, 10]. The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, distortion properties, extreme points, radii of starlikeness and convexity, Hadamard product and convolution and integral operators for the class.

2. Coefficient bounds

In this section we obtain a necessary and sufficient condition for function $u(z)$ is in the class $TS(\upsilon,\varrho,\mu,s)$.

We employ the technique adopted by Aqlan et al. [2] to find the coefficient estimates for our class.

Theorem 2.1. *The function u defined by (4) is in the class $TS(\upsilon,\varrho,\mu,s)$ if and only if*

$$\sum_{\eta=2}^{\infty}[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,s,\eta)|a_\eta|\leq 1-\upsilon, \tag{11}$$

where $-1\leq\upsilon<1,\varrho\geq 0$. The result is sharp.

Proof. We have $f\in TS(\upsilon,\varrho,\mu,s)$ if and only if the condition (10) satisfied. Upon the fact that

$$\Re(w)>\varrho|w-1|+\upsilon\Leftrightarrow\Re\{w(1+\varrho e^{i\theta})-\varrho e^{i\theta}\}>\upsilon,\ -\pi\leq\theta\leq\pi.$$

Equation (10) may be written as

$$\Re\left\{\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)}(1+\varrho e^{i\theta})-\varrho e^{i\theta}\right\}=\Re\left\{\frac{z(\mathcal{I}_{\mu,s}u(z))'(1+\varrho e^{i\theta})-\varrho e^{i\theta}\mathcal{I}_{\mu,s}u(z)}{\mathcal{I}_{\mu,s}u(z)}\right\}>\upsilon. \tag{12}$$

Now, we let

$$A(z)=z(\mathcal{I}_{\mu,s}u(z))'(1+\varrho e^{i\theta})-\varrho e^{i\theta}\mathcal{I}_{\mu,s}u(z)$$
$$B(z)=\mathcal{I}_{\mu,s}u(z).$$

Then (12) is equivalent to

$$|A(z)+(1-\upsilon)B(z)|>|A(z)-(1+\upsilon)B(z)|,\ \text{for } 0\leq\upsilon<1.$$

For $A(z)$ and $B(z)$ as above, we have

$$|A(z)+(1-\upsilon)B(z)|\geq(2-\upsilon)|z|-\sum_{\eta=2}^{\infty}[\eta+1-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)|a_\eta||z^\eta|$$

and similarly

$$|A(z)-(1+\upsilon)B(z)|\leq\upsilon|z|-\sum_{\eta=2}^{\infty}[\eta-1-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)|a_\eta||z^\eta|.$$

Therefore

$$|A(z)+(1-\upsilon)B(z)|-|A(z)-(1+\upsilon)B(z)|$$
$$\geq\ 2(1-\upsilon)-2\sum_{\eta=2}^{\infty}[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)|a_\eta|$$

or $\displaystyle\sum_{\eta=2}^{\infty}[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)|a_\eta|\leq(1-\upsilon),$

which yields (11).
On the other hand, we must have

$$\Re\left\{\frac{z(\mathcal{I}_{\mu,s}u(z))'}{\mathcal{I}_{\mu,s}u(z)}(1+\varrho e^{i\theta})-\varrho e^{i\theta}\right\}\geq\upsilon.$$

Upon choosing the values of z on the positive real axis where $0 \le |z| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum\limits_{\eta=2}^{\infty}[\eta-\upsilon+\varrho e^{i\theta}(\eta-1)]\phi(\mu,s,\eta)|a_\eta|\ r^\eta}{z - \sum\limits_{\eta=2}^{\infty}\phi(\mu,s,\eta)|a_\eta|\ r^\eta}\right\} \ge 0.$$

Since $\Re(-e^{i\theta}) \ge -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum\limits_{\eta=2}^{\infty}[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)|a_\eta|\ r^\eta}{z - \sum\limits_{\eta=2}^{\infty}\phi(\mu,s,\eta)|a_\eta|\ r^\eta}\right\} \ge 0.$$

Letting $r \to 1^-$, we get the desired result. Finally the result is sharp with the extremal function u given by

$$u(z) = z - \frac{1-\upsilon}{[\eta(1+\varrho)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}z^\eta \tag{13}$$

□

3. Growth and Distortion Theorems

Theorem 3.1. *Let the function u defined by* (4) *be in the class $TS(\upsilon,\varrho,\mu,s)$. Then for $|z| = r$*

$$r - \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(\mu,s,2)}r^2 \le |u(z)| \le r + \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(\mu,s,2)}r^2. \tag{14}$$

Equality holds for the function

$$u(z) = z - \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(\mu,s,2)}z^2. \tag{15}$$

Proof. We only prove the right hand side inequality in (14), since the other inequality can be justified using similar arguments. In view of Theorem 2.1, we have

$$\sum_{\eta=2}^{\infty}|a_\eta| \le \frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(\mu,s,2)}.. \tag{16}$$

Since ,

$$u(z) = z - \sum_{\eta=2}^{\infty}a_\eta z^\eta$$

$$|u(z)| = \left|z - \sum_{\eta=2}^{\infty}a_\eta z^\eta\right| \le r + \sum_{\eta=2}^{\infty}|a_\eta|r^\eta \le r + r^2\sum_{\eta=2}^{\infty}|a_\eta|$$

$$\le r + \sum_{\eta=2}^{\infty}\frac{1-\upsilon}{(2-\upsilon+\varrho)\phi(\mu,s,2)}r^2$$

which yields the right hand side inequality of (14). □

Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

Theorem 3.2. *Let the function u defined by* (4) *be in the class $TS(\upsilon,\varrho,\mu,s)$. Then for $|z|=r$*

$$1-\frac{2(1-\upsilon)}{(2-\upsilon+\varrho)\phi(\mu,s,2)}r\le|u'(z)|\le1+\frac{2(1-\upsilon)}{(2-\upsilon+\varrho)\phi(\mu,s,2)}r.$$

Equality holds for the function given by (15).

Proof. Since $f\in TS(\upsilon,\varrho,\mu,s)$ by Theorem 2.1, we have that

$$[2(1+\varrho)-(\upsilon+\varrho)]\,\phi(\mu,s,2)\sum_{\eta=2}^{\infty}\eta a_\eta\le[\eta(1+\varrho)-(\upsilon+\varrho)]\,\phi(\mu,s,\eta)|a_\eta|\le1-\upsilon$$

or

$$\sum_{\eta=2}^{\infty}\eta|a_\eta|\le\frac{2(1-\upsilon)}{(2-\upsilon+\varrho)\phi(\mu,s,2)}\ .$$

Thus from (16), we obtain

$$\begin{aligned}|u'(z)|&\le1+r\sum_{\eta=2}^{\infty}\eta|a_\eta|\\&\le1+\frac{2(1-\upsilon)}{(2-\upsilon+\varrho)\phi(\mu,s,2)}r\end{aligned}$$

which is right hand inequality of Theorem 3.2.

On the other hand, similarly

$$|u'(z)|\ge1-\frac{2(1-\upsilon)}{(2-\upsilon+\varrho)\phi(\mu,s,2)}r$$

and thus proof is completed. □

Theorem 3.3. *If $u\in TS(\upsilon,\varrho,\mu,s)$ then $u\in TS(\gamma)$, where*

$$\gamma=1-\frac{(\eta-1)(1-\upsilon)}{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)-(1-\upsilon)}.$$

Equality holds for the function given by (15).

Proof. It is sufficient to show that (11) implies

$$\sum_{\eta=2}^{\infty}(\eta-\gamma)|a_\eta|\le1-\gamma,$$

that is

$$\frac{\eta-\gamma}{1-\gamma}\le\frac{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}{(1-\upsilon)},$$

then

$$\gamma\le1-\frac{(\eta-1)(1-\upsilon)}{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)-(1-\upsilon)}.$$

The above inequality holds true for $\eta\in\mathbb{N}_0,\eta\ge2,\varrho\ge0$ and $0\le\upsilon<1$. □

4. Extreme points

Theorem 4.1. *Let* $u_1(z) = z$ *and*

$$u_\eta(z) = z - \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)} z^\eta, \tag{17}$$

for $\eta = 2, 3, \cdots$. *Then* $u(z) \in TS(\upsilon, \varrho, \mu, s)$ *if and only if* $u(z)$ *can be expressed in the form* $u(z) = \sum\limits_{\eta=1}^{\infty} \zeta_\eta u_\eta(z)$, *where* $\zeta_\eta \geq 0$ *and* $\sum\limits_{\eta=1}^{\infty} \zeta_\eta = 1$.

Proof. Suppose $u(z)$ can be expressed as in (17). Then

$$\begin{aligned} u(z) &= \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z) = \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) \\ &= \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ z - \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)} z^\eta \right\} \\ &= \zeta_1 z + \sum_{\eta=2}^{\infty} \zeta_\eta z - \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)} z^\eta \right\} \\ &= z - \sum_{\eta=2}^{\infty} \zeta_\eta \left\{ \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)} z^\eta \right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{\eta=2}^{\infty} \zeta_\eta \left(\frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)} \right) \left(\frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon} \right) \\ &= \sum_{\eta=2}^{\infty} \zeta_\eta = \sum_{\eta=1}^{\infty} \zeta_\eta - \zeta_1 = 1 - \zeta_1 \leq 1. \end{aligned}$$

So, by Theorem 2.1, $u \in TS(\upsilon, \varrho, \mu, s)$.

Conversely, we suppose $u \in TS(\upsilon, \varrho, \mu, s)$. Since

$$|a_\eta| \leq \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}, \ \eta \geq 2.$$

We may set

$$\zeta_\eta = \frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon} |a_\eta|, \ \eta \geq 2$$

and $\zeta_1 = 1 - \sum\limits_{\eta=2}^{\infty} \zeta_\eta$. Then □

$$\begin{aligned} u(z) &= z - \sum_{\eta=2}^{\infty} a_\eta z^\eta = z - \sum_{\eta=2}^{\infty} \zeta_\eta \frac{1-\upsilon}{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)} z^\eta \\ &= z - \sum_{\eta=2}^{\infty} \zeta_\eta [z - u_\eta(z)] = z - \sum_{\eta=2}^{\infty} \zeta_\eta z + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) \\ &= \zeta_1 u_1(z) + \sum_{\eta=2}^{\infty} \zeta_\eta u_\eta(z) = \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z). \end{aligned}$$

Corollary 4.1. *The extreme points of* $TS(\upsilon, \varrho, \mu, s)$ *are the functions* $u_1(z) = z$ *and*

$$u_\eta(z) = z - \frac{1-\upsilon}{[\eta(\varrho+1) - (\upsilon+\varrho)]\phi(\mu, s, \eta)} z^\eta, \ \eta \geq 2.$$

5. Radii of Close-to-convexity, Starlikeness and Convexity

A function $u \in TS(\upsilon, \varrho, \mu, s)$ is said to be close-to-convex of order δ if it satisfies

$$\Re\{u'(z)\} > \delta, \ (0 \leq \delta < 1; \ z \in E).$$

Also A function $u \in TS(\upsilon, \varrho, \mu, s)$ is said to be starlike of order δ if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta, \ (0 \leq \delta < 1; \ z \in E).$$

Further a function $u \in TS(\upsilon, \varrho, \mu, s)$ is said to be convex of order δ if and only if $zu'(z)$ is starlike of order δ that is if

$$\Re\left\{1 + \frac{zu'(z)}{u(z)}\right\} > \delta, \ (0 \leq \delta < 1; \ z \in E).$$

Theorem 5.1. *Let* $u \in TS(\upsilon, \varrho, \mu, s)$. *Then* u *is close-to-convex of order* δ *in* $|z| < R_1$, *where*

$$R_1 = \inf_{k \geq 2} \left[\frac{(1-\delta)[\eta - \upsilon + \varrho(\eta-1)]\phi(\mu, s, \eta)}{\eta(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u *is given by* (13).

Proof. It is sufficient to show that $|u'(z) - 1| \leq 1 - \delta$, for $|z| < R_1$. We have

$$|u'(z) - 1| = \left| -\sum_{\eta=2}^{\infty} \eta a_\eta z^{\eta-1} \right| \leq \sum_{\eta=2}^{\infty} \eta a_\eta |z|^{\eta-1}.$$

Thus $|u'(z) - 1| \leq 1 - \delta$ if

$$\sum_{\eta=2}^{\infty} \frac{\eta}{1-\delta} |a_\eta| |z|^{\eta-1} \leq 1. \tag{18}$$

But Theorem 2.1 confirms that

$$\sum_{\eta=2}^{\infty} \frac{[\eta(\varrho+1) - (\upsilon+\varrho)]\phi(\mu, s, \eta)}{1-\upsilon} |a_\eta| \leq 1. \tag{19}$$

Hence (18) will be true if

$$\frac{\eta |z|^{\eta-1}}{1-\delta} \leq \frac{[\eta(\varrho+1) - (\upsilon+\varrho)]\phi(\mu, s, \eta)}{1-\upsilon}.$$

We obtain

$$|z| \leq \left[\frac{(1-\delta)[\eta - \upsilon + \varrho(\eta-1)]\phi(\mu, s, \eta)}{\eta(1-\upsilon)}\right]^{\frac{1}{\eta-1}}, \eta \geq 2$$

as required. □

Theorem 5.2. *Let* $u \in TS(\upsilon, \varrho, \mu, s)$. *Then* u *is starlike of order* δ *in* $|z| < R_2$, *where*

$$R_2 = \inf_{k \geq 2} \left[\frac{(1-\delta)[\eta - \upsilon + \varrho(\eta-1)]\phi(\mu, s, \eta)}{(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u *is given by* (13).

Proof. We must show that $\left|\frac{zu'(z)}{u(z)} - 1\right| \leq 1-\delta$, for $|z| < R_2$. We have

$$\begin{aligned}\left|\frac{zu'(z)}{u(z)} - 1\right| &= \left|\frac{-\sum\limits_{\eta=2}^{\infty}(\eta-1)a_\eta z^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}a_\eta z^{\eta-1}}\right| \\ &\leq \frac{\sum\limits_{\eta=2}^{\infty}(\eta-1)|a_\eta||z|^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}|a_\eta||z|^{\eta-1}} \\ &\leq 1-\delta. \end{aligned} \tag{20}$$

Hence (20) holds true if

$$\sum_{\eta=2}^{\infty}(\eta-1)|a_\eta||z|^{\eta-1} \leq (1-\delta)\left(1-\sum_{\eta=2}^{\infty}|a_\eta||z|^{\eta-1}\right)$$

or equivalently,

$$\sum_{\eta=2}^{\infty}\frac{\eta-\delta}{1-\delta}|a_\eta||z|^{\eta-1} \leq 1. \tag{21}$$

Hence, by using (19) and (21) will be true if

$$\frac{\eta-\delta}{1-\delta}|z|^{\eta-1} \leq \frac{[\eta(\varrho+1)-(\upsilon+\varrho)]\phi(\mu,s,\eta)}{1-\upsilon}$$

$$\Rightarrow |z| \leq \left[\frac{(1-\delta)[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}{(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}, \eta \geq 2$$

which completes the proof. □

By using the same technique in the proof of Theorem 5.2, we can show that $\left|\frac{zu''(z)}{u'(z)} - 1\right| \leq 1-\delta$, for $|z| < R_3$, with the aid of Theorem 2.1. Thus we have the assertion of the following Theorem 5.3.

Theorem 5.3. *Let $u \in TS(\upsilon,\varrho,\mu,s)$. Then u is convex of order δ in $|z| < R_3$, where*

$$R_3 = \inf_{k\geq 2}\left[\frac{(1-\delta)[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}{\eta(\eta-\delta)(1-\upsilon)}\right]^{\frac{1}{\eta-1}}.$$

The result is sharp with the extremal function u is given by (13).

6. Inclusion theorem involving modified Hadamard products

For functions

$$u_j(z) = z - \sum_{\eta=2}^{\infty}|a_{\eta,j}|z^\eta,\ j=1,2 \tag{22}$$

in the class A, we define the modified Hadamard product $u_1 * u_2(z)$ of $u_1(z)$ and $u_2(z)$ given by

$$u_1 * u_2(z) = z - \sum_{\eta=2}^{\infty}|a_{\eta,1}||a_{\eta,2}|z^\eta.$$

We can prove the following.

Theorem 6.1. *Let the function* $u_j,\ j=1,2,$ *given by* (22) *be in the class* $TS(\upsilon,\varrho,\mu,s)$ *respectively. Then* $u_1 * u_2(z) \in TS(\upsilon,\varrho,\mu,s,\xi)$, *where*

$$\xi = 1 - \frac{(1-\upsilon)^2}{(\eta+1)(2-\upsilon)(2-\upsilon+\varrho)(1+\lambda)-(1-\upsilon)^2}.$$

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest ξ such that

$$\sum_{\eta=2}^{\infty} \frac{[\eta-\xi+\varrho(\eta-1)]\phi(\mu,s,\eta)}{1-\xi}|a_{\eta,1}||a_{\eta,2}| \leq 1.$$

Since $u_j \in TS(\upsilon,\varrho,\mu,s),\ j=1,2$ then we have

$$\sum_{\eta=2}^{\infty} \frac{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}{1-\upsilon}|a_{\eta,1}| \leq 1$$

$$\text{and } \sum_{\eta=2}^{\infty} \frac{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}{1-\upsilon}|a_{\eta,2}| \leq 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{\eta=2}^{\infty} \frac{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}{1-\upsilon}\sqrt{|a_{\eta,1}||a_{\eta,2}|} \leq 1.$$

Thus it is sufficient to show that

$$\frac{[\eta-\xi+\varrho(\eta-1)]\phi(\mu,s,\eta)}{1-\xi}|a_{\eta,1}||a_{\eta,2}| \leq \frac{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}{1-\upsilon}\sqrt{|a_{\eta,1}||a_{\eta,2}|},\ \eta \geq 2,$$

that is

$$\sqrt{|a_{\eta,1}||a_{\eta,2}|} \leq \frac{(1-\xi)[\eta-\upsilon+\varrho(\eta-1)]}{1-\upsilon)[\eta-\xi+\varrho(\eta-1)]}.$$

Note that

$$\sqrt{|a_{\eta,1}||a_{\eta,2}|} \leq \frac{(1-\upsilon)}{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)}.$$

Consequently, we need only to prove that

$$\frac{(1-\upsilon)}{[\eta-\upsilon+\varrho(\eta-1)]\phi(\mu,s,\eta)} \leq \frac{(1-\xi)[\eta-\upsilon+\varrho(\eta-1)]}{1-\upsilon)[\eta-\xi+\varrho(\eta-1)]},\ \eta \geq 2,$$

or, equivalently, that

$$\xi \leq 1 - \frac{(\eta-1)(1+\varrho)(1-\upsilon)^2}{[\eta-\upsilon+\varrho(\eta-1)]^2\phi(\mu,s,\eta)-(1-\upsilon)^2},\ \eta \geq 2.$$

Since

$$A(k) = 1 - \frac{(\eta-1)(1+\varrho)(1-\upsilon)^2}{[\eta-\upsilon+\varrho(\eta-1)]^2\phi(\mu,s,\eta)-(1-\upsilon)^2},\ \eta \geq 2$$

is an increasing function of $\eta, \eta \geq 2$, letting $\eta = 2$ in last equation, we obtain

$$\xi \leq A(2) = 1 - \frac{(1+\varrho)(1-\upsilon)^2}{[2-\upsilon+\varrho]^2\phi(\mu,s,\eta)-(1-\upsilon)^2}.$$

Finally, by taking the function given by (15), we can see that the result is sharp. □

7. Convolution and Integral Operators

Let $u(z)$ be defined by (4) and suppose that $g(z) = z - \sum_{\eta=2}^{\infty} |b_\eta| z^\eta$. Then,the Hadamard product (or convolution) of $u(z)$ and $g(z)$ defined here by

$$u(z) * g(z) = u * g(z) = z - \sum_{\eta=2}^{\infty} |a_\eta||b_\eta| z^\eta.$$

Theorem 7.1. *Let $u \in TS(\upsilon, \varrho, \mu, s)$ and $g(z) = z - \sum_{\eta=2}^{\infty} |b_\eta| z^\eta, 0 \leq |b_\eta| \leq 1$. Then $u * g \in TS(\upsilon, \varrho, \mu, s)$.*

Proof. In view of Theorem 2.1, we have

$$\begin{aligned}
&\sum_{\eta=2}^{\infty} [\eta - \upsilon + \varrho(\eta - 1)] \phi(\mu, s, \eta) |a_\eta||b_\eta| \\
\leq &\sum_{\eta=2}^{\infty} [\eta - \upsilon + \varrho(\eta - 1)] \phi(\mu, s, \eta) |a_\eta| \\
\leq &(1 - \upsilon).
\end{aligned}$$

□

Theorem 7.2. *Let $u \in TS(\upsilon, \varrho, \mu, s)$ and α be real number such that $\alpha > -1$. Then the function $F(z) = \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} u(t) dt$ also belongs to the class $TS(\upsilon, \varrho, \mu, s)$.*

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{\eta=2}^{\infty} |A_\eta| z^\eta, \text{ where } A_\eta = \left(\frac{\alpha+1}{\alpha+\eta}\right) |a_\eta|.$$

Since $\alpha > -1$, than $0 \leq A_\eta \leq |a_\eta|$. Which in view of Theorem 2.1, $F \in TS(\upsilon, \varrho, \mu, s)$. □

Acknowledgement. I warmly thank the referees for the careful reading of the paper and their comments.

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Venkateswarlu Bolineni for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.10, N.2.

Pinninti Thirupati Reddy for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.10, N.2.

Swapna Galla has received her master's degree from S. V. University in Mathematics, Andhra Pradesh. She received her M .Phil degree from VELS University in Chennai. Currently, she is a Ph.D. full-time research scholar in the department of Mathematics at GITAM University, Bengaluru under the guidance of Dr. B. Venkateswarlu. Her area of interest in research is Complex Analysis.